2-Large sets are sets of pointwise recurrence for nilsystems (draft)

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Abstract

A set $D \subset \mathbb{N}$ is r-large if for any r-coloring of the natural numbers, there are arbitrarily long monochromatic arithmetic progressions whose common difference comes from D. The set D is large if it is r-large for all r. In 1999 Brown, Graham, and Landman conjectured that if D is 2-large, then it is also large.

We prove that every 2-large set is a set of pointwise recurrence for nilsystems. In other words, we show that if a set is not large because of a (generalized) polynomial obstacle, then it is also not 2-large.

1 Introduction

The subject of the present paper on arithmetic progressions is the following notion of *largeness*, introduced by Brown, Graham, and Landman [1].

Definition 1.1 (r-large)**.** For $r \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, we say that D is r-large if for any r-coloring $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exists a color $1 \leq i \leq r$ so that C_i contains arbitrarily long arithmetic progressions with common difference in D.

We say that D is *large* if it is r-large for all r. Equivalently, D is large if and only if the restricted van der Waerden theorem with common difference in D is true. That $\mathbb{N}$ itself is large is the content of van der Waerden's theorem. Clearly if a set is large, then it is r-large for all $r \geq 2$, and an r-large set for $r \geq 2$ is clearly 2-large. Brown, Graham, and Landman conjectured that in fact all of these notions of largeness coincide.

Conjecture 1.2 ([1])**.** $D \subseteq \mathbb{N}$ is 2-large, then D is large.

Perhaps it is a little bit more intuitive to think about this conjecture in the contrapositive. If a set D is *not* large, then the restricted van der Waerden with common difference in D is false, and so some finite coloring blocks arbitrarily long arithmetic progressions. The question of Brown, Graham, and Landman is then whether two colors is sufficient to block such progressions.

Theorem 1.3. If $D \subseteq \mathbb{N}$ is $2-$large, then D is a set of pointwise recurrence for nilsystems.

2 Nilmanifolds

For our discussion about nilmanifolds, we use a mix of the presentations given in [3] and [4], and we refer the reader to those sources for additional details.

2.1 Basic properties

For the rest of this paper, we will use the following notation when talking about nilmanifolds. We use G to denote a finite dimensional nilpotent (real) Lie group with identity element id_G and G^0 the connected component containg the identity. Furthermore, we will have a standing assumption that G is generated by G^0 and finitely many commuting elements $\{\tau_i\}_{i=1}^k$. Moreover, we will assume that the group generated by $\{\tau_i\}_{i=1}^k$ is isomorphic to $\mathbb{Z}^k$ and has trivial intersection with G^0. These assumptions are standard in ergodic theory, but not in the general theory of Lie algebras and Lie groups. We will use Γ to denote a discrete cocompact subgroup of G, and we use $N := G/\Gamma$ to denote the set of left cosets of Γ in G. A **nilmanifold** is a space of the form $N = G/\Gamma$ with G and Γ satisfying the above conditions.

Now let G be a Lie group of (real) dimension m. A **filtration** $G_\bullet$ on G is a sequence of closed subgroups

$$G = G_0 = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_d \supseteq G_{d+1} = \{\mathrm{id}_G\}$$

such that $[G_i, G_j] \subseteq G_{i+j}$ for all integers $i, j > 0$. If G is connected, then we also require G_i to be connected for all i. The least integer d for which $G_{d+1} = \{\mathrm{id}_G\}$ is called the *degree* of the filtration $G_\bullet$.

As usual, let $[H, K]$ denote the subgroup generated by all commutators $[h, k] := hkh^{-1}k^{-1}$. Since G is nilpotent, G always possesses the lower central series filtration, namely, the filtration in which $G_i = \overline{[G, G_{i-1}]}$ for all $2 \leq i \leq c + 1$, where c is the nilpotency class of G. However, G may also possess other filtrations, so the degree of a given filtration $G_\bullet$ may differ from the nilpotency class of G. When we are given a filtration $G_\bullet$ of G, we use m to denote the (real) dimension of G, and m_i to denote the dimension of G_i for $0 \leq i \leq d + 1$.

Lemma 2.1 ([4, Corollary 10.26])**.** Every nilmanifold $N = G/\Gamma$ can be embdedded as a normal subnilmanifold of a nilmanifold $N' = G'/\Gamma'$ such that G' is connected and simply connected and such that every element of G is represented in G'.

A **nilsystem** is a pair (N, T_g) where N is a nilmanifold, $g \in G$, and $T_g : N \to N$ is given by $T_g(x\Gamma) = gx\Gamma$. It is well known that for nilsystems, transitivity, minimality, and unique ergodicity are equivalent. A

2.2 Lie algebras, logarithms, and exponentials

2.3 Mal'cev bases

We want to give real coordinates to a nilmanifold $N = G/\Gamma$, and this can be accompolished by a Mal'cev basis. We only define them in the case that G is connected and simply connected, as we will always be able to use Lemma 2.1 to reduce to this case.

