2-LARGE SETS, NILRECURRENCE, AND POLYNOMIAL APPROXIMATION

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ABSTRACT. A set of positive integers is 2-large if every two-coloring contains arbitrarily long monochromatic arithmetic progressions with common difference in that set. We prove that every 2-large set is a set of multiple pointwise recurrence for all nilsystems. This proves a conjecture of Alweiss and answers a question of Host, Kra, and Maass with two colors. We also obtain a finite quantitative theorem: the two-color hypothesis at a single progression length k forces simultaneous approximations $\|P_i(s)\|_{\mathbb{R}/\mathbb{Z}} \ll_{m,D} k^{-d_i}$ for special generalized polynomials P_i of weights $1 \le d_i \le D$, with s an allowed difference. The constants are uniform in their real coefficients. Both results follow from a structural theorem for monochromatic polynomial orbits. A fixed two-coloring forces every sufficiently long such orbit to have only a smooth factor and a rational factor with bounded coset period. The geometric input is a coloring of rational subnilmanifolds with logarithmic dependence on their complexity.

1. INTRODUCTION

1.1. **Largeness and nilrecurrence.** Van der Waerden's theorem [9] says that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. Brown, Graham, and Landman [3] asked what happens when the common difference is required to belong to a prescribed set.

Definition 1.1. Let $r \ge 2$ and $k \ge 1$. A set $S \subseteq \mathbb{N}$ is (r,k)-*large* if every r-coloring of $\mathbb{N}$ has a monochromatic k-term arithmetic progression with common difference in S. It is r-*large* if this holds for every k, and *large* if it is r-large for every $r \ge 2$.

Thus S is large exactly when van der Waerden's theorem remains true with all common differences restricted to S.

Conjecture 1.2 (Brown–Graham–Landman [3])**.** *Every* 2*-large set is large.*

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It is useful to think about the contrapositive. If some finite coloring blocks long progressions with differences in S, must there also be a two-coloring that does so? A basic class of obstructions comes from rotations on tori. Host, Kra, and Maass [7, Lemma 7.4] observed that failure of one-dimensional Bohr recurrence already obstructs 2-largeness: if some $\alpha \in \mathbb{R}$ and $\delta > 0$ satisfy $\|\alpha s\|_{\mathbb{R}/\mathbb{Z}} > \delta$ for every $s \in S$, then S is not 2-large. Farhangi and Grytczuk [4, Theorem 2] subsequently gave a quantitative formulation in terms of the loneliness constant. The first author [1] proved that every 2-large set is Bohr recurrent, extending the two-color obstruction to simultaneous linear conditions in arbitrary dimension.

We prove the corresponding result for translations on nilmanifolds. A *nilsystem* is a compact homogeneous space $X = G/\Gamma$, where G is a nilpotent Lie group and Γ is a discrete cocompact subgroup, together with a translation $T : x \mapsto gx$. The group need not be connected or simply connected. Nilsystems are central to the study of multiple recurrence [6], and their return-time sets admit a description in terms of generalized polynomials [8].

Theorem 1.3 (Nilrecurrence)**.** *Let $S \subseteq \mathbb{N}$ be 2-large. For every nilsystem (X, T), every $x \in X$, every $\ell \geq 1$, and every $\eta > 0$, there is $s \in S$ such that*

$$\max_{1 \leq j \leq \ell} d_X(T^{js}x, x) < \eta. \tag{1.1}$$

In particular, S is a set of multiple pointwise recurrence for nilsystems.

This proves Conjecture 3.6 of [2]. It also answers Question 7.8 of Host, Kra, and Maass [7]: if (X, T) is a minimal nilsystem and U is a nonempty open set with $x_0 \notin \overline{U}$, then

$$\{n \in \mathbb{N} : T^n x_0 \in U\}$$

is not 2-large. We give the short deduction in Corollary 8.1.

The distinction between pointwise and open-set recurrence matters here. Host, Kra, and Maass proved that Bohr recurrence implies ordinary open-set recurrence for minimal nilsystems [7, Theorem 4.1]. Pointwise recurrence is strictly stronger, even in this class [7, Corollary 5.10]. It is pointwise recurrence that is equivalent to meeting every nil-Bohr neighborhood of the appropriate step, and that implies multiple pointwise recurrence [7, Corollary 5.3]. Theorem 1.3 establishes this higher-order conclusion.

Two consequences are worth recording. First, the exponents $s, 2s, \ldots$ in (1.1) can be replaced by any finite family of integer polynomials vanishing at zero; this follows from the standard lifting of polynomial orbits to linear nilorbits (Corollary 8.2). Second, if B is any finite-step nil-Bohr$_0$ set, then

$$S \text{ is } 2r\text{-large} \quad \Longrightarrow \quad S \cap B \text{ is } r\text{-large} \qquad (r \geq 2).$$

The color cost is independent of the dimension and nilpotency step (Corollary 8.3).

1.2. **Finite generalized-polynomial approximation.** The proof also gives a quantitative statement under the hypothesis at a single progression length. We use the nearest-integer convention of Huang, Shao, and Ye [8]: $\langle t\rangle$ is a nearest integer to t, with ties resolved toward the smaller integer. Put

$$\mathsf{L}(t) = t, \qquad \mathsf{L}(t_1, \dots, t_r) = t_1\langle \mathsf{L}(t_2, \dots, t_r)\rangle.$$

A *special generalized polynomial* is an expression $\mathsf{L}(\alpha_1 n^{j_1}, \dots, \alpha_r n^{j_r})$ with $j_1, \dots, j_r \geq 1$. Its *weight* is $j_1 + \cdots + j_r$. We retain this assigned weight even if a special choice of coefficients simplifies the expression. The expressions of weight at most D are the class SGP_D of [8, Definition 4.1.2].

Theorem 1.4 (Finite generalized-polynomial approximation)**.** *For every pair of positive integers m, D there are constants $C = C(m, D)$ and $k_0 = k_0(m, D)$ with the following property. Let $k \geq k_0$, let $S \subseteq \mathbb{N}$ be $(2, k)$-large, and let $P_1, \dots, P_m$ be special generalized polynomials of respective weights $1 \leq d_1, \dots, d_m \leq D$. Then some $s \in S$ satisfies*

$$\|P_i(s)\|_{\mathbb{R}/\mathbb{Z}} \leq Ck^{-d_i} \qquad (1 \leq i \leq m). \tag{1.2}$$

More strongly, for each prescribed family $P_1, \dots, P_m$ there is a single two-coloring of $\mathbb{N}$ such that the difference of every monochromatic k-term progression satisfies (1.2), simultaneously for all $k \geq k_0$.

For instance, one can force

$$\|\alpha s\|_{\mathbb{R}/\mathbb{Z}} \leq Ck^{-1}, \qquad \|\beta s^2\|_{\mathbb{R}/\mathbb{Z}} \leq Ck^{-2}, \qquad \|\gamma s\langle \delta s\rangle\|_{\mathbb{R}/\mathbb{Z}} \leq Ck^{-2}$$

using only two colors. The constants are uniform in the real coefficients and in S. The conclusion bounds the approximation error; it gives no bound on the size of s.

The scalar linear obstruction is due to Host, Kra, and Maass [7, Lemma 7.4]; the later quantitative formulation of Farhangi and Grytczuk [4, Theorem 2] gives the $O(k^{-1})$ approximation in this case. Host, Kra, and Maass also gave finite-color obstructions for $\{n : \|\alpha n^d\|_{\mathbb{R}/\mathbb{Z}} > \delta\}$, with the number of colors depending on δ [7, Proposition 7.9]. Theorem 1.4 gives simultaneous weight-dependent rates for nested expressions with the number of colors fixed at two. Its restriction to ordinary monomials gives $\|\alpha_i s^{d_i}\|_{\mathbb{R}/\mathbb{Z}} \leq Ck^{-d_i}$; its qualitative linear consequence recovers the Bohr recurrence theorem of [1]. For general normalized generalized polynomials we obtain simultaneous qualitative approximation and a finite threshold depending on the expressions and the tolerance (Corollary 8.4). A bound depending only on degree cannot hold for arbitrary expressions, as Proposition 8.5 shows.

1.3. **Monochromatic polynomial orbits.** The common input to the two main results is a structural theorem that does not refer to a set of differences. Fix a nilmanifold with a connected, simply connected presenting group, a rational filtration, and an adapted Mal'cev basis. We construct one Borel two-coloring of the nilmanifold such that every sufficiently long monochromatic

polynomial orbit has a factorization

$$p(n) = \varepsilon(n)\gamma(n).$$

Here ε is smooth on the scale of the orbit length and $\gamma(n)\Gamma$ has bounded period. The bounds are independent of the polynomial, its coefficients, its initial point, and the length of the orbit. The precise statement is Theorem 5.1.

To prove this, apply the Green–Tao factorization theorem to p:

$$p = \varepsilon g'\gamma.$$

The middle factor is totally equidistributed on a rational subnilmanifold. Our coloring makes the full orbit visit both colors whenever that subnilmanifold has positive dimension. The proof must account for the rational conjugation caused by γ and for the varying intrinsic metric on the subgroup. Lemmas 4.1 and 4.2 make these losses explicit. Factoring the full translated sequence is essential: a large left translation can magnify a smooth error. We illustrate this in Section 5.1.

