2-Large sets are sets of pointwise recurrence for nilsystems (draft)

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Abstract

A set $S \subset \mathbb{N}$ is r-large if for any r-coloring of the natural numbers, there are arbitrarily long monochromatic arithmetic progressions whose common difference comes from S. The set D is large if it is r-large for all r. In 1999 Brown, Graham, and Landman conjectured that if S is 2-large, then it is also large.

We prove that every 2-large set is a set of pointwise recurrence for nilsystems. In other words, we show that if a set is not large because of a (generalized) polynomial obstacle, then it is also not 2-large.

1 Introduction

The subject of the present paper on arithmetic progressions is the following notion of *largeness*, introduced by Brown, Graham, and Landman [4].

Definition 1.1 (r-large)**.** For $r \in \mathbb{N}$ and $S \subseteq \mathbb{N}$, we say that S is r-large if for any r-coloring $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exists a color $1 \leq i \leq r$ so that C_i contains arbitrarily long arithmetic progressions with common difference in S.

We say that S is *large* if it is r-large for all r. Equivalently, S is large if and only if the restricted van der Waerden theorem with common difference in S is true. That $\mathbb{N}$ itself is large is the content of van der Waerden's theorem. Clearly if a set is large, then it is r-large for all $r \geq 2$, and an r-large set for $r \geq 2$ is clearly 2-large. Brown, Graham, and Landman conjectured that in fact all of these notions of largeness coincide.

Conjecture 1.2 ([4])**.** $D \subseteq \mathbb{N}$ is 2-large, then D is large.

Perhaps it is a little bit more intuitive to think about this conjecture in the contrapositive. If a set S is *not* large, then the restricted van der Waerden with common difference in S is false, and so some finite coloring blocks arbitrarily long arithmetic progressions. The question of Brown, Graham, and Landman is then whether two colors is sufficient to block such progressions.

One basic class of non-large sets are those coming from Bohr sets. For instance if S is the set of d so that $1/10 < \{d\alpha\} < 9/10$ or $1/10 < \{d\beta\} < 9/10$, then if we color x based on $\{x\alpha\}$ and

$\{x\beta\}$ up to ε error then even $x, x+d$ will not have the same color, so S will not be large. The first author [1] recently proved that such sets will not be 2-large.

A more general and very important class of non-large sets are those coming from generalized polynomials. For example, let S be the set of d so that $1/4 < \{d^2\alpha\} < 3/4$. Since $(x+2d)^2 - 2(x+d)^2 + x^2 = 2d^2$, if we color x by $x^2/2$ modulo α up to some ϵ error, then $x, x+d, x+2d$ cannot all have the same color in this coloring if $d \in S$. Equivalently, these sets come from nilsystems, which we will formally define in the next section.

Definition 1.3 (Pointwise recurrence for nilsystems)**.** A set $S \subset \mathbb{N}$ is a set of pointwise recurrence for nilsystems if for every d-step nil-system (X, T) there is a point $x \in X$ and a subsequence $S' \subset S$ so that for all $1 \leq d' \leq d$, $\lim_{n \in S'} T^{d'n} x = x$.

This has an equivalent combinatorial definition in terms of generalized polynomials. Let the nearest integer to x be $[x] := \lfloor x + 1/2 \rfloor$.

Definition 1.4 (Special Generalized Polynomials, after [9])**.** We define L so that $L(x) = x$ and $L(x_1, \ldots, x_\ell) = x_1[L(x_2, \ldots, x_\ell)]$.

Then the special generalized polynomials of degree at most d are the expressions

$$L(n^{j_1} a_1, \ldots, n^{j_\ell} a_\ell)$$

such that $j_i \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $1 \leq i \leq \ell$, and $\sum_{i=1}^{\ell} j_i \leq d$.

In particular, we must have $\ell \leq d$.

Definition 1.5 (Pointwise recurrence for nilsystems, combinatorial definition)**.** A set $S \subset \mathbb{N}$ is a set of pointwise recurrence for nilsystems if for every finite family of special generalized polynomials $P_1, \cdots, P_k$ with degree at most d and every ε, there exists $s \in S$ so that $||P_i(s)||_{\mathbb{R}/\mathbb{Z}} < \varepsilon$ for all $1 \leq i \leq k$.

Huang, Shao, and Ye [9] prove the equivalence of these definitions for a set of pointwise recurrence for nilsystems. One can check for instance that the set of α so that $1/4 < n^2\alpha < 3/4$ is not such a set, as then $||L(n^2\alpha)||_{\mathbb{R}/\mathbb{Z}} > 1/4$.

We also note that Huang, Shao, and Ye [9] also asked whether if S is a set of pointwise recurrence for nilsystems, then S is large. Recently, the first author [2] found an example of a non-large set which is nonetheless a set of pointwise recurrence for nilsystems.

We are now ready to state the main theorem of our paper.

Theorem 1.6. If $S \subseteq \mathbb{N}$ is $2-$large, then S is a set of pointwise recurrence for nilsystems.

The proof extends that of [1], but we must work with nilmanifolds rather than with only tori.

2 Nilmanifold Preliminaries

For our discussion about nilmanifolds, we use a mix of the presentations given in [6] and [8], and we refer the reader to those sources for additional details.