Definition 2.2. Let $N = G/\Gamma$ be a nilmanifold with G a m-dimensional connected and simply connected Lie group. Let $G_\bullet = \{G_i\}_{i=0}^{d+1}$ be a filtration on G and let $\mathfrak{g}$ denote the (real) Lie algebra of G. A basis $\mathcal{X} = \{X_1, \cdots, X_m\}$ for $\mathfrak{g}$ over $\mathbb{R}$ is called a **Mal'cev basis** for G/Γ adapted to $G_\bullet$ if the following four conditions are satisfied:

(i) For each $j = 0, \cdots, m-1$ the subspace $\mathfrak{h}_j := \text{Span}(X_{j+1}, \cdots, X_m)$ is a Lie algebra ideal in $\mathfrak{g}$, and hence $H_k := \exp \mathfrak{h}_j$ is a normal Lie subgroup of G.

(ii) For every $0 \leq i \leq d$ we have $G_i = H_{m-m_i}$ (recall that $m_i = \dim G_i$);

(iii) Each $g \in G$ can be written uniquely as $g = \exp(t_1 X_1)\exp(t_2 X_2)\cdots\exp(t_m X_m)$, for $t_i \in \mathbb{R}$;

(iv) Γ consists precisely of those elements which, when written in the above form, have all $t_i \in \mathbb{Z}$.

It is a well known result of Mal'cev [6] that any nilmanifold G/Γ can be equipped with a Mal'cev basis adapted to the lower central series filtration. More generally there is a Mal'cev basis adapted to any filtration $G_\bullet$ consisting of rational subgroups. We refer to the t_i as the **Mal'cev coordinates** of g, and we define the **Mal'cev coordinate map** $\psi = \psi_{\mathcal{X}} : G \to \mathbb{R}^m$ by

$$\psi(g) = (t_1, \cdots, t_m). \tag{1}$$

A nice property of Mal'cev coordinates is that they show the polynomial nature of the group operation of G. More specifically, there exist polynomials $\{P_i\}_{i=1}^m$ with $P_1 = 0$, taking integer values on integer inputs, such that for every $g, h \in G$ with $\psi(g) = (t_1, \cdots, t_m)$ and $\psi(h) = (s_1, \cdots, s_m)$ we have

$$\psi(gh) = (t_1 + s_1, t_2 + s_2 + P_2(t_1, s_1), \cdots, t_m + s_m + P_m(t_1, \cdots, t_{m-1}, s_1, \cdots, s_{m-1})). \tag{2}$$

Furthermore, there exist polynomials $\{\tilde{P}_i\}_{i=1}^m$ with $\tilde{P}_1 = 0$, taking integer values on integer inputs, such that for every $g \in G$ with $g = (t_1, \cdots, t_m)$ we have

$$\psi(g^{-1}) = (-t_1, -t_2 + \tilde{P}_2(t_1), \cdots, -t_m + \tilde{P}_m(t_1, \cdots, t_{m-1})). \tag{3}$$

Furthermore, if $G_\bullet$ is the lower central series filtration, and m_1 is the real dimension of $G_2 = [G, G]$, then $P_i = \tilde{P}_i = 0$ for $1 \leq i \leq m - m_1$. More generally, for $0 \leq j \leq d$ and $m - m_j < i \leq m - m_{j+1}$ we have $\deg_t(P_i), \deg_s(P_i), \deg_t(\tilde{P}_i), \deg_s(\tilde{P}_i) \leq j$, where $\deg_t(P)$ denotes the degree of P when

viewed as a polynomial in the variables $\{t_i\}_{i=1}^m$ and the s_j are held constant. We will only need to make use of the simpler bounds that $\deg_t(P_i), \deg_s(P_i), \deg_t(\tilde{P}_i), \deg_s(\tilde{P}_i) \leq i$ for all $2 \leq i \leq m$. From this, we deduce that for any $g \in G$ with $\psi(g) = (t_1, \cdots, t_m)$, we have $\psi(g^n) = (Q_1(t_1; n) := nt_1, Q_2(t_1, t_2; n), \cdots, Q_m(t_1, \cdots, t_m; n))$ for some polynomials $\{Q_i\}_{i=1}^m$ satisfying $\deg_t(Q_i), \deg_n(Q_i) \leq i$. For the sake of notational convenience, we will write $P_i(t_1, \cdots, t_m, s_1, \cdots, s_m), \tilde{P}(t_1, \cdots, t_m)$ and $Q_i(t_1, \cdots, t_m; n)$ for all $1 \leq i \leq m$ with the understanding that P_i does not use t_j and s_j for $j > 1$, and similarly for $\tilde{P}_i$ and Q_i. We will also be using the fact that ψ is a diffeomorphism (cf. [2, Proposition 1.2.7]).