The geometric construction is a probabilistic coloring of a fine Voronoi partition. Translated rational subnilmanifolds of complexity at most Q have a Hausdorff δ-net of size $Q^{O_X(1)}\delta^{-O_X(1)}$. Each positive-dimensional model contains $\gg_X \delta^{-1}$ separated test regions. A random coloring fails to supply a robust ball of either color with probability at most $\exp(-c_X/\delta)$. A union bound gives monochromatic balls of both colors, of radius $c_X/(1+\log Q)$, on every relevant translate (Theorem 3.2).

Once the structural theorem is available, the two applications separate. For generalized-polynomial approximation, we work in weighted upper-unitriangular groups. Taking the highest coefficient allowed by each matrix-entry weight removes the initial point and makes the smooth factor small at the corresponding power of the orbit length. A fixed dilation clears the rational denominators, and integer matrix reduction recovers the nested nearest-integer expressions. For nilrecurrence, a realization theorem of Huang, Shao, and Ye converts failure of recurrence into uniform separation of short orbit segments. Taking a root of the translating element clears the same bounded period, contradicting the slow motion of a monochromatic orbit.

Sections 2–5 establish the structural theorem. Section 6 proves the finite approximation theorem, Section 7 proves nilrecurrence, and Section 8 gives the consequences. The results establish the nilsystem case of the Brown–Graham–Landman conjecture. They do not settle the conjecture in full: nil-Bohr recurrence alone does not imply topological multiple recurrence [2].

2. Quantitative preliminaries

We write $\mathbb{N} = \{1, 2, \ldots\}$, $[N] = \{1, \ldots, N\}$, and $\|t\|_{\mathbb{R}/\mathbb{Z}} = \min_{a\in\mathbb{Z}} |t-a|$. Coordinate norms are sup norms. All progressions in $[N]$ are nonempty finite arithmetic progressions contained in $[N]$.

2.1. **Filtered nilmanifolds and complexity.** In Sections 2–6, a fixed filtered nilmanifold consists of a connected, simply connected nilpotent Lie group G, a lattice Γ, a filtration

$$G = G_0 = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_{D+1} = \{\mathrm{id}\}$$

by connected rational subgroups satisfying $[G_i, G_j] \subseteq G_{i+j}$, and a rational Mal'cev basis $\mathcal{X} = (X_1, \dots, X_m)$ adapted to the filtration. Write $X = G/\Gamma$. Constants denoted $O_X(1)$ are allowed to depend on all these fixed data, including the filtration degree and the basis. No constant in this notation depends on the coefficients of a polynomial sequence or its initial point.

The Mal'cev coordinate map is defined by

$$g = \exp(t_1 X_1) \cdots \exp(t_m X_m) \quad \Longleftrightarrow \quad \psi_{\mathcal{X}}(g) = (t_1, \dots, t_m).$$

We use the right-invariant metric $d_G = d_{\mathcal{X}}$ and its quotient metric d_X, as in [5, Definitions 2.1 and 2.2]. In particular,

$$d_X(g\Gamma, h\Gamma) \le d_G(g, h), \qquad d_G(gu, hu) = d_G(g, h). \tag{2.1}$$

The coordinate maps, group operations, and changes between controlled rational bases have polynomial bounds on bounded coordinate boxes; we use the precise estimates in [5, Appendix A].

A connected subgroup is *Q-rational* if its Lie algebra has a basis of rational $\mathcal{X}$-combinations whose coefficients have height at most Q. Here the height of $a/b \in \mathbb{Q}$, in lowest terms, is $\max(|a|, |b|)$. Such a subgroup is rational in the usual sense: $H \cap \Gamma$ is a lattice in H. Write $\mathrm{comp}(H)$ for the least $Q \ge 2$ for which it is Q-rational. A Mal'cev basis for $H/(H \cap \Gamma)$ has *complexity at most Q* if it is adapted to $H_i = H \cap G_i$ and each of its vectors is a rational $\mathcal{X}$-combination of height at most Q. We say that a translated rational subnilmanifold Z has complexity at most Q if it has a presentation

$$Z = xH\Gamma/\Gamma, \qquad \mathrm{comp}(H) \le Q, \qquad |\psi_{\mathcal{X}}(x)| \le Q.$$

An element $u \in G$ is *Q-rational* if $u^q \in \Gamma$ for some integer $1 \le q \le Q$. A point of G/Γ is Q-rational if it has a Q-rational representative. A sequence $\gamma : \mathbb{Z} \to G$ is Q-rational if every point $\gamma(n)\Gamma$ is Q-rational, as in [5, Definition 1.17]. Lemma A.11 there relates rational elements to rational Mal'cev coordinates, with polynomial losses. Proposition A.10 supplies adapted Mal'cev bases for rational subgroups with the same type of losses. Intersections $H \cap G_i$ are connected and rational: the exponential map identifies them with intersections of rational Lie algebras. Rational linear algebra also controls their complexity polynomially in $\mathrm{comp}(H)$.

We will repeatedly reduce a rational coset to the fixed Mal'cev fundamental domain. If u is the representative of an M-rational coset with $\psi_{\mathcal{X}}(u) \in [0,1)^m$, then u is $M^{O_X(1)}$-rational. To see the uniformity of this assertion, use Lemma A.11 to bound the denominators of the coordinates of a rational representative. The triangular coordinate reduction in Lemma A.14, together with the polynomial product law, preserves a common denominator of size

$M^{O_X(1)}$. The reduced coordinates also have absolute value at most one, so a second application of Lemma A.11 proves the claim.

2.2. **Polynomial sequences and equidistribution.** For $h \in \mathbb{Z}$, put $\partial_h p(n) = p(n+h)p(n)^{-1}$. A sequence $p : \mathbb{Z} \to G$ belongs to $\mathrm{poly}(\mathbb{Z}, G_\bullet)$ if every j-fold discrete derivative takes values in G_j, for every $j \geq 0$ [5, Definition 1.8]. The sequences $n \mapsto a^n g$ belong to this class for the lower central series filtration. For the filtration $G_0 = \cdots = G_D = \mathbb{R}^m$, $G_{D+1} = \{0\}$, they are exactly the vector-valued real polynomials of degree at most D.

For a compact metric space Z and $F : Z \to \mathbb{C}$, set

$$\|F\|_{\mathrm{Lip}} = \|F\|_\infty + \sup_{z \neq z'} \frac{|F(z) - F(z')|}{d_Z(z, z')}.$$

A sequence $(z_n)_{n=1}^N$ in a compact nilmanifold Z is δ-equidistributed if

$$\left| \frac{1}{N} \sum_{n=1}^{N} F(z_n) - \int_Z F \, dm_Z \right| \leq \delta \|F\|_{\mathrm{Lip}}$$

for every Lipschitz F, where m_Z is Haar probability measure. It is *totally δ-equidistributed* if this inequality holds on every progression $P \subseteq [N]$ with $|P| \geq \delta N$, with $|P|^{-1} \sum_{n \in P}$ in place of the full average.

A sequence $\varepsilon : \mathbb{Z} \to G$ is (M, N)-*smooth* if

$$d_G(\varepsilon(n), \mathrm{id}) \leq M, \qquad d_G(\varepsilon(n), \varepsilon(n-1)) \leq M/N \quad (n \in [N]). \tag{2.2}$$

These are precisely the index conventions of [5, Definition 1.18]. In particular, we will freeze smooth factors at index 1.

For a polynomial $f(n) = \sum_{j=0}^{D} \alpha_j \binom{n}{j}$ with values in $\mathbb{R}/\mathbb{Z}$, write

$$\|f\|_{C^\infty[N]} = \max_{1 \leq j \leq D} N^j \|\alpha_j\|_{\mathbb{R}/\mathbb{Z}}.$$

A *horizontal character* of $H/(H \cap \Gamma)$ is a continuous homomorphism $\chi : H \to \mathbb{R}/\mathbb{Z}$ that vanishes on $H \cap \Gamma$. Relative to a Mal'cev basis, it has the form $\chi(h) = k \cdot \psi_H(h) \bmod 1$ for an integer row vector k. Its modulus $|\chi|$ is the Euclidean norm of that vector.

2.3. **Factorization.** We use the following form of the Green–Tao theorem.

Theorem 2.1 (Green–Tao factorization)**.** *Fix $A > 0$. There are constants B, C depending only on A and the fixed filtered nilmanifold such that the following holds. For every sufficiently large K, every $N \geq 2$, and every $p \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$, there is an integer*

$$K \leq M \leq CK^B$$

and a factorization

$$p(n) = \varepsilon(n) g'(n) \gamma(n) \qquad (n \in \mathbb{Z})$$

into members of $\mathrm{poly}(\mathbb{Z}, G_\bullet)$ with the following properties.

(i) ε is (M, N)-smooth.