2.1 Basic properties

For the rest of this paper, we will use the following standard notation when talking about nilmanifolds. We use G to denote a finite dimensional nilpotent (real) Lie group with identity element id_G and let G^0 be the connected component containing the identity. Furthermore, we will have a standing assumption that G is generated by G^0 and finitely many commuting elements $\{\tau_i\}_{i=1}^k$. Moreover, we will assume that the group generated by $\{\tau_i\}_{i=1}^k$ is isomorphic to $\mathbb{Z}^k$ and has trivial intersection with G^0. These assumptions are standard in ergodic theory, but not in the general theory of Lie algebras and Lie groups. We will use Γ to denote a discrete cocompact subgroup of G, and we use $N := G/\Gamma$ to denote the set of left cosets of Γ in G. A **nilmanifold** is a space of the form $N = G/\Gamma$ with G and Γ satisfying the above conditions.

Now let G be a Lie group of (real) dimension m. As usual, let $[H, K]$ denote the subgroup generated by all commutators $[h, k] := hkh^{-1}k^{-1}$ for $h \in H, k \in K$.

A **filtration** $G_\bullet$ on G is a sequence of closed subgroups

$$G = G_0 = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_d \supseteq G_{d+1} = \{\mathrm{id}_G\}$$

such that $[G_i, G_j] \subseteq G_{i+j}$ for all integers $i, j > 0$. If G is connected, then we also require G_i to be connected for all i. The least integer d for which $G_{d+1} = \{\mathrm{id}_G\}$ is called the *degree* of the filtration.

Since G is nilpotent, G always possesses the lower central series filtration, namely, the filtration in which $G_i = \overline{[G, G_{i-1}]}$ for all $2 \leq i \leq c+1$, where c is the nilpotency class of G. However, G may also possess other filtrations, so the degree of a given filtration $G_\bullet$ may differ from the nilpotency class of G. When we are given a filtration $G_\bullet$ of G, we use m to denote the (real) dimension of G, and m_i to denote the dimension of G_i for $0 \leq i \leq d+1$.

Lemma 2.1 ([8, Corollary 10.26])**.** Every nilmanifold $N = G/\Gamma$ can be embdedded as a normal subnilmanifold of a nilmanifold $N' = G'/\Gamma'$ such that G' is connected and simply connected and such that every element of G is represented in G'.

A **nilsystem** is a pair (N, T_g) where N is a nilmanifold, $g \in G$, and $T_g : N \to N$ is given by $T_g(x\Gamma) = gx\Gamma$. It is well known that for nilsystems, transitivity, minimality, and unique ergodicity are equivalent. A

2.2 Lie algebras, logarithms, and exponentials

2.3 Mal'cev bases

Often, we will want to give real coordinates to a nilmanifold $N = G/\Gamma$. This can be accomplished by identifying a *Mal'cev basis*. We only define these in the case where G is connected and simply connected, as we will always be able to use Lemma 2.1 to reduce to this case.

Definition 2.2. [[6, Definition 2.1], Mal'cev basis] Let $N = G/\Gamma$ be a nilmanifold with G a m-dimensional connected and simply connected Lie group. Let $G_\bullet = \{G_i\}_{i=0}^{d+1}$ be a filtration on G

and let $\mathfrak{g}$ denote the (real) Lie algebra of G. A basis $\mathcal{X} = \{X_1, \cdots, X_m\}$ for $\mathfrak{g}$ over $\mathbb{R}$ is called a Mal'cev basis for G/Γ adapted to $G_\bullet$ if the following four conditions are satisfied:

(i) For each $j = 0, \cdots, m-1$ the subspace $\mathfrak{h}_j := \mathrm{Span}(X_{j+1}, \cdots, X_m)$ is a Lie algebra ideal in $\mathfrak{g}$, and hence $H_k := \exp \mathfrak{h}_j$ is a normal Lie subgroup of G.

(ii) For every $0 \leq i \leq d$ we have $G_i = H_{m-m_i}$ (recall that $m_i = \dim G_i$); I am a bit confused by this one

(iii) Each $g \in G$ can be written uniquely as $g = \exp(t_1 X_1)\exp(t_2 X_2)\cdots\exp(t_m X_m)$, for $t_i \in \mathbb{R}$;

(iv) Γ consists precisely of those elements which, when written in the above form, have all $t_i \in \mathbb{Z}$.

It is a well known result of Mal'cev [11] that any nilmanifold G/Γ can be equipped with a Mal'cev basis adapted to the lower central series filtration. More generally there is a Mal'cev basis adapted to any filtration $G_\bullet$ consisting of rational subgroups. We refer to the t_i as the **Mal'cev coordinates** of g, and we define the **Mal'cev coordinate map** $\psi = \psi_{\mathcal{X}} : G \to \mathbb{R}^m$ by

$$\psi(g) = (t_1, \cdots, t_m). \tag{1}$$

A nice property of Mal'cev coordinates is that they show the polynomial nature of the group operation of G. More specifically, there exist polynomials $\{P_i\}_{i=1}^m$ with $P_1 = 0$, taking integer values on integer inputs, such that for every $g, h \in G$ with $\psi(g) = (t_1, \cdots, t_m)$ and $\psi(h) = (s_1, \cdots, s_m)$ we have

$$\psi(gh) = (t_1 + s_1, t_2 + s_2 + P_2(t_1, s_1), \cdots, t_m + s_m + P_m(t_1, \cdots, t_{m-1}, s_1, \cdots, s_{m-1})). \tag{2}$$