Definition 2.3 (Metrics on G and G/Γ)**.** Let G/Γ be a nilmanifold with Mal'cev basis $\mathcal{X}$. We define $d = d_{\mathcal{X}} : G \times G \to \mathbb{R}_{\geq 0}$ to be the largest metric such that $d(x, y) \leq |\psi(xy^{-1})|$ for all $x, y \in G$, where $|\cdot|$ denotes the ℓ^∞ norm on $\mathbb{R}^m$. More explicitly, we have

$$d(x,y) = \inf\left\{\sum_{i=0}^{n-1} \min\left(|\psi(x_{i-1}x_i^{-1})|, |\psi(x_i x_{i-1}^{-1})|\right) : x_0, \cdots, x_n \in G; x_0 = x; x_n = y\right\}.$$

This descends to a metric on G/Γ by setting

$$d(x\Gamma, y\Gamma) := \inf\{d(x', y') \ : \ x', y' \in G; x' \equiv x \pmod{\Gamma}; y' \equiv y \pmod{\Gamma}\}.$$

It turns out that this is indeed a metric on G/Γ. We see that d is also right-invariant, hence

$$d(x\Gamma, y\Gamma) = \inf_{\gamma \in \Gamma} d(x, y\gamma) = \min_{\gamma \in \Gamma} d(x, y\gamma). \tag{4}$$

When the letter d is used for a metric, it will always denote the metric $d_{\mathcal{X}}$ relative to some basis $\mathcal{X}$ that is already under discussion. The symbol d' will be used for the metric defined using some other basis $\mathcal{X}'$. In the proof of Theorem 3.2 we will be considering many nilmanifolds and their respective Mal'cev bases, so then we will explicitly write $d_{\mathcal{X}}$ in each situation.

Remark 2.4 (Haar measure and Lebesgue measure). We see that for each $g \in G$, $\phi_{\mathcal{X}}(g)$ acts on $\mathbb{R}^m = \psi_{\mathcal{X}}(G)$ as an upper triangular matrix with 1s on the diagonal and polynomial functions in the entries above the diagonal. Since each $\psi_{\mathcal{X}}(g)$ has determinant 1, regardless of where it is acting on $\mathbb{R}^m$, we see that the m-dimensional Lebesgue measure μ on $\mathbb{R}^m$ is invariant under the action of $\psi_{\mathcal{X}}(g)$ for all $g \in G$. Since G is a nilpotent Lie group, it is unimodular, so the Haar measure μ of G normalized so that $\mu(\psi_{\mathcal{X}}^{-1}([0,1)^m)) = 1$ is the pushforward of μ from $\mathbb{R}^m$, which is why we choose to abuse notation and write μ for both measures.

The **height** of a rational number $\frac{p}{q}$ with $\gcd(p, q) = 1$ is q, and by convention we set the height of 0 to be 1. The Lie algebra basis $\mathcal{X}$ is Q**-rational** if for every $1 \leq i, j \leq m$ we have $[X_i, X_j] = \sum_{k=1}^m c_{i,j,k} X_k$ where every $c_{i,j,k}$ is a rational number of height at most Q. We will frequently use this rationality parameter Q of $\mathcal{X}$ in our inequalities, so we place a standing assumption that $Q \geq 2$ throughout the article. If $\mathcal{X}$ is a Mal'cev basis of a nilmanifold G/Γ adapted to some filtration $G_\bullet$, then $\mathcal{X}$ is Q-rational for some $Q \in \mathbb{N}$.

Remark 2.5 (Asymptotic Notation). We use $a \ll b$ to mean that there exists a constant $C > 0$ for which $a \leq Cb$. Similarly, we use $a = O(b)$ to mean that there is a constant C for which $C^{-1}b \leq a \leq Cb$. In the discussion that follows, the constants C that are implied by our usage of $\ll$ and $O(1)$ will be allowed to depend on the global structure of the nilmanifold G/Γ, but not the local structure. For example, the constant C may depend on the dimension m, the nilpotence degree s, and the degree d of the filtration $G_\bullet$, but not on the Mal'cev basis $\mathcal{X}$ or on the points $g, g_1, g_2, x, y \in G$. For a given result, we will always assume without loss of generality that $C = C(m, s, d)$ is increasing as a function of m, s, and d, i.e., $C(m, s, d) \leq C(m', s', d')$ if $m \leq m'$, $s \leq s'$, or $d \leq d'$. The constants generated by the Mal'cev basis $\mathcal{X}$ and the polynomials $P_i, \tilde{P}_i$, and Q_i will be absorbed into the $Q^{O(1)}$ terms. We will also differ from the conventions of [3] in that we will write $a \leq Q^{O(1)}b$ instead of $a \ll Q^{O(1)}b$, as we will always assume that $Q \geq 2$, so the constant from $\ll$ can be absorbed into the $Q^{O(1)}$ term.