(ii) *There is a connected rational subgroup $H \le G$ such that $g' \in \mathrm{poly}(\mathbb{Z}, H_\bullet)$, where $H_i = H \cap G_i$. The nilmanifold $H/(H\cap\Gamma)$ has an adapted Mal'cev basis $\mathcal{X}_H$ of complexity at most M, and $(g'(n)(H\cap\Gamma))_{n=1}^N$ is totally M^{-A}-equidistributed in its associated metric.*

(iii) *γ is M-rational, and $\gamma(n)\Gamma$ is periodic with some period $q \le M$.*

This follows from [5, Theorem 1.19], with the adapted basis provided by the construction in its Section 10 and Proposition A.10. The filtration is fixed but need not be the lower central series.

3. Coloring rational subnilmanifolds

Throughout this section only the nilmanifold $X = G/\Gamma$ and its fixed Mal'cev basis enter the constants. We assume $\dim X > 0$. For $Q \ge 2$, let $\mathcal{F}_Q$ be the family of all sets

$$Z = xH\Gamma/\Gamma, \qquad \dim H > 0, \qquad \mathrm{comp}(H) \le Q, \qquad |\psi_{\mathcal{X}}(x)| \le Q,$$

where H is connected and rational. Hausdorff distances are taken in X.

Lemma 3.1 (Size of the family)**.** *There are constants $A, B, C, \delta_X > 0$, depending only on X, such that $\mathcal{F}_Q$ has a Hausdorff δ-net of cardinality at most*

$$CQ^A\delta^{-B} \qquad (Q \ge 2,\ 0 < \delta \le \delta_X), \tag{3.1}$$

whose members belong to $\mathcal{F}_Q$. Moreover, for every fixed $T > 0$ there are $a_T, \delta_T > 0$, depending only on X, T, such that every positive-dimensional connected translated rational subnilmanifold contains at least a_T/δ points at pairwise distance at least $T\delta$, whenever $0 < \delta \le \delta_T$.

Proof. A Q-rational subgroup has a basis for its Lie algebra consisting of at most $m = \dim G$ rational vectors of height at most Q. There are only $Q^{O_X(1)}$ such lists. Each list determines its linear span, and a connected subgroup of the simply connected nilpotent group G is determined by its Lie algebra. Thus there are at most $Q^{O_X(1)}$ possible subgroups.

For fixed H, right invariance gives the useful estimate

$$d_{\mathrm{Haus}}(xH\Gamma/\Gamma, yH\Gamma/\Gamma) \le d_G(x, y). \tag{3.2}$$

Indeed, pair $xh\Gamma$ with $yh\Gamma$ for every $h \in H$ and use (2.1); the same pairing works in the other direction. On the coordinate box $|\psi_{\mathcal{X}}(x)| \le Q$, [5, Lemma A.4] gives

$$d_G(x, y) \le C_1 Q^{C_2} |\psi_{\mathcal{X}}(x) - \psi_{\mathcal{X}}(y)|,$$

where C_1, C_2 depend only on X. A rectangular coordinate grid of mesh at most $\delta/(C_1Q^{C_2})$, with points chosen in $[-Q, Q]^m$, therefore yields the required Hausdorff net for these translates. The number of grid points is at most $C_3Q^{(C_2+1)m}\delta^{-m}$. Taking the union over the possible subgroups proves (3.1).

For the second assertion, there is a uniform $r_X > 0$ such that every positive-dimensional connected translated rational subnilmanifold $Z = xH\Gamma/\Gamma$ has diameter at least r_X. Choose a nonidentity $\lambda \in H \cap \Gamma$. The loop

$$t \longmapsto x \exp(t \log \lambda)\Gamma, \qquad 0 \le t \le 1,$$

lifts in the universal cover G to a path from x to $x\lambda$. It is therefore not null-homotopic. By compactness, every sufficiently small metric ball in X is contained in a contractible coordinate neighborhood. The loop, and hence Z, cannot have arbitrarily small diameter.

Choose $z, z' \in Z$ with $d_X(z, z') \ge r_X$. Since Z is connected, the image of $w \mapsto d_X(z, w)$ contains $[0, r_X]$. Choose points at distance levels spaced by $T\delta$. The reverse triangle inequality makes these points pairwise $T\delta$-separated, proving the assertion. □

Theorem 3.2 (Logarithmic robust coloring)**.** *There is $c_X > 0$ such that for every $Q \ge 2$ there is a Borel two-coloring $\kappa_Q : X \to \{0, 1\}$ with the following property: every $Z \in \mathcal{F}_Q$ contains points z_0, z_1 such that*

$$\kappa_Q \equiv i \quad \text{on } B_{c_X/(1+\log Q)}(z_i), \qquad i = 0, 1. \tag{3.3}$$

Proof. Fix a small constant $\eta > 0$. For $0 < \delta \ll_X 1$, choose a maximal δ-separated set in X and form its Borel Voronoi partition, resolving ties by a fixed ordering of the centers. Every cell lies in a δ-ball about its center and has diameter at most 2δ. The fixed Mal'cev metric is bi-Lipschitz equivalent to a Riemannian metric on the compact manifold X, and hence is doubling. Thus any ball of radius $4\eta\delta$ meets at most L_X cells, with L_X independent of δ.

Take a Hausdorff $\eta\delta$-net $\mathcal{Z}$ for $\mathcal{F}_Q$. By Lemma 3.1,

$$|\mathcal{Z}| \le C_X Q^A \delta^{-B}.$$

On each model choose at least a_X/δ points at pairwise distance at least $T\delta$, where $T > 8\eta + 2$ is fixed. For each chosen point collect all Voronoi cells meeting its $4\eta\delta$-ball. These collections are disjoint: a cell meeting two of the balls would place their centers within $(8\eta + 2)\delta$.

Color each cell independently with a fair binary color. For a fixed model and a prescribed color i, a selected ball is entirely color i with probability at least 2^{-L_X}, and these events are independent over the selected balls. The probability that this model has no such ball is at most

$$(1 - 2^{-L_X})^{a_X/\delta} \le \exp(-b_X/\delta).$$

Consequently the probability of failure on some model, for either color, is at most

$$C_X Q^A \delta^{-B} \exp(-b_X/\delta). \tag{3.4}$$

Put $\delta = a/(1 + \log Q)$. If $a > 0$ is sufficiently small in terms of X, this is less than one for every $Q \ge 2$. Indeed, taking logarithms, the negative term $-b_X(1+\log Q)/a$ dominates $A \log Q + B \log(1+\log Q) + B \log(1/a) + \log C_X$. We also take a small enough for all the preceding geometric estimates.

Fix a coloring outside this exceptional event. If $Z \in \mathcal{F}_Q$ is within $\eta\delta$ of a model, move the centers of the two monochromatic model balls to points of Z, by distance at most $\eta\delta$. Each moved center still has a monochromatic $2\eta\delta$-ball of the same color. Taking $c_X = 2\eta a$ proves (3.3). □

We will use the following consequence with polynomial complexity.

Corollary 3.3. *There is $c_X > 0$ such that for every $D' > 0$ and all sufficiently large integers K, there is a Borel two-coloring κ_K with monochromatic c_X/K-balls of both colors centered on every member of $\mathcal{F}_{K^{D'}}$.*

Proof. Apply Theorem 3.2 with $Q = K^{D'}$ and use $1 + D' \log K \le K$, valid for sufficiently large K. □

4. Equidistribution under rational conjugation

Two losses of uniformity have to be recorded explicitly. First, the periodic term in the factorization changes the lattice by a rational conjugation. Second, density is needed in the ambient metric on X, not merely in an intrinsic metric whose complexity may vary. The next two lemmas isolate these points.

Lemma 4.1 (Rational conjugation)**.** *For every $E > 0$ and $D > 0$ there is a constant $A_* = A_*(X, E, D)$ with the following property. Let $M \ge 2$, and let H be a connected rational subgroup equipped with a Mal'cev basis $\mathcal{X}_H$ of complexity at most M^D, and let $u \in G$ be an M^D-rational element lying in the fixed Mal'cev fundamental domain. Suppose that $g' \in \mathrm{poly}(\mathbb{Z}, H_\bullet)$ for the induced rational filtration and that*

$$(g'(n)(H \cap \Gamma))_{n=1}^{N}$$

is totally M^{-A_*}-equidistributed on $H/(H \cap \Gamma)$ with respect to $d_{\mathcal{X}_H}$. Put

$$H^u = u^{-1}Hu, \qquad H_i^u = u^{-1}(H \cap G_i)u.$$

Then $H_i^u = H^u \cap G_i$, and there is a Mal'cev basis $\mathcal{X}_{H^u}$, adapted to $H_\bullet^u$ and of complexity $M^{O_{X,D}(1)}$, for which the conjugated sequence

$$(u^{-1}g'(n)u(H^u \cap \Gamma))_{n=1}^{N}$$

is totally M^{-E}-equidistributed on $H^u/(H^u \cap \Gamma)$ with respect to $d_{\mathcal{X}_{H^u}}$. It is enough to replace A_* by any larger constant.