Furthermore, there exist polynomials $\{\tilde{P}_i\}_{i=1}^m$ with $\tilde{P}_1 = 0$, taking integer values on integer inputs, such that for every $g \in G$ with $g = (t_1, \cdots, t_m)$ we have

$$\psi(g^{-1}) = (-t_1, -t_2 + \tilde{P}_2(t_1), \cdots, -t_m + \tilde{P}_m(t_1, \cdots, t_{m-1})). \tag{3}$$

Furthermore, if $G_\bullet$ is the lower central series filtration, and m_1 is the real dimension of $G_2 = [G, G]$, then $P_i = \tilde{P}_i = 0$ for $1 \leq i \leq m - m_1$. More generally, for $0 \leq j \leq d$ and $m - m_j < i \leq m - m_{j+1}$ we have $\deg_t(P_i), \deg_s(P_i), \deg_t(\tilde{P}_i), \deg_s(\tilde{P}_i) \leq j$, where $\deg_t(P)$ denotes the degree of P when viewed as a polynomial in the variables $\{t_i\}_{i=1}^m$ and the s_j are held constant. We will only need to make use of the simpler bounds that $\deg_t(P_i), \deg_s(P_i), \deg_t(\tilde{P}_i), \deg_s(\tilde{P}_i) \leq i$ for all $2 \leq i \leq m$. From this, we deduce that for any $g \in G$ with $\psi(g) = (t_1, \cdots, t_m)$, we have $\psi(g^n) = (Q_1(t_1; n) := nt_1, Q_2(t_1, t_2; n), \cdots, Q_m(t_1, \cdots, t_m; n))$ for some polynomials $\{Q_i\}_{i=1}^m$ satisfying $\deg_t(Q_i), \deg_n(Q_i) \leq i$. For the sake of notational convenience, we will write $P_i(t_1, \cdots, t_m, s_1, \cdots, s_m), \tilde{P}(t_1, \cdots, t_m)$ and $Q_i(t_1, \cdots, t_m; n)$ for all $1 \leq i \leq m$ with the understanding that P_i does not use t_j and s_j for $j > 1$, and similarly for $\tilde{P}_i$ and Q_i. We will also be using the fact that ψ is a diffeomorphism (cf. [5, Proposition 1.2.7]).

Definition 2.3 ([6, Definition 2.2], Metrics on G and G/Γ)**.** Let G/Γ be a nilmanifold with Mal'cev basis $\mathcal{X}$. We define $d = d_{\mathcal{X}} : G \times G \to \mathbb{R}_{\geq 0}$ to be maximal such that $d(x,y) \leq |\psi(xy^{-1})|$ for all $x, y \in G$, where $|\cdot|$ denotes the ℓ^∞ norm on $\mathbb{R}^m$. More explicitly, we have

$$d(x,y) = \inf\left\{\sum_{i=0}^{n-1} \min\left(|\psi(x_{i-1}x_i^{-1})|, |\psi(x_i x_{i-1}^{-1})|\right) : x_0, \cdots, x_n \in G; x_0 = x; x_n = y\right\}.$$

This descends to a metric on G/Γ by setting

$$d(x\Gamma, y\Gamma) := \inf\{d(x',y') \ : \ x', y' \in G; x' \equiv x \pmod{\Gamma}; y' \equiv y \pmod{\Gamma}\}.$$

It turns out that this is indeed a metric on G/Γ (cf. [6, Lemma A.15]). We see that d is also right-invariant, hence

$$d(x\Gamma, y\Gamma) = \inf_{\gamma\in\Gamma} d(x, y\gamma) = \min_{\gamma\in\Gamma} d(x, y\gamma). \tag{4}$$

When the letter d is used for a metric, it will always denote the metric $d_{\mathcal{X}}$ relative to some basis $\mathcal{X}$ that is already under discussion. The symbol d' will be used for the metric defined using some other basis $\mathcal{X}'$. In the proof of Theorem 4.2 we will be considering many nilmanifolds and their respective Mal'cev bases, so then we will explicitly write $d_{\mathcal{X}}$ in each situation.

Remark 2.4 (Haar measure and Lebesgue measure)*.* We see that for each $g \in G$, $\psi_{\mathcal{X}}(g)$ acts on $\mathbb{R}^m = \psi_{\mathcal{X}}(G)$ as an upper triangular matrix with 1s on the diagonal and polynomial functions in the entries above the diagonal. Since each $\psi_{\mathcal{X}}(g)$ has determinant 1, regardless of where it is acting on $\mathbb{R}^m$, we see that the m-dimensional Lebesgue measure μ on $\mathbb{R}^m$ is invariant under the action of $\psi_{\mathcal{X}}(g)$ for all $g \in G$. Since G is a nilpotent Lie group, it is unimodular, so the Haar measure μ of G normalized so that $\mu(\psi_{\mathcal{X}}^{-1}([0,1)^m)) = 1$ is the pushforward of μ from $\mathbb{R}^m$, which is why we choose to abuse notation and write μ for both measures.