Lemma 2.6 ([3, Lemma A.4])**.** Suppose that $Q \geq 2$. Suppose that $\mathcal{X}, \mathcal{X}'$ are two Q-rational Mal'cev bases for G/Γ. Suppose that each X_i' is given by a Q-rational combination of the X_i and vice versa. Then for all $x, y \in G$ with $|\psi'(x)|, |\psi'(y)| \leq Q$, we have the bound

$$d(x, y) \ll Q^{O(1)}|\psi'(x) - \psi'(y)|,$$

and for all $x, y \in G$ with $d(x, \mathrm{id}_G), d(y, \mathrm{id}_G) \leq Q$, we have the bound

$$|\psi'(x) - \psi'(y)| \ll Q^{O(1)}d(x, y).$$

Lemma 2.7 (Approximate left-invariance of d, [3, Lemma A.5])**.** Suppose $Q \geq 2$ and that $\mathcal{X}$ is a Q-rational basis for G/Γ adapted to some filtration $G_\bullet$. suppose that $g, x, y \in G$ are elements with $|\psi(x)|, |\psi(y)|, |\psi(g)| \leq Q$. Then we have the bound

$$d(gx, gy) \ll Q^{O(1)}d(x, y).$$

Lemma 2.8 (Comparison lemma [3, Lemma A.6])**.** Suppose that $G' \subseteq G$ is a closed subgroup and that $\mathcal{X}, \mathcal{X}'$ are bases for $\mathfrak{g}, \mathfrak{g}'$ respectively which have the nesting property (A.1). Let $Q \geq 2$, and suppose that each X_i' is a Q-rational combination of the X_i. Then we have the bounds

$$d'(x, y) \ll Q^{O(1)}d(x, y)$$

uniformly for all $x, y \in G'$ with $|\psi(x)|, |\psi(y)| \leq Q$ and

$$d(x, y) \ll Q^{O(1)}d'(x, y)$$

uniformly for all $x, y \in G'$ with $|\psi'(x)|, |\psi'(y)| \leq Q$.

Lemma 2.9 (Nilmanifolds are bounded [3, Lemma A.16])**.** Let $Q \geq 2$, and suppose that $\mathcal{X}$ is a Q-rational Mal'cev basis for a nilmanifold G/Γ with respect to some filtration $G_\bullet$. Then $d(x\Gamma, y\Gamma) \ll Q^{O(1)}$ uniformly in $x, y \in G$.

Lemma 2.10. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For $C > 0$ let $R_C := \psi_{\mathcal{X}}^{-1}([-Q^C, Q^C])$. There exists $0 < C_0 \leq Q^{O(1)}$ such that for any $x, y \in G$, there exist $\gamma_1, \gamma_2 \in \Gamma$ with $d_{\mathcal{X}}(x\Gamma, y\Gamma) = d_{\mathcal{X}}(x\gamma_1, y\gamma_2)$ and $x\gamma_1, y\gamma_2 \in R_{C_0}$. Furthermore, we may assume without loss of generality that $x\gamma_1 \in \psi_{\mathcal{X}}^{-1}([0,1)^m)$.

Proof. We first pick $\gamma_1 \in \Gamma$ for which $x\gamma_1 \in \psi_{\mathcal{X}}^{-1}([0,1)^m)$. Let $\gamma_2 \in \Gamma$ be such that $d_{\mathcal{X}}(x\Gamma, y\Gamma) = d_{\mathcal{X}}(x\gamma_1, y\gamma_2)$. Using Lemma 2.9 we see that

$$d_{\mathcal{X}}(y\gamma_2, \mathrm{Id}_G) \leq d_{\mathcal{X}}(x\gamma_1, y\gamma_2) + d_{\mathcal{X}}(x\gamma_1, \mathrm{Id}_G) \leq d_{\mathcal{X}}(x\Gamma, y\Gamma) + |\psi_{\mathcal{X}}(x\gamma_1)| \leq Q^{O(1)}. \tag{5}$$

The desired result now follows from Lemma 2.6. □

When the nilmanifold G/Γ is endowed with a Mal'cev basis $\mathcal{X}$, we use d_∞ to denote the pull back of the ℓ^∞ metric on $\mathbb{R}^m$ to G via $\psi_{\mathcal{X}}$. More concretely, we have $d_\infty(x, y) = |\psi_{\mathcal{X}}(x) - \psi_{\mathcal{X}}(y)|$ for all $x, y \in G$.

Lemma 2.11 (Comparison of metrics on nilmanifolds [3, Lemma A.17])**.** Let $Q \geq 2$. Suppose that $G' \subseteq G$ is a closed subgroup and that $\mathcal{X}, \mathcal{X}'$ are Q-rational Mal'cev bases for G/Γ and G'/Γ' respectively such that each X_i' is a Q-rational combination of the X_i. Let d and d' be the metrics induced on G/Γ and G'/Γ' respectively. Then for any $x, y \in G'$, we have

$$d'(x\Gamma', y\Gamma') \ll Q^{O(1)} d(x\Gamma, y\Gamma) \text{ and } d(x\Gamma, y\Gamma) \ll Q^{O(1)} d'(x\Gamma', y\Gamma').$$

2.4 Equidistribution in nilmanifolds

Definition 2.12. Let G/Γ be a nilmanifold with unique normalized Haar measure μ and Mal'cev basis $\mathcal{X}$.