Proof. Put $\Lambda = H \cap \Gamma$ and $\Lambda^u = H^u \cap \Gamma$. Every G_i is normal in G, since $[G_i, G] \subseteq G_i$. Consequently

$$u^{-1}(H \cap G_i)u = H^u \cap G_i.$$

Conjugation is a homomorphism and commutes with discrete derivatives, so $u^{-1}g'u$ belongs to $\mathrm{poly}(\mathbb{Z}, H_\bullet^u)$. Rational linear algebra, the Baker–Campbell–Hausdorff formula, and Lemmas A.11 and A.13 of [5] show that H^u and every H_i^u are $M^{O_{X,D}(1)}$-rational. Proposition A.10 of [5] therefore gives a

Mal'cev basis $\mathcal{X}_{H^u}$ adapted to $H^u_\bullet$ and of the asserted complexity. Thus the quantitative Leibman theorem is applicable below with uniform polynomial losses. The same results and the Baker–Campbell–Hausdorff formula show that the matrix of $\mathrm{Ad}(u^{-1})$ between the two controlled bases has rational entries of height $M^{O_{X,D}(1)}$. Rational linear algebra also bounds the structure constants of both bases by $M^{O_{X,D}(1)}$.

We make the lattice comparison explicit. If $\mathcal{X} = (X_1, \dots, X_m)$ is the fixed strong Mal'cev basis, put $a_i = \exp X_i$; these are the associated strong Mal'cev generators of Γ. Each $u^{-1}a_i u$ is $M^{O_{X,D}(1)}$-rational by Lemma A.11 of [5]. We may therefore choose a common integer $r \le M^{O_{X,D}(1)}$ such that $(u^{-1}a_i u)^r \in \Gamma$ for every i. The subgroup $\Delta = \langle a_1^r, \dots, a_m^r \rangle$ has index at most r^m, hence $M^{O_{X,D}(1)}$, in Γ. Indeed, along the normal series of Mal'cev tail subgroups Γ_i, the image of $\Delta \cap \Gamma_i$ in each infinite cyclic quotient Γ_i/Γ_{i+1} contains the rth power of its generator. Multiplying the resulting index bounds gives r^m. Moreover, $u^{-1}\Delta u \subseteq \Gamma$. Applying the same argument to u^{-1} shows that Γ and $u\Gamma u^{-1}$ are commensurable, with both indices bounded by $M^{O_{X,D}(1)}$. Consequently, if

$$\Lambda_1 = u^{-1}\Lambda u, \qquad \Lambda_0 = \Lambda_1 \cap \Lambda^u,$$

then $[\Lambda_1 : \Lambda_0]$ and $[\Lambda^u : \Lambda_0]$ are bounded by $M^{O_{X,D}(1)}$.

It follows that there is $C_0 = C_0(X, D) \ge 2$, which we take large enough to bound the rationality of $\mathcal{X}_{H^u}$, with the following property. Whenever η is a horizontal character of H^u/Λ^u of modulus at most T, the finite group $\eta(\Lambda_1)$ has order at most M^{C_0}. If q denotes its order, then

$$\chi(h) := q\eta(u^{-1}hu) \tag{4.1}$$

is a horizontal character of H/Λ of modulus at most $M^{C_0}T$; the modulus bound follows from the controlled rational change of basis. More explicitly, if A is the matrix of $\mathrm{Ad}(u^{-1})$ in the two controlled bases, then the coefficient rows satisfy $k_\chi = qk_\eta A$. The character χ is nontrivial whenever η is: since H^u is connected, a nontrivial horizontal character has image $\mathbb{R}/\mathbb{Z}$, and multiplication by q cannot annihilate it.

Replacing E by $\max(E, C_0)$ only strengthens the desired conclusion, so we may assume $E \ge C_0$. Suppose that the conjugated sequence is not totally M^{-E}-equidistributed. There is then a progression

$$Q = \{a + bm : 1 \le m \le N'\} \subseteq [1, N], \qquad N' \ge M^{-E}N,$$

on which it is not M^{-E}-equidistributed. Taking $E \ge C_0$, the quantitative Leibman theorem [5, Theorem 2.9] applies with the basis $\mathcal{X}_{H^u}$. It supplies a nontrivial horizontal character η such that, for some $C_1 = C_1(X, D, E)$,

$$|\eta| \le M^{C_1}, \qquad \|\eta(u^{-1}g'(a + b\cdot)u)\|_{C^\infty[N']} \le M^{C_1}.$$

Pulling back as in (4.1) and increasing C_1 gives

$$0 < |\chi| \le M^{C_1}, \qquad \|\chi(g'(a + b\cdot))\|_{C^\infty[N']} \le M^{C_1}. \tag{4.2}$$

By the slow-variation estimate [5, Lemma 2.8], the phase in (4.2) varies by at most $O_X(M^{C_1}/N')$ in one step. Hence some consecutive block $I \subseteq [1, N']$ of length

$$|I| = \max\left\{1, \left\lfloor \frac{N'}{C_2 M^{C_1}} \right\rfloor\right\}$$

has all its values in an arc of length less than $1/20$, where C_2 depends only on the degree and dimension. If $N' \geq 2C_2M^{C_1}$, then $|I| \geq N'/(2C_2M^{C_1})$; otherwise $N < 2C_2M^{E+C_1}$. In either case $|I| \geq M^{-A_*}N$ once $A_* > E + C_1 + O_X(1)$. Thus, after taking A_* still larger, total M^{-A_*}-equidistribution applies on the progression $\{a + bm : m \in I\}$.

The function $F = \exp(2\pi i\chi)$ has integral zero on H/Λ and Lipschitz norm at most $M^{C_1+O_X(1)}$. Equidistribution therefore makes the modulus of its average on this last progression less than $1/2$. On the other hand, all its values lie in an arc of angular length less than $\pi/10$, so that modulus is greater than $9/10$. This contradiction proves total M^{-E}-equidistribution of the conjugated sequence. □

Lemma 4.2 (Ambient density)**.** *For every $c > 0$, $E_0 > 0$, and $D > 0$ there is $E \geq E_0$ with the following property. Let $2 \leq K \leq M$, let $H \leq G$ be a positive-dimensional connected rational subgroup with a Mal'cev basis of complexity at most M^D, and let $x \in G$ have Mal'cev coordinates of size at most M^D. If $(v(n)(H \cap \Gamma))_{n=1}^N$ is totally M^{-E}-equidistributed on $H/(H \cap \Gamma)$, then for every progression $P \subseteq [N]$ with $|P| \geq M^{-E}N$, the sequence $(xv(n)\Gamma)_{n\in P}$ comes within c/K of every point of $xH\Gamma/\Gamma$.*

Proof. We may assume $0 < c \leq 1$, since decreasing c strengthens the conclusion. Put $\Lambda = H \cap \Gamma$, let d_H be the metric on H/Λ associated with the chosen Mal'cev basis, and write m_H for Haar probability measure. The map

$$\Phi_x : H/\Lambda \longrightarrow X, \qquad h\Lambda \longmapsto xh\Gamma,$$

is injective. There is $C_3 = C_3(X, D)$ such that

$$\begin{aligned} \operatorname{Lip}(\Phi_x) &\leq M^{C_3}, \\ m_H(B_{d_H}(w, r)) &\geq M^{-C_3} r^{\dim H} \quad (w \in H/\Lambda,\ 0 < r \leq 1). \end{aligned} \tag{4.3}$$

Indeed, Proposition A.10 of [5] gives the controlled change of basis for H, and Lemma A.17 compares its intrinsic quotient metric with the ambient quotient metric. After representatives are reduced to their Mal'cev fundamental boxes by Lemma A.14, their ambient coordinates are $M^{O_{X,D}(1)}$; Lemma A.5, applied to left multiplication by x, then gives the first estimate in (4.3).

For the volume estimate, write $w = h_0\Lambda$ with h_0 reduced to the Mal'cev fundamental box. Lemma A.5 in H, applied also to h_0^{-1}, gives

$$h_0 B_{d_H}(\Lambda, M^{-C_4}r) \subseteq B_{d_H}(w, r)$$

for some $C_4 = C_4(X, D)$. Haar invariance therefore reduces the estimate to a ball at Λ. If $C_4' = C_4'(X, D)$ is sufficiently larger than C_4, a one-sided Mal'cev coordinate cube of side $c_0M^{-C_4'}r$ at the identity is injective modulo Λ by

Lemma A.14 and lies in $B_{d_H}(\Lambda, M^{-C_4}r)$ by the metric–coordinate comparison of Lemma A.4. Its measure is $(c_0 M^{-C_4'}r)^{\dim H}$, since normalized Haar measure is Lebesgue measure on the Mal'cev fundamental box. Enlarging C_3 proves the second estimate in (4.3); the translated cube also handles centers lying on a seam of that box.