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The **height** of a rational number $\frac{p}{q}$ with $\gcd(p,q) = 1$ is q, and by convention we set the height of 0 to be 1. The Lie algebra basis $\mathcal{X}$ is Q**-rational** if for every $1 \leq i, j \leq m$ we have $[X_i, X_j] = \sum_{k=1}^m c_{i,j,k} X_k$ where every $c_{i,j,k}$ is a rational number of height at most Q. We will frequently use this rationality parameter Q of $\mathcal{X}$ in our inequalities, so we place a standing assumption that $Q \geq 2$ in any instance that we write $Q^{O(1)}$. If $\mathcal{X}$ is a Mal'cev basis of a nilmanifold G/Γ adapted to some filtration $G_\bullet$, then $\mathcal{X}$ is Q-rational for some $Q \in \mathbb{N}$.

Definition 2.5. Suppose that a nilmanifold G/Γ is given together with a Mal'cev basis $\mathcal{X} = \{X_1, \cdots, X_m\}$ and that $Q > 0$. Suppose that $G' \subseteq G$ is a closed connected subgroup. We say that G' is Q**-rational (relative to** $\mathcal{X}$**)** if the Lie algebra $\mathfrak{g}'$ of G' has a basis $\mathcal{X}' = \{X_1', \cdots, X_{m'}'\}$ consisting of linear combinations $\sum_{i=1}^m a_i X_i$, where a_i are rational numbers with height at most Q for all i. If we can further assume that $\mathcal{X}'$ is a Q-rational Mal'cev basis for $G'/(\Gamma \cap G')$, then we

call G' **strongly Q-rational (relative to $\mathcal{X}$)**. If G' is Q-rational for some $Q > 0$, then we say that G' is **rational**. We say that G' is **proper** if $G' \neq G$.

It is worth noting that G' is a rational subgroup of G if and only if $G' \cap \Gamma$ is cocompact in G' (cf. [6, Definition 1.10]). In particular, the quantitative rationality of G' with respect to G and Γ depends on the choice of Mal'cev basis $\mathcal{X}$, but the qualitative property of whether or not G' is rational is independent of $\mathcal{X}$. [6, Proposition A.10] tells us that G' is a rational subgroup of G with respect to $\mathcal{X}$ if and only

Remark 2.6 (Asymptotic Notation). We use $a \ll b$ to mean that there exists a constant $C > 0$ for which $a \leq Cb$. Similarly, we use $a = O(b)$ to mean that there is a constant C for which $C^{-1}b \leq a \leq Cb$. In the discussion that follows, the constants C that are implied by our usage of $\ll$ and $O(1)$ will be allowed to depend on the global structure of the nilmanifold G/Γ, but not the local structure. For example, the constant C may depend on the dimension m, the nilpotence degree s, and the degree d of the filtration $G_\bullet$, but not on the Mal'cev basis $\mathcal{X}$ or on the points $g, g_1, g_2, x, y \in G$. For a given result, we will always assume without loss of generality that $C = C(m, s, d)$ is increasing as a function of m, s, and d, i.e., $C(m, s, d) \leq C(m', s', d')$ if $m \leq m'$, $s \leq s'$, or $d \leq d'$. The constants generated by the Mal'cev basis $\mathcal{X}$ and the polynomials $P_i, \tilde{P}_i$, and Q_i will be absorbed into the $Q^{O(1)}$ terms. We will also differ from the conventions of [6] in that we will write $a \leq Q^{O(1)}b$ instead of $a \ll Q^{O(1)}b$, as we will always assume that $Q \geq 2$, so the constant from $\ll$ can be absorbed into the $Q^{O(1)}$ term.

Lemma 2.7 ([6, Lemma A.4])**.** Suppose that $Q \geq 2$. Suppose that $\mathcal{X}, \mathcal{X}'$ are two Q-rational Mal'cev bases for G/Γ. Suppose that each X_i' is given by a Q-rational combination of the X_i and vice versa. Then for all $x, y \in G$ with $|\psi'(x)|, |\psi'(y)| \leq Q$, we have the bound

$$d(x, y) \ll Q^{O(1)}|\psi'(x) - \psi'(y)|,$$

and for all $x, y \in G$ with $d(x, \mathrm{id}_G), d(y, \mathrm{id}_G) \leq Q$, we have the bound

$$|\psi'(x) - \psi'(y)| \ll Q^{O(1)}d(x, y).$$

Lemma 2.8 (Approximate left-invariance of d, [6, Lemma A.5])**.** Suppose $Q \geq 2$ and that $\mathcal{X}$ is a Q-rational basis for G/Γ adapted to some filtration $G_\bullet$. suppose that $g, x, y \in G$ are elements with $|\psi(x)|, |\psi(y)|, |\psi(g)| \leq Q$. Then we have the bound

$$d(gx, gy) \ll Q^{O(1)}d(x, y).$$

Lemma 2.9 (Comparison lemma [6, Lemma A.6])**.** Suppose that $G' \subseteq G$ is a closed subgroup and that $\mathcal{X}, \mathcal{X}'$ are bases for $\mathfrak{g}, \mathfrak{g}'$ respectively which have the nesting property (A.1). Let $Q \geq 2$, and suppose that each X_i' is a Q-rational combination of the X_i. Then we have the bounds

$$d'(x, y) \ll Q^{O(1)}d(x, y)$$

uniformly for all $x, y \in G'$ with $|\psi(x)|, |\psi(y)| \leq Q$ and

$$d(x, y) \ll Q^{O(1)} d'(x, y)$$

uniformly for all $x, y \in G'$ with $|\psi'(x)|, |\psi'(y)| \leq Q$.