(i) Given a length $N > 0$ and an error tolerance $\delta > 0$, a finite sequence $(p(n)\Gamma)_{n=1}^N$ is said to be **δ-equidistributed** if we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} F(p(n)\Gamma) - \int_{G/\Gamma} F d\mu \right| \leq \delta ||F||_{\mathrm{Lip}}$$

for all Lipschitz functions $F : G/\Gamma \to \mathbb{C}$, where

$$||F||_{\mathrm{Lip}} := ||F||_\infty + \sup_{x,y \in G/\Gamma, x \neq y} \frac{|F(x) - F(y)|}{d_{\mathcal{X}}(x, y)}. \tag{6}$$

(ii) The finite sequence $(p(n)\Gamma)_{n=1}^N$ is **totally δ-equidistributed** if we have

$$\left| \frac{1}{|P|} \sum_{n \in P} F(p(n)\Gamma) - \int_{G/\Gamma} F d\mu \right| \leq \delta ||F||_{\mathrm{Lip}}$$

for all Lipschitz functions $F : G/\Gamma \to \mathbb{C}$ and all arithmetic progressions $P \subseteq [1, N]$ of length at least δN.

Lemma 2.13. Let G/Γ be a nilmanifold of dimension m endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For $g \in G$ let $B_r(g\Gamma)$ denote the ball of radius r in G/Γ with respect to the $d_{\mathcal{X}}$ metric.

(i) For $0 < r < r_0 \asymp Q^{-O(1)}$ and $g \in G$, we have

$$Q^{-O(1)} r^m \leq \mu(B_r(g\Gamma)) \leq Q^{O(1)} r^m. \tag{7}$$

(ii) If $\delta < r_0$ and $(p(n)\Gamma)_{n=1}^N$ is $Q^{-O(1)}\delta^{m+1}$-equidistributed, then $(p(n)\Gamma)_{n=1}^N$ visit every ball of radius δ in G/Γ.

Proof of (i). Firstly, we will show that the injectivity radius of the quotient map q given by $q(x) = x\Gamma$ is $Q^{-O(1)}$. Indeed, suppose that $0 < r < \operatorname{diam}(G/\Gamma), x, y \in \psi^{-1}([0,1)^m)$, and $\gamma_1 \neq \gamma_2 \in \Gamma$ are such that $d(x, y\gamma_1), d(x, y\gamma_2) < r$. We see that $d(y^{-1}, \mathrm{Id}_G) = d(y, \mathrm{Id}_G) \leq 1$, so Lemma 2.6 tells us that $|\psi(y^{-1})| < Q^{O(1)}$. Since $d(y\gamma_1, y\gamma_2) < 2r$, by Lemma 2.7 we have $d(\gamma_1, \gamma_2) < 2rQ^{O(1)}$. Similarly, we see that $d(y\gamma_i, y) \leq d(y\gamma_i, x) + d(x, y) \leq r + \operatorname{diam}(G/\Gamma) \leq 2Q^{O(1)}$, hence we may use Lemma 2.7 again to see that $d(\gamma_i, \mathrm{Id}_G) < Q^{O(1)}$ for $i = 1, 2$. We may now use Lemma 2.6 to see that $d(\gamma_1, \gamma_2) \geq Q^{-O(1)}|\psi(\gamma_1) - \psi(\gamma_2)| \geq Q^{-O(1)}$, hence $r \geq Q^{-O(1)}$ as claimed.

Now let r_0' denote the injectivity radius of q. We see that for $r \in (0, r_0')$ and $x \in \psi^{-1}([0,1)^m)$ we have that q is a diffeomorphism between $B_r(x)$ and $B_r(x\Gamma)$, hence $\mu(B_r(x)) = \mu(B_r(x\Gamma))$. Using Lemma **??**, we set $r_0 = Q^{-O(1)} r_0'$ and we see that for $r \in (0, r_0)$ and $x \in \psi^{-1}([0,1)^m)$ we have

$$B_{Q^{-O(1)}r}(\psi(x)) \subseteq \psi(B_r(x)) \subseteq B_{Q^{O(1)}r}(\psi(x)). \tag{8}$$

As discussed in Remark 2.4, ψ pushesforward the Haar measure of G to the Lebesgue measure of $\mathbb{R}^m$, so $Q^{-O(1)} r^m \leq \mu(B_r(x)) \leq Q^{O(1)} r^m$, from which the desired result follows. □

Proof of (ii). For $g \in G$ let $F_g : G/\Gamma \to \mathbb{C}$ be given by

$$F_g(x\Gamma) = \begin{cases} \delta - d_{\mathcal{X}}(x\Gamma, g\Gamma) & \text{if } d_{\mathcal{X}}(x\Gamma, g\Gamma) \leq \delta \\ 0 & \text{else} \end{cases}. \tag{9}$$