Fix $z = \Phi_x(w) \in xH\Gamma/\Gamma$ and put $r = c/K \le 1$. The first estimate in (4.3) shows that $\Phi_x^{-1}(B_X(z, r/2))$ contains the d_H-ball about w of radius $M^{-C_3}r/2$. Thus this inverse image has measure at least $M^{-C_6}K^{-\dim H}$ for some $C_6 = C_6(X, D, c)$. The function

$$F_z(v) = \min\{1, \max\{0, 2 - 2Kd_X(\Phi_x(v), z)/c\}\}$$

is one on this inverse image, vanishes outside $\Phi_x^{-1}(B_X(z, r))$, and satisfies $\|F_z\|_{\mathrm{Lip}} \le M^{C_5}K$ for some $C_5 = C_5(X, D, c)$. Its integral is at least $M^{-C_6}K^{-\dim H}$.

Since $K \le M$ and $\dim H \le \dim X$, we may choose E sufficiently large, in terms of E_0, X, D, and c, that

$$M^{-E}M^{C_5}K < \tfrac{1}{2}M^{-C_6}K^{-\dim H}.$$

Total M^{-E}-equidistribution on every progression P of length at least $M^{-E}N$ then makes the average of the tent function positive. The sequence meets every ambient c/K-ball, as required. □

5. Structure of monochromatic polynomial orbits

We now combine the coloring and equidistribution lemmas. The conclusion holds for every polynomial sequence in the fixed filtration.

Theorem 5.1 (Monochromatic orbit structure)**.** *For every fixed filtered nilmanifold as in Section 2, there are integers $R, N_0 \ge 2$ and a Borel two-coloring $\kappa : X \to \{0, 1\}$ such that the following holds. If $N \ge N_0$, $p \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$, and*

$$p(1)\Gamma, \dots, p(N)\Gamma$$

are monochromatic, then there is a factorization

$$p(n) = \varepsilon(n)\gamma(n) \qquad (n \in \mathbb{Z}) \tag{5.1}$$

with $\varepsilon, \gamma \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$, ε (R, N)-smooth, and γ R-rational. Moreover, $\gamma(n)\Gamma$ has a period $q \le R$. The coloring and the constants depend only on the fixed filtered nilmanifold.

Proof. If G is trivial there is nothing to prove, so assume $\dim G > 0$. Choose an integer $A_0 \ge 3$, and let c_X be the constant in Corollary 3.3. Fix $D_0 = D_0(X)$ large enough for the following bounds. In a Green–Tao factorization with parameter M, the adapted basis of H, the rationality of the reduced representative u of $\gamma(1)\Gamma$, and the coordinates of $\varepsilon(1)u$ are all bounded by M^{D_0}. The last bound follows from smoothness, the metric–coordinate comparison, and the polynomial product law. Choose $D_1 = D_1(X, D_0)$ large

enough also to bound the complexity of the conjugated basis in Lemma 4.1 by M^{D_1}.

Apply Lemma 4.2 with $c = c_X/3$, $E_0 = A_0 + 2$, and $D = D_1$, obtaining $E \geq A_0 + 2$. Lemma 4.1, with this E and $D = D_0$, gives A_*. Apply Theorem 2.1 with $A \geq A_*$, and let B, C be its constants. Choose $D' > 0$ so large that

$$M^{D_1} \leq K^{D'} \qquad (K \leq M \leq CK^B)$$

for all sufficiently large K.

Now fix one sufficiently large integer K. In particular, require $K^{-2} < c_X/(3K)$ and the conclusion of Corollary 3.3 for the exponent D'. Let κ be the resulting coloring and set

$$R = \lceil CK^B \rceil, \qquad N_0 > 4R^{A_0+1}.$$

This order of choices is important: D_1 is independent of E, and D' is chosen after the factorization exponent. All parameters, including the coloring, are fixed before p and N are given.

Suppose $N \geq N_0$ and the displayed orbit is monochromatic. Factor

$$p(n) = \varepsilon(n)g'(n)\gamma(n)$$

as in Theorem 2.1, with $K \leq M \leq R$. If $H = \{\mathrm{id}\}$, then $g' = \mathrm{id}$ and the required conclusion follows. Suppose instead that $\dim H > 0$.

Let $q \leq M$ be a period of $\gamma(n)\Gamma$ and put

$$J = \left\lfloor \frac{N-1}{qM^{A_0}} \right\rfloor, \qquad P = \{1, 1+q, \ldots, 1+Jq\}.$$

Then $P \subseteq [N]$ and

$$|P| \geq \frac{N}{2M^{A_0+1}} \geq M^{-E}N, \qquad n - 1 \leq \frac{N}{M^{A_0}} \quad (n \in P). \tag{5.2}$$

Let u be the representative of $\gamma(1)\Gamma$ in the fixed Mal'cev fundamental domain. Set

$$H^u = u^{-1}Hu, \qquad Z = \varepsilon(1)uH^u\Gamma/\Gamma.$$

The choices of D_0, D_1, D' imply $Z \in \mathcal{F}_{K^{D'}}$. There are therefore $z_0, z_1 \in Z$ with

$$\kappa \equiv i \quad \text{on } B_{c_X/K}(z_i), \qquad i = 0, 1.$$

The sequence on $H/(H \cap \Gamma)$ is totally M^{-A_*}-equidistributed. Lemma 4.1 makes its conjugate totally M^{-E}-equidistributed on $H^u/(H^u \cap \Gamma)$. By (5.2) and Lemma 4.2, there are $n_0, n_1 \in P$ such that

$$d_X(\varepsilon(1)u(u^{-1}g'(n_i)u)\Gamma, z_i) < \frac{c_X}{3K}, \qquad i = 0, 1.$$

For $n \in P$, periodicity gives $\gamma(n)\Gamma = u\Gamma$, hence

$$p(n)\Gamma = \varepsilon(n)u(u^{-1}g'(n)u)\Gamma.$$

Right invariance and smoothness now give

$$d_X(p(n)\Gamma, \varepsilon(1)u(u^{-1}g'(n)u)\Gamma) \le d_G(\varepsilon(n), \varepsilon(1)) \le \frac{(n-1)M}{N} \le M^{1-A_0} \le K^{-2} < \frac{c_X}{3K}.$$

Thus $p(n_i)\Gamma \in B_{c_X/K}(z_i)$ for $i = 0, 1$, contradicting monochromaticity. Therefore H is trivial, as required. □

The bounded period gives a useful uniform consequence.

Corollary 5.2 (Slow motion on a fixed progression)**.** *Let κ, R, N_0 be as in Theorem 5.1 and put $L = \operatorname{lcm}(1, \dots, R)$. For every monochromatic polynomial orbit as in that theorem,*

$$\operatorname{diam}\{p(n + jL)\Gamma : 0 \le j \le v\} \le \frac{vLR}{N} \tag{5.3}$$

whenever $1 \le n \le n + vL \le N$.

Proof. Since $q \mid L$, all the cosets $\gamma(n + jL)\Gamma$ equal $\gamma(n)\Gamma$. Compare the representatives $\varepsilon(n + jL)\gamma(n)$ using right invariance, and telescope the smoothness bound over indices in $[N]$. □

The estimate concerns motion between nearby indices in a fixed residue class. It does not assert that the whole orbit stays $O(N^{-1})$-close to one periodic orbit: the smooth factor may move an order-one distance over $[N]$.

5.1. **Why the initial term stays inside the factorization.** Left translations on a nilmanifold need not have a common Lipschitz bound. This already occurs in the Heisenberg group

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$$

with lattice $\Gamma = \mathbb{Z}^3$. For $A \in \mathbb{N}$, let

$$u_A = (A, 0, 0) \in \Gamma, \qquad v_A = (0, 1/(4A), 0).$$

Then

$$u_A v_A u_A^{-1} = (0, 1/(4A), 1/4).$$

Consequently $v_A\Gamma \to \Gamma$, whereas

$$u_A v_A\Gamma \longrightarrow (0, 0, 1/4)\Gamma \ne \Gamma, \qquad u_A\Gamma = \Gamma.$$

A smooth error cannot therefore be carried through an arbitrary initial left translation with uniform control. We apply factorization directly to the full polynomial sequence, including its initial term.

6. Finite generalized-polynomial approximation

The small error needed for a special generalized polynomial is a weighted matrix coefficient of the smooth factor in Theorem 5.1. We first record the polynomial estimate and the matrix algebra which allow us to extract it. In the one-dimensional abelian case, this reduces to taking a higher difference of a bounded ordinary polynomial.

6.1. **Bounded polynomials and weighted matrices.** For a real polynomial E, write $\Delta_h E(t) = E(t+h) - E(t)$.

Lemma 6.1 (Differences of bounded polynomials)**.** *For every $D \geq 1$ there is $C_D > 0$ with the following property. Let $N \geq D+1$, and let $E \in \mathbb{R}[t]$ have degree at most D and $|E(n)| \leq R$ for all $n \in [N]$. Then*

$$|\Delta_h^j E(t)| \leq C_D R (h/N)^j$$

for $1 \leq j \leq D$, $h > 0$, and $1 \leq t \leq t + jh \leq N$.