Lemma 2.10 ([6, Proposition A.10])**.** Suppose that $\mathcal{X} = \{X_1, \cdots, X_m\}$ is a Q-rational Mal'cev basis for G/Γ adapted to a filtration $G_\bullet$. Suppose that $G' \subseteq G$ is a Q-rational subgroup of G and, furthermore, that $G'_\bullet$ is a filtration of G' in which each of the groups G'_i is Q-rationa (with respect to the basis $\mathcal{X}$). Write $\Gamma' := \Gamma \cap G'$. Then G'/Γ' has a Mal'cev basis $\mathcal{X}' = \{X_1, \cdots, X_m\}$ adapted to $G'_\bullet$ in which each X'_i is a $Q^{O(1)}$-rational combination of the X_i.

Remark 2.11. We will only be interested in applying Lemma 2.10 in the case that the filtration $G'_\bullet$ is given by $G_\bullet \cap G'$, i.e., when $G'_i = G_i \cap G'$. In general, the intersection of a Q_1-rational subgroup with a Q_2-rational subgroup is a $(Q_1 Q_2)^{O(1)}$-rational subgroup, hence each G'_i is a $Q^{O(1)}$-rational subgroup. Consequently, we will assume throughout this paper that a rational subgroup G' of G is endowed with the filtration $G'_\bullet := G_\bullet \cap G'$, and a Mal'cev basis $\mathcal{X}'$ adapted to $G'_\bullet$.

Lemma 2.12 ([6, Lemma A.14])**.** Let G/Γ be a nilmanifold endowed with a Mal'cev basis $\mathcal{X}$. For any $g \in G$, we may write $g = \{g\}[g]$ in a unique way, where $\psi_{\mathcal{X}}(\{g\}) \in [0, 1)^m$ and $[g] \in \Gamma$.

Lemma 2.13 (Nilmanifolds are bounded [6, Lemma A.16])**.** Let $Q \geq 2$, and suppose that $\mathcal{X}$ is a Q-rational Mal'cev basis for a nilmanifold G/Γ with respect to some filtration $G_\bullet$. Then $d(x\Gamma, y\Gamma) \ll Q^{O(1)}$ uniformly in $x, y \in G$.

Lemma 2.14. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For $C > 0$ let $R_C := \psi_{\mathcal{X}}^{-1}([-Q^C, Q^C])$. There exists $0 < C_0 \leq Q^{O(1)}$ such that for any $x, y \in G$, there exist $\gamma_1, \gamma_2 \in \Gamma$ with $d_{\mathcal{X}}(x\Gamma, y\Gamma) = d_{\mathcal{X}}(x\gamma_1, y\gamma_2)$ and $x\gamma_1, y\gamma_2 \in R_{C_0}$. Furthermore, we may assume without loss of generality that $x\gamma_1 \in \psi_{\mathcal{X}}^{-1}([0, 1)^m)$.

Proof. We first pick $\gamma_1 \in \Gamma$ for which $x\gamma_1 \in \psi_{\mathcal{X}}^{-1}([0, 1)^m)$. Let $\gamma_2 \in \Gamma$ be such that $d_{\mathcal{X}}(x\Gamma, y\Gamma) = d_{\mathcal{X}}(x\gamma_1, y\gamma_2)$. Using Lemma 2.13 we see that

$$d_{\mathcal{X}}(y\gamma_2, \mathrm{Id}_G) \leq d_{\mathcal{X}}(x\gamma_1, y\gamma_2) + d_{\mathcal{X}}(x\gamma_1, \mathrm{Id}_G) \leq d_{\mathcal{X}}(x\Gamma, y\Gamma) + |\psi_{\mathcal{X}}(x\gamma_1)| \leq Q^{O(1)}. \tag{5}$$

The desired result now follows from Lemma 2.7. □

When the nilmanifold G/Γ is endowed with a Mal'cev basis $\mathcal{X}$, we use d_∞ to denote the pull back of the ℓ^∞ metric on $\mathbb{R}^m$ to G via $\psi_{\mathcal{X}}$. More concretely, we have $d_\infty(x, y) = |\psi_{\mathcal{X}}(x) - \psi_{\mathcal{X}}(y)|$ for all $x, y \in G$.

Lemma 2.15 (Comparison of metrics on nilmanifolds [6, Lemma A.17])**.** Let $Q \geq 2$. Suppose that $G' \subseteq G$ is a closed subgroup and that $\mathcal{X}, \mathcal{X}'$ are Q-rational Mal'cev bases for G/Γ and G'/Γ' respectively such that each X'_i is a Q-rational combination of the X_i. Let d and d' be the metrics induced on G/Γ and G'/Γ' respectively. Then for any $x, y \in G'$, we have

$$d'(x\Gamma', y\Gamma') \ll Q^{O(1)} d(x\Gamma, y\Gamma) \text{ and } d(x\Gamma, y\Gamma) \ll Q^{O(1)} d'(x\Gamma', y\Gamma').$$

2.4 Equidistribution in nilmanifolds

Definition 2.16. Let G/Γ be a nilmanifold with unique normalized Haar measure μ and Mal'cev basis $\mathcal{X}$.