We see that for all $g \in G$ we have

$$\int_{G/\Gamma} F_g d\mu \geq \frac{\delta}{2} \mu(B_{\frac{\delta}{2}}(g\Gamma)) \geq Q^{-A} \left(\frac{\delta}{2}\right)^{m+1}, \text{ and } ||F_g||_{\mathrm{Lip}} = 1 + \delta, \tag{10}$$

where the constant $A > 0$ is given to us by part (i). The desired result now follows from the observation that

$$\int_{G/\Gamma} F_g d\mu \geq Q^{-A-m-2} \left(\frac{\delta}{6}\right)^{m+1} ||F_g||_{\mathrm{Lip}}. \tag{11}$$

□

Definition 2.14 (cf. [3, Definition 1.18])**.** Let G/Γ be a nilmanifold with a Mal'cev basis $\mathcal{X}$. Let $(\epsilon(n))_{n\in\mathbb{Z}}$ be a sequence in G, and let $M, N \geq 1$. We say that $(\epsilon(n))_{n\in\mathbb{Z}}$ is (M,N)**-smooth** if we have $d(\epsilon(n), \mathrm{id}_G) \leq M$ and $d(\epsilon(n), \epsilon(n-1)) \leq M/N$ for all $1 \leq n \leq N$.

Theorem 2.15 ([3, Theorem 1.19])**.** Let $m, d \geq 0$, and let $M_0, N \geq 1$ and $A > 0$ be real numbers. Suppose that G/Γ is an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d. Suppose that $\mathcal{X}$ is an M_0-rational Mal'cev basis adapted to $G_\bullet$ and that $p \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. Then there is an integer M with $M_0 \leq M \ll M_0^{O_{A,m,d}(1)}$, a rational subgroup $G' \subseteq G$, a Mal'cev basis $\mathcal{X}'$ for G'/Γ' in which each element is an M-rational combination of the elements of $\mathcal{X}$, and a decomposition $p = \epsilon g'\gamma$ into polynomial sequences $\epsilon, g', \gamma \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ with the following properties:

(i) $\epsilon : \mathbb{Z} \to G$ is (M,N)-smooth;

(ii) $g' : \mathbb{Z} \to G'$ takes values in G', and the finite sequence $(g'(n)\Gamma')_{n=1}^N$ is totally M^{-A}-equidistributed in G'/Γ', using the metric $d_{\mathcal{X}'}$ on G'/Γ';

(iii) $\gamma : \mathbb{Z} \to G$ is M-rational, and $(\gamma(n)\Gamma)_{n\in\mathbb{Z}}$ is periodic with period at most M.

Remark 2.16. Since $\epsilon(n)g'(n)\gamma(n) = \epsilon(n)g''(n)\gamma'(n)$ where $g''(n) = g'(n)\gamma(0)$ and $\gamma'(n) = \gamma(0)^{-1}\gamma(n)$, we will assume without loss of generality that $\gamma(0) = \mathrm{Id}_G$. While $\gamma(0)^{-1}$ is not necessarily M-rational, it need only be $M^{O(1)}$-rational, this will not be of concern to us, as we will only care about the period of γ.

We mention that [5] has quantitative improvements of the equidistribution results of [3], but we do not state them here since we do not need them for our purposes.

3 The proof

Lemma 3.1. Let $m, n \in \mathbb{N}$, $\rho \in (0,1)$, and $B \subseteq [0,1]^m$ a path between two distinct vertices of the cube. Let μ^d denote the normalized d-dimensional Lebesgue measure on $[0,1]^d$. Let $\mathrm{Grid}(\rho, n) \subseteq [0,1]^m$ denote the set of points that are within a distance of ρ/n of the boundary of a subcube of $[0,1]^m$ whose vertices consist of rational numbers with denominator n. For $\theta \in [0,1]^m$, let $\mathrm{Grid}(\rho, n, \theta) = \theta + \mathrm{Grid}(\rho, n) \pmod 1$. We have

$$\mu^d\left(\{\theta \mid B \text{ intersects at least } n \text{ connected components of } \mathrm{Grid}(\rho, n, \theta)^c \pmod 1\}\right) \geq 1 - mn\rho.$$

Proof. Since B is a path between two distinct vertices v_1 and v_2 of $[0,1]^m$, we may assume without loss of generality that $v_1 = (0, \cdots)$ and $v_2 = (1, \cdots)$, i.e., that v_1 and v_2 differ in the first coordinate. Consequently, we may pick points $\{p_i\}_{i=0}^{n-1}$ on B for which the first coordinate of p_i is $\frac{i+\frac{1}{2}}{n}$. We see that for $i \neq j$, we can never have p_i and p_j lie in the same connected component of $\mathrm{Grid}(\rho, n, \theta)^c \pmod 1$. Let $\theta = (\theta_1, \cdots, \theta_m)$. We see that for each $1 \leq j \leq m$, there is a probability of ρ that

the jth coordinate of $p_i + \theta \pmod 1$ is within ρ/n of a rational number with denominator n, so with probability $1 - m\rho$ we have that $p_i + \theta \in \mathrm{Grid}(\rho, n, \theta)^c$. We use a union bound once again to obtain the desired result. □

Theorem 3.2. If $D \subseteq \mathbb{N}$ is 2−large, then D is a set of pointwise recurrence for basic nilsystems.