Proof. Put $b = \lfloor (N-1)/D \rfloor$ and $F(u) = E(1+bu)$. The points $1, 1+b, \ldots, 1+Db$ are in $[N]$, so $|F(i)| \leq R$ for $0 \leq i \leq D$. Lagrange interpolation at these fixed integer nodes gives

$$\max_{0 \leq j \leq D} \sup_{0 \leq u \leq 2D} |F^{(j)}(u)| \leq C_D R.$$

Since $b \geq N/(4D)$ and $(N-1)/b \leq 2D$, rescaling gives

$$\sup_{1 \leq t \leq N} |E^{(j)}(t)| \leq C_D R N^{-j}.$$

Finally,

$$\Delta_h^j E(t) = \int_{[0,h]^j} E^{(j)}(t + u_1 + \cdots + u_j)\, du_1 \cdots du_j,$$

which proves the claim. □

In particular, if $\deg E \leq w \leq D$ and $|E(n)| \leq R$ on $[N]$, the derivative estimate in this proof gives

$$|[t^w]E(t)| \leq C_D R N^{-w}. \tag{6.1}$$

The coefficient is zero when the degree is smaller than w.

Fix positive integers $j_1, \ldots, j_r$ with sum $d \leq D$. Set

$$G = \mathrm{UT}_{r+1}(\mathbb{R}), \qquad \Gamma = \mathrm{UT}_{r+1}(\mathbb{Z}), \qquad w_{uv} = j_u + \cdots + j_{v-1} \quad (u < v),$$

where UT denotes the group of upper-unitriangular matrices. For $\ell \geq 1$, let G_ℓ consist of the matrices whose entries of weight below ℓ vanish, and set $G_0 = G$. These are connected rational subgroups, $G_1 = G$, and $[G_\ell, G_q] \subseteq G_{\ell+q}$. In particular, this is a filtration of degree at most D.

Lemma 6.2 (Weighted coordinates)**.** *A sequence $F : \mathbb{Z} \to G$ belongs to $\mathrm{poly}(\mathbb{Z}, G_\bullet)$ if and only if each entry $F_{uv}(t)$, $u < v$, is an ordinary polynomial of degree at most w_{uv}. For such a sequence define $\mathcal{H}(F) \in G$ by*

$$\mathcal{H}(F)_{uv} = [t^{w_{uv}}] F_{uv}(t) \quad (u < v).$$

Then $\mathcal{H}(F_1 F_2) = \mathcal{H}(F_1)\mathcal{H}(F_2)$, and $\mathcal{H}(u) = I$ for every constant sequence $u \in G$. If $F(t) \in \Gamma$ for every $t \in \mathbb{Z}$, then

$$w_{uv}!\, \mathcal{H}(F)_{uv} \in \mathbb{Z}.$$

Proof. Order the elementary matrices E_{uv} by nondecreasing weight, with ties resolved arbitrarily. Every nonzero bracket has strictly larger weight than either input, so every tail spans an ideal. Since $\exp(tE_{uv}) = I + tE_{uv}$, the matrix entries and the Mal'cev coordinates of the second kind are related, in both directions, by triangular polynomial maps with integer coefficients and unit linear part. Thus this basis is also integral relative to Γ. Each monomial contributing to entry (u, v) has total weight w_{uv}. The coordinate description of polynomial sequences in [5, Lemma 6.7] therefore gives the first assertion.

For the second assertion, use

$$(F_1F_2)_{uv} = (F_1)_{uv} + (F_2)_{uv} + \sum_{u<b<v} (F_1)_{ub}(F_2)_{bv}.$$

Because $w_{ub} + w_{bv} = w_{uv}$, the coefficient of degree w_{uv} in a product in the sum is the product of the coefficients of degrees w_{ub} and w_{bv}. All weights are positive, so constant sequences map to I. Finally, if F_{uv} is integer-valued on $\mathbb{Z}$, then $\Delta_1^{w_{uv}} F_{uv}(0) = w_{uv}![t^{w_{uv}}]F_{uv}(t)$ is an integer. □

For $b > 0$, the map $\delta_b : G \to G$ which multiplies entry (u, v) by $b^{w_{uv}}$ is an automorphism. The last assertion of the lemma implies

$$F(\mathbb{Z}) \subseteq \Gamma \quad \Longrightarrow \quad \delta_{D!}(\mathcal{H}(F)) \in \Gamma, \tag{6.2}$$

since $w_{uv} \le D$ and $(D!)^{w_{uv}}/w_{uv}!$ is an integer.

6.2. **Integer reduction and nested brackets.** Let $A(t_1, \dots, t_r)$ denote the upper-unitriangular matrix with first superdiagonal $t_1, \dots, t_r$ and all higher entries zero.

Lemma 6.3 (Nested rounding)**.** *Suppose*

$$A(t_1, \dots, t_r) = E\Lambda, \qquad E \in G, \quad \Lambda \in \Gamma,$$

and $|E_{u,r+1}| < 1/2$ *for* $1 \le u \le r$*. Then*

$$\mathsf{L}(t_1, \dots, t_r) \equiv (-1)^{r-1} E_{1,r+1} \pmod{\mathbb{Z}}.$$

Proof. Put $B = \Lambda^{-1} \in \Gamma$. The identity $E = AB$ gives

$$E_{u,r+1} = B_{u,r+1} + t_u B_{u+1,r+1}, \qquad B_{r+1,r+1} = 1. \tag{6.3}$$

Let $Q_u = \mathsf{L}(t_u, \dots, t_r)$. We claim, by downward induction, that

$$Q_u = (-1)^{r-u+1} B_{u,r+1} + (-1)^{r-u} E_{u,r+1}, \qquad \langle Q_u \rangle = (-1)^{r-u+1} B_{u,r+1}.$$

For $u = r$ this follows directly from (6.3). If it holds at $u + 1$, then $Q_u = t_u \langle Q_{u+1} \rangle = (-1)^{r-u} t_u B_{u+1,r+1}$, and (6.3) proves the displayed identity at u. The bound $|E_{u,r+1}| < 1/2$ identifies the nearest integer uniquely at each step, so no rounding tie occurs. Taking $u = 1$ proves the lemma. □

6.3. **Proof of the finite theorem.**

Proof of Theorem 1.4. First fix one weight pattern $(j_1, \dots, j_r)$, and write

$$P(n) = \mathsf{L}(\alpha_1 n^{j_1}, \dots, \alpha_r n^{j_r}), \qquad \mathcal{A}(x) = A(\alpha_1 x^{j_1}, \dots, \alpha_r x^{j_r}).$$

Use the weighted group and lattice above, with their fixed adapted basis. Let κ, R, N_0 be supplied by Theorem 5.1, and set

$$L = \operatorname{lcm}(1, \dots, R), \qquad M = LD!.$$

Color the positive integers by

$$c(n) = \kappa(\mathcal{A}(n/M)\Gamma). \tag{6.4}$$

This coloring is fixed before the progression length is chosen.

Suppose $a, a+s, \dots, a+(k-1)s$ is monochromatic, with $k \geq \max(N_0, D+1)$. By Lemma 6.2,

$$p(t) = \mathcal{A}((a + s(t-1))/M)$$

is a polynomial sequence in the fixed filtration. The structural theorem gives $p(t) = \varepsilon(t)\gamma(t)$ for every $t \in \mathbb{Z}$, where ε is (R,k)-smooth and $\gamma(t)\Gamma$ has period dividing L.

Put $u = \gamma(1)$ and $\lambda(z) = u^{-1}\gamma(1 + Lz)$. Then $\lambda(z) \in \Gamma$ for every integer z, and

$$\mathcal{A}((a + sLz)/M) = \varepsilon(1 + Lz)u\lambda(z). \tag{6.5}$$

All entries of these polynomial matrices have degrees at most their weights. Apply $\mathcal{H}$ in the variable z. The initial point a and the constant matrix u disappear, and Lemma 6.2 gives

$$\mathcal{A}(s/D!) = E_*\Lambda_*, \qquad E_* = \mathcal{H}(\varepsilon(1 + Lz)), \quad \Lambda_* = \mathcal{H}(\lambda).$$

Smoothness bounds $d_G(\varepsilon(t), I)$ by R on $[k]$. The metric–coordinate comparison therefore bounds every matrix entry of $\varepsilon(t)$ on this interval by a constant depending only on the fixed filtered nilmanifold and R. Applying (6.1) to these ordinary polynomials yields

$$|(E_*)_{uv}| = L^{w_{uv}}\big|[t^{w_{uv}}]\varepsilon_{uv}(t)\big| \ll_{D,G_\bullet,\Gamma} k^{-w_{uv}}.$$

Here L and R are fixed structural constants. No bound on ε outside $[k]$ is used.

Apply $\delta_{D!}$ to the factorization. By (6.2), it becomes

$$\mathcal{A}(s) = E\Lambda, \qquad \Lambda \in \Gamma, \qquad |E_{uv}| \ll_{D,G_\bullet,\Gamma} k^{-w_{uv}}.$$

For sufficiently large k, all these entries are smaller than $1/2$ in absolute value. Lemma 6.3 now gives

$$\|P(s)\|_{\mathbb{R}/\mathbb{Z}} \leq |E_{1,r+1}| \ll_{D,G_\bullet,\Gamma} k^{-d}.$$

For m expressions, take the product of their weighted groups, lattices, and filtrations, and apply the structural theorem once to that product. Use its common constants R, L, M in all the components. The same argument gives the asserted bounds simultaneously. There are only finitely many positive-integer weight patterns of total weight at most D. Maximizing the

constants and thresholds over their m-tuples makes them depend only on m, D, not on the real coefficients. The product version of (6.4) is one coloring for all k above the resulting threshold. Finally, $(2,k)$-largeness supplies a monochromatic progression whose difference lies in S. □

One can apply the theorem to every coefficient of several ordinary polynomials at once. This records the distinction between the degree-dependent monomial bounds and the bound for a polynomial with terms of different degrees.