(i) Given a length $N > 0$ and an error tolerance $\delta > 0$, a finite sequence $(p(n)\Gamma)_{n=1}^N$ is said to be **δ-equidistributed** if we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} F(p(n)\Gamma) - \int_{G/\Gamma} F d\mu \right| \leq \delta ||F||_{\text{Lip}}$$

for all Lipschitz functions $F : G/\Gamma \to \mathbb{C}$, where

$$||F||_{\text{Lip}} := ||F||_\infty + \sup_{x,y \in G/\Gamma, x \neq y} \frac{|F(x) - F(y)|}{d_{\mathcal{X}}(x, y)}. \tag{6}$$

(ii) The finite sequence $(p(n)\Gamma)_{n=1}^N$ is **totally δ-equidistributed** if we have

$$\left| \frac{1}{|P|} \sum_{n \in P} F(p(n)\Gamma) - \int_{G/\Gamma} F d\mu \right| \leq \delta ||F||_{\text{Lip}}$$

for all Lipschitz functions $F : G/\Gamma \to \mathbb{C}$ and all arithmetic progressions $P \subseteq [1, N]$ of length at least δN.

Lemma 2.17. Let G/Γ be a nilmanifold of dimension m endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For $g \in G$ let $B_r(g\Gamma)$ denote the ball of radius r in G/Γ with respect to the $d_{\mathcal{X}}$ metric.

(i) For $0 < r < r_0 \asymp Q^{-O(1)}$ and $g \in G$, we have

$$Q^{-O(1)} r^m \leq \mu(B_r(g\Gamma)) \leq Q^{O(1)} r^m. \tag{7}$$

(ii) If $\delta < r_0$ and $(p(n)\Gamma)_{n=1}^N$ is $Q^{-O(1)}\delta^{m+1}$-equidistributed, then $(p(n)\Gamma)_{n=1}^N$ visit every ball of radius δ in G/Γ.

Proof of (i). Firstly, we will show that the injectivity radius of the quotient map q given by $q(x) = x\Gamma$ is $Q^{-O(1)}$. Indeed, suppose that $0 < r < \text{diam}(G/\Gamma), x, y \in \psi^{-1}([0,1)^m)$, and $\gamma_1 \neq \gamma_2 \in \Gamma$ are such that $d(x, y\gamma_1), d(x, y\gamma_2) < r$. We see that $d(y^{-1}, \text{Id}_G) = d(y, \text{Id}_G) \leq 1$, so Lemma 2.7 tells us that $|\psi(y^{-1})| < Q^{O(1)}$. Since $d(y\gamma_1, y\gamma_2) < 2r$, by Lemma 2.8 we have $d(\gamma_1, \gamma_2) < 2rQ^{O(1)}$. Similarly, we see that $d(y\gamma_i, y) \leq d(y\gamma_i, x) + d(x, y) \leq r + Q^{O(1)}|\psi(x) - \psi(y)| \leq 2Q^{O(1)}$, hence we may use Lemma 2.8 again to see that $d(\gamma_i, \text{Id}_G) < Q^{O(1)}$ for $i = 1, 2$. We may now use Lemma 2.7 to see that $d(\gamma_1, \gamma_2) \geq Q^{-O(1)}|\psi(\gamma_1) - \psi(\gamma_2)| \geq Q^{-O(1)}$, hence $r \geq Q^{-O(1)}$ as claimed.

Now let r_0' denote the injectivity radius of q. We see that for $r \in (0, r_0')$ and $x \in \psi^{-1}([0,1)^m)$ we have that q is a diffeomorphism between $B_r(x)$ and $B_r(x\Gamma)$, hence $\mu(B_r(x)) = \mu(B_r(x\Gamma))$. Using Lemma 2.7, we set $r_0 = Q^{-O(1)}r_0'$ and we see that for $r \in (0, r_0)$ and $x \in \psi^{-1}([0,1)^m)$ we have

$$B_{Q^{-O(1)}r}(\psi(x)) \subseteq \psi(B_r(x)) \subseteq B_{Q^{O(1)}r}(\psi(x)). \tag{8}$$

As discussed in Remark 2.4, ψ pushesforward the Haar measure of G to the Lebesgue measure of $\mathbb{R}^m$, so $Q^{-O(1)}r^m \leq \mu(B_r(x)) \leq Q^{O(1)}r^m$, from which the desired result follows. □

Proof of (ii). For $g \in G$ let $F_g : G/\Gamma \to \mathbb{C}$ be given by

$$F_g(x\Gamma) = \begin{cases} \delta - d_{\mathcal{X}}(x\Gamma, g\Gamma) & \text{if } d_{\mathcal{X}}(x\Gamma, g\Gamma) \leq \delta \\ 0 & \text{else} \end{cases}. \tag{9}$$

We see that for all $g \in G$ we have

$$\int_{G/\Gamma} F_g d\mu \geq \frac{\delta}{2}\mu(B_{\frac{\delta}{2}}(g\Gamma)) \geq Q^{-A}\left(\frac{\delta}{2}\right)^{m+1}, \text{ and } ||F_g||_{\text{Lip}} = 1 + \delta, \tag{10}$$

where the constant $A > 0$ is given to us by part (i). The desired result now follows from the observation that

$$\int_{G/\Gamma} F_g d\mu \geq Q^{-A-m-2}\left(\frac{\delta}{6}\right)^{m+1} ||F_g||_{\text{Lip}}. \tag{11}$$