Proof. Since this proof is longer than any of the proofs appearing in Section 2, we will not use big O notation for the sake of clarity. Instead, let us pick $C = C(m, s, d)$ to be large enough that all of the results of Section 2 hold with C in place of $O(1)$.

Let us fix a nilmanifold G/Γ, a Q-rational Mal'cev basis $\mathcal{X}$ adapted to the lower central series filtration, an $\epsilon > 0$, and a $g \in G$. Let $D = \{n \in \mathbb{N} \mid d(g^n\Gamma, \Gamma) \geq \epsilon\}$. Let us also fix a small $\delta > 0$ and a large $N \in \mathbb{N}$ that will be specified later. By enlarging $C = C(m, s, d)$ if necessary, we will further assume that if $G' \subseteq G$ is a closed connected rational subgroup and $\mathcal{X}'$ is a Mal'cev basis adapted to $G'/(G' \cap \Gamma)$ for which every element of $\mathcal{X}'$ is a M-rational combination of elements of $\mathcal{X}$ for some $M \geq 2$, we have that $\mathcal{X}'$ is a M^C-rational Mal'cev basis.

We apply Theorem 2.15 with $A = (3C + 1)(m + 1) + 3m^2(m + 1) - (m + 1)\log_Q(\frac{\delta^3}{m})$, and $M_0 = Q$. Let $\{G_i'\}_{i=1}^{I_M}$ be an enumeration of the rational subgroups of G that possess a Mal'cev basis $\mathcal{X}_i'$ for G_i'/Γ_i' in which $\Gamma_i' = G_i' \cap \Gamma$ and each element of $\mathcal{X}_i'$ is a M-rational combination of elements of $\mathcal{X}$. We assume without loss of generality that $M \geq Q \geq 3$ and observe that $I_M \leq (M^m(2M + 1)^m)^m \leq M^{3m^2}$. We assume without loss of generality that $G_1' = \mathrm{id}_G$ and $G_{I_M} = G$. Consider the fundamental domain $F = \psi_{\mathcal{X}}^{-1}([0, 1)^m)$ for G/Γ, and for $1 \leq i \leq I_M$ let $F_i \subseteq G_i \cap F$ be a fundamental domain for G_i/Γ_i.

We assume that $\delta^{-1} \in \mathbb{N}$, and we partition $[0, 1)^m$ into boxes of side length δ. Let $B_r(x\Gamma_i')$ denote the ball of radius r in the $d_{\mathcal{X}_i'}$ metric centered at $x\Gamma_i'$. For each $1 \leq i \leq I_M$, let μ_i be the Haar measure of G_i' normalized so that $\mu_i(F_i) = 1$, and let ν_i be the pushforward of μ_i under $\psi_{\mathcal{X}}$ to a probability measure on $[0, 1)^m$. For each $i > 1$, we know that the dimension m_i' of G_i' is at least 1. Since G_i'/Γ_i' is isomorphic to $G_i'\Gamma/\Gamma$, it is naturally identified with the submanifold $B_i := \psi_{\mathcal{X}}(G_i'\Gamma/\Gamma)$ of dimension m_i' in $[0, 1)^m$. In fact, we see that for $i > 1$ we have that $\overline{B_i}$ contains a path connecting two distinct vertices of $[0, 1]^m$. We apply Lemma 3.1 with $n = \delta^{-1}$, $\rho = \frac{M^{-3m^2}}{m}\delta^2$, and the paths B_i to see that there exists $\theta \in [0, 1]^m$ for which each B_i intersects at least n distinct connected components of $\mathrm{Grid}(\rho, \delta^{-1}, \theta)^c \pmod 1$ which we enumerate as $\{B_{i,j}\}_{j=1}^{\delta^{-1}}$.

We will now use the probabilistic method Rewrite this sentence for better exposition. We see that there are $2^{\delta^{-m}}$ ways to 2-color the cells of $\mathrm{Grid}(0, \delta^{-1}, \theta)^c$. For each $2 \leq i \leq I_M$, there are $2^{\delta^{-m}-\delta^{-1}+1}$ "bad" colorings in which the collection $\{B_{i,j}\}_{j=1}^{\delta^{-1}}$ is monochromatic. In total, this yields at most $(I_M - 1)2^{\delta^{-m}-\delta^{-1}+1} \leq M^{3m^2}2^{\delta^{-m}-\delta^{-1}+1}$ bad colorings, i.e., 2-colorings in which there exist some $2 \leq i \leq I_M$ for which $\{B_{i,j}\}_{j=1}^{\delta^{-1}}$ is monochromatic. Since $M \leq Q^{p(A,m,s,d)}$ for some polynomial function p, and $A \asymp -\ln(\delta)$, we see that for $\delta > 0$ small enough we have that $M^{3m^2} < 2^{\delta^{-1}-1}$. In particular, for such δ there exists a 2-coloring χ' of the connected components of $\mathrm{Grid}(\rho, \delta^{-1}, \theta)^c$ such that for every $2 \leq i \leq I_M$, the collection $\{B_{i,j}\}_{j=1}^{\delta^{-1}}$ is not monochromatic. Now we produce a 2-coloring χ of $\mathbb{N}$ by $\chi(n) = \chi'(g_0^n)$, and we observe that χ depends only on δ and g.