Corollary 6.4 (Coefficientwise approximation)**.** *For every $v, D \geq 1$ there are C, k_0 depending only on v, D such that the following holds. Let*

$$P_i(n) = \sum_{j=1}^{D} \alpha_{ij} n^j, \qquad \alpha_{ij} \in \mathbb{R}, \quad 1 \leq i \leq v,$$

and let S be $(2,k)$-large with $k \geq k_0$. There is $s \in S$ such that

$$\|\alpha_{ij} s^j\|_{\mathbb{R}/\mathbb{Z}} \leq C k^{-j} \quad (1 \leq i \leq v,\ 1 \leq j \leq D).$$

In particular, if P_i has no terms of degree below $\nu_i \geq 1$, then

$$\|P_i(s)\|_{\mathbb{R}/\mathbb{Z}} \leq DC k^{-\nu_i}.$$

Proof. Use Theorem 1.4 for the vD monomials $\alpha_{ij} n^j$, and then apply the triangle inequality on $\mathbb{R}/\mathbb{Z}$. □

7. Nilrecurrence

We return to nilsystems without connectedness assumptions. We call a nilmanifold or nilsystem d-step if its presenting group is nilpotent of step at most d.

Let (X,T) be a topological dynamical system and let $U \subseteq X$ be open. For $d \geq 1$ write

$$N_T^d(U) := \{n \in \mathbb{N} : U \cap T^{-n}U \cap \cdots \cap T^{-dn}U \neq \varnothing\}.$$

Definition 7.1. A set $A \subseteq \mathbb{N}$ is a Nil_d-Bohr_0 set if there are a d-step nilsystem (X,T), a point $x_0 \in X$, and a neighborhood V of x_0 such that

$$\{n \in \mathbb{N} : T^n x_0 \in V\} \subseteq A.$$

The dual family Nil_d-Bohr_0^* consists of the sets which meet every Nil_d-Bohr_0 set.

We use the realization theorem of Huang, Shao, and Ye and the resulting recurrence equivalences of Host, Kra, and Maass, in the form stated in [7, Theorem 5.2 and Corollary 5.3]:

(i) S is pointwise recurrent for d-step nilsystems if and only if $S \in \mathrm{Nil}_d$-Bohr_0^*;
(ii) if A is a Nil_d-Bohr_0 set, then there are a minimal d-step nilsystem (Y,T) and a nonempty open set $U \subseteq Y$ such that $N_T^d(U) \subseteq A$.

The second statement turns a failure of recurrence into a witness which is uniform in the initial point.

Lemma 7.2 (Uniform separation)**.** *Let T be a minimal homeomorphism of a compact metric space Y, let $U \subseteq Y$ be nonempty and open, and let $S \subseteq \mathbb{N}$ satisfy*

$$S \cap N_T^d(U) = \varnothing.$$

There is $\rho > 0$ such that, for every $s \in S$ and every $y \in Y$,

$$\operatorname{diam}\{y, T^s y, \dots, T^{ds} y\} \geq \rho.$$

Proof. Minimality implies that $\{T^r U : r \in \mathbb{Z}\}$ covers Y. Choose a finite subcover and let ρ be a Lebesgue number for it. If $\{y, T^s y, \dots, T^{ds} y\}$ had diameter smaller than ρ, all of these points would belong to some $T^r U$. Applying T^{-r} would give

$$U \cap T^{-s} U \cap \cdots \cap T^{-ds} U \neq \varnothing,$$

contrary to $s \notin N_T^d(U)$. □

The quantitative factorization theorem is stated for connected, simply connected nilpotent groups. We shall also need roots of the translating element. The following observation makes both reductions without losing uniform separation.

Lemma 7.3 (Root phases)**.** *Assume that the conclusion of Lemma 7.2 holds for a nilsystem (Y, T) and a set S. There are a nilmanifold $X = G/\Gamma$ with G connected and simply connected, an element $g \in G$ representing T on an embedded copy of Y, a base point $x_0 = \Gamma \in Y \subseteq X$, and $\rho_0 > 0$ such that*

$$\operatorname{diam}_X\{\exp(t \log g) g^{a+js} x_0 : 0 \leq j \leq d\} \geq \rho_0 \tag{7.1}$$

for every $t \in [0, 1]$, $a \in \mathbb{Z}$, and $s \in S$.

Proof. The conclusion is immediate if $S = \varnothing$, so assume $S \neq \varnothing$. By [6, Chapter 10, Corollary 26], Y embeds as a normal subnilmanifold of a nilmanifold G/Γ with G connected and simply connected, and the action of T is represented by some $g \in G$. The exponential map is a diffeomorphism on G [5, Section 2], so $\exp(t \log g)$ is defined for every real t. If the chosen base point is $v\Gamma$, replace Γ by the conjugate lattice $v\Gamma v^{-1}$ and use the equivariant identification

$$G/(v\Gamma v^{-1}) \longrightarrow G/\Gamma, \qquad k(v\Gamma v^{-1}) \longmapsto kv\Gamma.$$

After this harmless change of presentation the base point is Γ.

For $0 \leq t \leq 1$, let

$$\phi_t : Y \longrightarrow X, \qquad \phi_t(y) = \exp(t \log g) y.$$

Each ϕ_t is an embedding. The set

$$\{(t, y, z) : 0 \leq t \leq 1,\ d_Y(y, z) \geq \rho\}$$

is compact, and the continuous function $(t, y, z) \mapsto d_X(\phi_t(y), \phi_t(z))$ is positive on it. Its minimum ρ_0 is therefore positive. Since $\exp(t \log g)$ commutes with g, the separation supplied by Lemma 7.2, applied at $g^a x_0$, gives (7.1). □

The structural theorem gives a short two-color obstruction from this uniform separation.

Proposition 7.4. *Let $X = G/\Gamma$ have a connected, simply connected presenting group, and give it a fixed rational Mal'cev metric. Let $g \in G$ and $S \subseteq \mathbb{N}$. Suppose that $d \geq 1$ and $\rho > 0$ satisfy*

$$\operatorname{diam}\{\exp(t \log g) g^{b+js}\Gamma : 0 \leq j \leq d\} \geq \rho \tag{7.2}$$

for every $t \in [0, 1]$, $b \in \mathbb{Z}$, and $s \in S$. Let R, N_0 be the constants of Theorem 5.1 for the lower central series filtration, and put $L = \operatorname{lcm}(1, \ldots, R)$. Then S is not $(2, N)$-large for any integer

$$N \geq \max\{N_0, 1 + dL\} \quad \text{with} \quad N > dLR/\rho.$$

In particular, S is not 2-large.

Proof. Let κ be the structural coloring and take $h = \exp(L^{-1} \log g)$, so $h^L = g$. Color $\mathbb{N}$ by $c(n) = \kappa(h^n\Gamma)$. Suppose $a, a + s, \ldots, a + (N-1)s$ is monochromatic with $s \in S$. The sequence $p(n) = h^{a+s(n-1)}$ belongs to the polynomial class for the lower central series filtration. Corollary 5.2 gives

$$\operatorname{diam}\{p(1 + jL)\Gamma : 0 \leq j \leq d\} \leq dLR/N < \rho.$$

Write $a = bL + r$ with $0 \leq r < L$. Then

$$p(1 + jL)\Gamma = \exp((r/L) \log g) g^{b+js}\Gamma,$$

contradicting (7.2). □

Proof of Theorem 1.3. We first prove pointwise recurrence, by contraposition. If S is not pointwise recurrent for nilsystems, then for some $d \geq 1$ there is a Nil_d-Bohr_0 set A disjoint from S. The realization theorem gives a minimal d-step nilsystem (Y, T) and a nonempty open $U \subseteq Y$ such that

$$N_T^d(U) \subseteq A.$$

Thus $S \cap N_T^d(U) = \varnothing$. Lemma 7.2 gives uniform separation on Y, and Lemma 7.3 gives a connected, simply connected ambient presentation satisfying (7.2). Proposition 7.4 shows that S is not 2-large.

This proves pointwise recurrence for every nilsystem. To obtain (1.1), apply this assertion to the product nilsystem

$$(X^\ell, T \times T^2 \times \cdots \times T^\ell)$$

at the point $(x, \ldots, x)$. □

8. Consequences

8.1. **Hitting times and polynomial orbits.** We first give the promised answer to the question of Host, Kra, and Maass.

Corollary 8.1. *Let (X,T) be a minimal nilsystem, let $U \subseteq X$ be nonempty and open, and let $x_0 \notin \overline{U}$. Then the set*

$$N(x_0,U) = \{n \in \mathbb{N} : T^n x_0 \in U\}$$

is not 2-large.