□

Definition 2.18. Suppose that G/Γ is a nilmanifold with a Mal'cev basis $\mathcal{X}$. Suppose that $\eta : G \to \mathbb{R}/\mathbb{Z}$ is a **horizontal character**, that is to say a homomorphism from G to $\mathbb{R}\mathbb{Z}$ which annihilates Γ. Then, when written in coordinates relative to $\mathcal{X}$, properties (iii) and (iv) of Definition 2.2 imples that $\eta(g) = k \cdot \psi_{\mathcal{X}}(g)$ for some unique $k \in \mathbb{Z}^m$. We write $|\eta| := |k|$.

Theorem 2.19 ([6, Theorem 1.16])**.** Let $m, d \geq 0$, $0 < \delta < \frac{1}{2}$, and $N \geq 1$. Let G/Γ be an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d and a $\frac{1}{\delta}$-rational Mal'cev basis $\mathcal{X}$ adapted to this filtration. Suppose that $g \in \text{poly}(\mathbb{Z}, G_\bullet)$. Then at least one of the following statements is true:

(i) $(g(n)\Gamma)_{n=1}^N$ is δ-equidistributed in G/Γ.

(ii) There exists a non-trivial horizontal character $\eta : G \to \mathbb{R}/\mathbb{Z}$ with $|\eta| \ll \delta^{-O_{m,d}(1)}$ such that $||\eta \circ g(n) - \eta \circ g(n-1)||_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O_{m,d}(1)}/N$ for all $n \in [1, N]$.

Definition 2.20 (cf. [6, Definition 1.18])**.** Let G/Γ be a nilmanifold with a Mal'cev basis $\mathcal{X}$. Let $(\epsilon(n))_{n\in\mathbb{Z}}$ be a sequence in G, and let $M, N \geq 1$. We say that $(\epsilon(n))_{n\in\mathbb{Z}}$ is (M, N)**-smooth** if we have $d(\epsilon(n), \text{id}_G) \leq M$ and $d(\epsilon(n), \epsilon(n-1)) \leq M/N$ for all $1 \leq n \leq N$.

Theorem 2.21 ([6, Theorem 1.19])**.** Let $m, d \geq 0$, and let $M_0, N \geq 1$ and $A > 0$ be real numbers. Suppose that G/Γ is an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d. Suppose that $\mathcal{X}$ is an M_0-rational Mal'cev basis adapted to $G_\bullet$ and that $p \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. Then there is an integer M with $M_0 \leq M \ll M_0^{O_{A,m,d}(1)}$, a rational subgroup $G' \subseteq G$, a Mal'cev basis $\mathcal{X}'$ for G'/Γ' in which each element is an M-rational combination of the elements of $\mathcal{X}$, and a decomposition $p = \epsilon g' \gamma$ into polynomial sequences $\epsilon, g', \gamma \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ with the following properties:

(i) $\epsilon : \mathbb{Z} \to G$ is (M, N)-smooth;

(ii) $g' : \mathbb{Z} \to G'$ takes values in G', and the finite sequence $(g'(n)\Gamma')_{n=1}^N$ is totally M^{-A}-equidistributed in G'/Γ', using the metric $d_{\mathcal{X}'}$ on G'/Γ';

(iii) $\gamma : \mathbb{Z} \to G$ is M-rational, and $(\gamma(n)\Gamma)_{n \in \mathbb{Z}}$ is periodic with period at most M.

We mention that [10] has quantitative improvements of the equidistribution results of [6], but we do not state them here since we do not need them for our purposes.

3 Polynomial covering number

Given a compact metric space (X, d), we define the Hausdorff distance d_H on the collection of closed subsets of X by

$$d_H(C_1, C_2) = \max\left(\sup_{c_1 \in C_1} \inf_{c_2 \in C_2} d(c_1, c_2), \sup_{c_2 \in C_2} \inf_{c_1 \in C_1} d(c_2, c_1) \right). \tag{12}$$

Given a collection $\mathcal{C}$ of closed subsets of X, we let $N_H(\mathcal{C}, \epsilon)$ denote the minimum numbers of ϵ-balls in the d_H metric needed to cover $\mathcal{C}$. The goal of this section is to prove the following result.

Theorem 3.1. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$, and $H \leq G$ a connected subgroup. The set $\{g_1 H g_2 \Gamma/\Gamma\}_{g_1, g_2 \in G}$ can be covered by at most $Q^{O(1)} n^{O(1)}$ balls of radius $1/n$ in the Hausdorff metric induced on the closed subsets of G/Γ by $d_{\mathcal{X}}$.

We first require some auxillary results. The next lemma is essentially a reformulation of Theorem 2.19. We recall from Remark 2.11 that once we fix a filtration and a Q-rational Mal'cev basis $\mathcal{X}$ of the ambient Nilmanifold G/Γ, all subnilmanifolds G'/Γ' with G' a Q-rational subgroup of G are endowed with a canonical filtration $G'_\bullet$ consisting of $Q^{O(1)}$-rational subgroups. Furthermore, we can pick a (non-canonical) $Q^{O(1)}$-rational Mal'cev basis $\mathcal{X}'$ adapted to G'/Γ' in which each element of $\mathcal{X}'$ is a $Q^{O(1)}$-rational combination of elements of $\mathcal{X}$. While the Mal'cev basis $\mathcal{X}'$ is not uniquely determined, it does not matter which one we pick.