Now suppose that $d_0 \in D$, $a \in \mathbb{N}$, $g_0 \in G$, let $g := \exp(\frac{1}{M!}\log(g_0))$, and observe that $g_0 = g^{M!}$. Let $1 \leq i \leq I_M$ be such that when we apply Theorem 2.15 to $p(n) = g_0^{a+d_0 n}$, we obtain $p(n) = \epsilon g' \gamma$ with g' taking values in G_i'. We claim that for $N > M(M!)\epsilon^{-1}$, we cannot have $i = 1$. Indeed, if we assume for the sake of contradiction that $i = 1$, then $G = g' = \mathrm{Id}_G$, hence $g^{d_0} = g_0^{d_0 M!} = p(M!) = \epsilon(M!)g'^M\gamma(M!) = \epsilon(M!)$. We have that $d(g^{d_0}\Gamma, \Gamma) \geq \epsilon$ by assumption, which contradicts the fact that $d(\epsilon(M!), \mathrm{Id}_G) \leq M(M!)/N < \epsilon$.

We now take $N'' = M^{A+1}Q^C\delta^{-m-2} + M(M!)\epsilon^{-1}$. Let $M_1 \leq M$ be the period of γ, $d' = M_1 d$ and $N' = N''/M$ and note that $\{g'(M_1 n)\}_{n=1}^{N'}$ is $M^{-(3C+1)(m+1)}\rho^{m+1}\delta^{m+1}$-equidistributed in G_i'/Γ_i' by our choice of A. Lemma 2.13(iii) tells us that $\{g'(n)\Gamma_i'\}_{n=1}^{N'}$ visits every ball in G_i'/Γ_i' of radius $M^{-2C-1}\rho\delta$. Let $q_i : G_i' \to G_i'/\Gamma_i'$ denote the quotient map $x \mapsto x\Gamma_i'$. We see that the image of a ball of radius r centered at $x \in F_i \subseteq G_i'$ under q_i contains the ball of radius r centered at $x\Gamma_i'$, so $\{\{g'(M_1 n)\}\}_{n=1}^{N'}$ visits every ball of radius $M^{-2C-1}\rho\delta$ in F_i. Lemma 2.8 tells us that when we view F_i as a subset of G endowed with the $d_{\mathcal{X}}$ metric instead of the $d_{\mathcal{X}_i'}$ metric, we have that $\{\{g'(M_1 n)\}\}_{n=1}^{N'}$ visits every ball of radius $M^{-C-1}\rho\delta$. We see that for $x \in F$ and $r > 0$, $\psi_{\mathcal{X}}^{-1}(B_{Q^C r}(\psi_{\mathcal{X}}(x)) \supseteq B_r(x)$, so $\{\psi_{\mathcal{X}}(\{g'(M_1 n)\})\}_{n=1}^{N'}$ visits every balls of radius $M^{-1}\rho\delta$ (in the ℓ^∞ norm) in $[0,1)^m$ centered at a point on $\psi_{\mathcal{X}_i'}(F_i)$. In particular, we have that $\{\psi_{\mathcal{X}}(\{g'(M_1 n)\})\}_{n=1}^{N'}$ visits every balls of radius $M^{-1}\rho\delta$ centered at a point in $B_{i,j} \cap \psi_{\mathcal{X}}(F_i)$ for all $1 \leq j \leq n$. Since G is a topological group, let $r_1 > 0$ be such that for any $u \in B_{r_1}(\mathrm{Id}_G)$ and $f \in F$ we have $d_{\mathcal{X}}(uf, f) < M^{-1}\rho\delta$, hence we also have $d_{\mathcal{X}}(uf\Gamma, f\Gamma) < M^{-1}\rho\delta$. We let N be such that $MN''/N < r_1$, as we then have that $d_{\mathcal{X}}(\epsilon(n), \mathrm{Id}_G) < r_1$ for $1 \leq n \leq N''$. It follows that $d_{\mathcal{X}}(\epsilon(M_1 n)g'(M_1 n), g'(M_1 n)) < M^{-1}\rho\delta$, hence for each $1 \leq j \leq \delta^{-1}$, there is a point in the orbit $\{\psi_{\mathcal{X}}(\{g_0^{d_0 n}\})\}_{n=1}^{N}$ that is the same connected component of $\mathrm{Grid}(\rho, \delta^{-1}, \theta)$ as $B_{i,j}$

□

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