Proof. Put $\delta = d_X(x_0,\overline{U}) > 0$. If $N(x_0,U)$ were 2-large, Theorem 1.3, with $\ell = 1$, would give $n \in N(x_0,U)$ such that $d_X(T^n x_0, x_0) < \delta$. This is impossible because $T^n x_0 \in U$. □

Polynomial-orbit lifting gives the following consequence without a new coloring argument.

Corollary 8.2 (Polynomial nilrecurrence)**.** *Let S be 2-large. Given finitely many nilsystems (X_i,T_i), points $x_i \in X_i$, integer polynomials $P_i \in \mathbb{Z}[n]$ with $P_i(0) = 0$, and $\eta > 0$, there is $s \in S$ such that*

$$d_{X_i}(T_i^{P_i(s)} x_i, x_i) < \eta \qquad \text{for every } i.$$

More generally, if $g_i : \mathbb{Z} \to G_i$ are polynomial sequences in finite filtrations of nilpotent Lie groups that are rational with respect to lattices Γ_i, and $x_i \in X_i = G_i/\Gamma_i$, then some $s \in S$ satisfies

$$d_{X_i}(g_i(s)x_i, g_i(0)x_i) < \eta \qquad \text{for every } i.$$

Proof. A polynomial orbit on a nilmanifold is a continuous image of a linear orbit on another nilmanifold. We use the construction in [6, Chapter 14, Proposition 13 and Section 3.1]. If $x = v\Gamma$, write $g(n)x = g(n)v\Gamma$; multiplication by a constant preserves the polynomial class. For a polynomial sequence of points $\psi(n) = g(n)x$, that construction gives a nilsystem (Y,σ) containing ψ as a point and a continuous evaluation map $\pi : Y \to X$ such that

$$\pi(\sigma^n \psi) = \psi(n) \qquad (n \in \mathbb{Z}).$$

The filtration is rational and finite, so this construction gives a compact nilmanifold and the resulting nilsystem has finite step. The construction does not require connectedness.

Apply this construction to each prescribed orbit, and take the product of the resulting nilsystems. Theorem 1.3, with $\ell = 1$, gives $s \in S$ returning the product point to a neighborhood on which all evaluation maps are within η of their values at that point. This proves the general assertion. In particular, the sequences $T_i^{P_i(n)} x_i$ are polynomial orbits; they can also be lifted by the polynomial-sampling construction in [6, Chapter 14, Theorem 15]. Since $P_i(0) = 0$, their values at zero are x_i. □

We make no assertion that the lifting preserves the original step or complexity. Only finiteness of the step is needed.

8.2. **Localization in nil-Bohr neighborhoods.** Finite intersections of nil-Bohr$_0$ sets of finite step are again of this form, by taking products of their defining nilsystems. Thus the next corollary also permits any finite collection of nilpotent return conditions at the same color cost.

Corollary 8.3 (Nil-Bohr localization)**.** *Let $r \geq 2$, let $S \subseteq \mathbb{N}$ be $2r$-large, and let B be a Nil_d-Bohr$_0$ set for some finite d. Then $S \cap B$ is r-large.*

Proof. By Theorem 1.3, every 2-large set meets B. Consequently $\mathbb{N} \setminus B$ is not 2-large. There are a two-coloring c_0 of $\mathbb{N}$ and an integer $\ell \geq 1$ such that no monochromatic ℓ-term progression for c_0 has difference outside B.

Fix an r-coloring c_1 and a desired progression length k. The product coloring (c_0, c_1) uses at most $2r$ colors. Since S is $2r$-large, it has a monochromatic progression of length $\max(k, \ell)$ with common difference $s \in S$. Its first ℓ terms and the choice of c_0 force $s \in B$; its first k terms are monochromatic for c_1. Thus $S \cap B$ is r-large. □

For ordinary Bohr neighborhoods, Proposition 7.7 of [7] gives a color cost depending on the dimension of the torus. The dimension-free factor of two in that case already follows from the Bohr recurrence theorem of [1], by the same product-coloring proof. Corollary 8.3 extends this conclusion to every finite nilpotency step.

8.3. **Generalized polynomials and expression complexity.** We now pass from the special expressions in Theorem 1.4 to all normalized generalized polynomials. We use the class GP_d of [8, Definition 4.1.1]. Equivalently, start with the monomials an^j, $a \in \mathbb{R}$, $j \geq 1$, assigned weight j, and allow finite sums, products, real scalar multiplication, and the nearest-integer operation fixed in the introduction. A sum has the maximum of its input weights, a product has their sum, and scalar multiplication and rounding preserve weight. The class GP_d consists of functions admitting such an expression of weight at most d. To compare this with the definition in [8], expand products outside brackets: each term is a scalar multiple of a monomial times a product of rounded expressions, with the same total weight. No nonzero constant functions or additive constant shifts are introduced. In particular, every member vanishes at zero.

Corollary 8.4 (Generalized-polynomial approximation)**.** *Let $P_1, \dots, P_v \in \mathrm{GP}_d$ and $\eta > 0$. Every 2-large set $S \subseteq \mathbb{N}$ contains an s such that*

$$\|P_i(s)\|_{\mathbb{R}/\mathbb{Z}} < \eta \qquad (1 \leq i \leq v).$$

More precisely, there is an integer $K = K(P_1, \dots, P_v, \eta)$ such that the same conclusion holds for every $(2, K)$-large set S.

Proof. By Theorem 4.2.11 and Theorem B of [8], the set

$$\{n \in \mathbb{N} : \|P_i(n)\|_{\mathbb{R}/\mathbb{Z}} < \eta \text{ for } 1 \leq i \leq v\}$$

is a Nil_d-Bohr$_0$ set, say B. Theorem 1.3 shows that every 2-large set meets B. Thus $\mathbb{N} \setminus B$ is not 2-large: some two-coloring has no monochromatic

K-term progression with difference outside B. Applying $(2, K)$-largeness to this coloring gives a difference in $S \cap B$. □

For the special expressions, Theorem 1.4 gives explicit powers of K with constants depending only on their number and maximum weight. The next observation explains why the same uniformity is unavailable for arbitrary generalized polynomials. Their degree does not bound the number of summands or bracket operations.

Proposition 8.5. *For every $N \geq 1$ there is $P \in \mathrm{GP}_1$ such that $P(n) = 1/4$ for $1 \leq n \leq N$. Consequently, there is no function $\omega(k) \to 0$ such that every $(2, k)$-large set S and every $P \in \mathrm{GP}_1$ admit an $s \in S$ with $\|P(s)\|_{\mathbb{R}/\mathbb{Z}} \leq \omega(k)$.*

Proof. For $1 \leq j \leq N$ put

$$h_j(n) = \left\langle \frac{n}{2j-1} \right\rangle.$$

Then $h_j(i) = 0$ for $i < j$ and $h_j(j) = 1$. The integer matrix $(h_j(i))_{1 \leq i,j \leq N}$ is therefore lower triangular with diagonal entries one. There are integers $b_1, \ldots, b_N$ such that $\sum_{j=1}^{N} b_j h_j(i) = 1$ for every $1 \leq i \leq N$. The function $P = \frac{1}{4} \sum_j b_j h_j$ belongs to GP_1 and has the required values.

For the last assertion, take k with $\omega(k) < 1/4$ and let $N \geq W(2, k)$, where $W(2, k)$ is the van der Waerden number. The set $S = [N]$ is $(2, k)$-large, since every binary coloring of $[N]$ contains a k-term progression whose difference lies in $[N]$. The function just constructed contradicts the proposed bound. □

The normalization is also essential for the qualitative statement. It cannot be replaced by the condition $P(0) = 0$ on an arbitrary floor-polynomial. For irrational β, the expression

$$P(n) = \tfrac{1}{4}(\lfloor \beta n \rfloor + \lfloor -\beta n \rfloor)$$

vanishes at zero but equals $-1/4$ at every positive integer. This expression is not in the normalized class used above.

8.4. **Further questions.** The finite theorem gives explicit powers of the progression length, but its constants come from a qualitative choice of the coloring scale and repeated polynomial losses in factorization. It would be useful to determine the dependence of $C(m, D)$ and $k_0(m, D)$ on m, D, even when $D = 2$. We do not claim that the exponents k^{-d_i} are optimal under the $(2, k)$-largeness hypothesis. For general normalized generalized polynomials, Proposition 8.5 shows that quantitative bounds must also account for expression complexity. Obtaining useful rates with this additional parameter is a natural extension of the finite theorem.

The logarithmic radius in Theorem 3.2 comes from balancing polynomial covering numbers against exponentially small failure probabilities. Determining the largest radius that can hold uniformly over all translates of complexity at most Q is a separate geometric question.

Finally, the argument is specific to finite-step nilsystems. It does not prove recurrence for arbitrary distal systems or settle the Brown–Graham–Landman conjecture. The examples in [2] show that nil-Bohr recurrence alone does not imply topological multiple recurrence. Extending the two-color obstruction to systems outside the nilpotent setting would therefore require an additional idea.

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