Lemma 3.2. Let G/Γ be a nilmanifold endowed with a Mal'cev basis $\mathcal{X}$. Let $L \leq G$ be a closed connected Q-rational subgroup, let $\Lambda = L \cap \Gamma$, let $\mathcal{X}_L$ be a $Q^{O(1)}$-rational Mal'cev basis for L/Λ adapted to $L \cap G_\bullet$ and for which each element of $\mathcal{X}_L$ is a $Q^{O(1)}$-rational combination of elements of $\mathcal{X}$, and let $K \leq L$ be a closed connected rational subgroup. For every $0 < \rho \leq \frac{1}{2}$, one of the following alternatives holds:

(i) $K\Lambda/\Lambda$ is ρ-dense in L/Λ with respect to $d_{\mathcal{X}_L}$.

(ii) There is a proper connected rational subgroup $L_1 < L$ such that $K \leq L_1$ and L_1 is a $Q^{O(1)}\rho^{-O(1)}$-rational subgroup of G.

Proof. Assume that $K\Lambda/\Lambda$ is not ρ-dense in L/Λ with respect to $d_{\mathcal{X}_L}$. Let $\ell \in L$ be such that $B_r(\ell\Lambda) \cap K\Lambda/\Lambda = \emptyset$. Lemma 2.17(ii) tells us that any sequence which fails to visit $B_r(\ell\Lambda)$ is not $Q^{-O(1)}\rho^{m_L+1}$-equidistributed. In particular, any orbit that is contained in $K\Lambda/\Lambda$ will fail to be $Q^{-O(1)}\rho^{m_L+1}$-equidistributed. Now pick some $b \in K$ for which $\{b^n K\Lambda/\Lambda\}_{n=1}^{\infty}$ is dense in $K\Lambda/\Lambda$. We apply Theorem 2.19 with $\delta = Q^{-O(1)}\rho^{m_L+1}$ and some $N \in \mathbb{N}$ to obtain a horizontal character $\eta_N : L \to \mathbb{R}/\mathbb{Z}$ with $|\eta_N| \ll Q^{O(1)}\rho^{-(m_L+1)O(1)}$. Since there are only finitely many horizontal characters of height at most $Q^{O(1)}\rho^{-(m_L+1)O(1)}$, there is some such η for which $\eta_N = \eta$ for infinitely many N. For each such N, we see that $|\eta(b)| = |\eta_N(b)| \ll Q^{-O(1)}\rho^{-(m_L+1)O(1)}/N$, hence $\eta(b) = 0$. Since η annihilates Λ, and $b^n\Lambda$ is dense in $K\Lambda/\Lambda$, we see that η annihilates K. It follows that $L_1 := (\ker(\eta))^\circ$ contains K. We see that L_1 is a $|\eta|$-rational subgroup of L with respect to $\mathcal{X}_L$, hence it is a $Q^{O(1)} - \rho^{-O(1)}$-rational subgroup of G with respect to $\mathcal{X}$. □

Lemma 3.3. Suppose that G/Γ is a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For every connected rational subgroup $K \leq G$ and every $0 < \rho \leq \frac{1}{2}$ there is a connected $Q^{O(1)}\rho^{-O(1)}$-rational subgroup $L \leq G$ for which $K \leq L$ and $d_H(K\Gamma/\Gamma, L\Gamma/\Gamma) \leq \rho$.

Proof. Let $L_0 = G$. If $K\Gamma/\Gamma$ is $Q^{-O(1)}\rho$-dense in G/Γ, then we are done. Otherwise, we induct on i as follows. We apply Lemma 3.2 to obtain a connected rational $Q^{O(1)}\rho^{-O(1)}$-rational subgroup L_{i+1} of L_i for which $K \leq L_{i+1}$. Since the dimension of L_{i+1} is at most the dimension of L_i minus 1, we see that we can only iterate this process at most $m-1$ times before we obtain some L_i for which $K(\Gamma \cap L_i)/(\Gamma \cap L_i)$ is $Q^{-O(1)}\rho$-dense in $L_i/(\Gamma \cap L_i)$ with respect to the metric $d_{\mathcal{X}_{L_i}}$. We see that L_i is a $Q^{O(1)}$-rational subgroup of G[1] so Lemma 2.15 tells us that $d_{\mathcal{X}_{L_i}}$ on $L_i/(L_i \cap \Gamma)$ is $Q^{O(1)}$-comparable to $d_{\mathcal{X}}$ on G/Γ, from which the desired result follows. □

Lemma 3.4. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. Let $F = \psi_{\mathcal{X}}^{-1}([0,1)^m)$ be a fundamental domain for G/Γ. Consider

$$\mathcal{A} := \{uK\Gamma/\Gamma \ : \ u \in F, K \leq G \text{ a connected rational subgroup}\}. \tag{13}$$

We have $N_H(\mathcal{A}, \frac{1}{n}) \leq Q^{O(1)}n^{O(1)}$.

Proof. Lemma 2.8 tells us that for any $u \in F$ and any closed $A, B \subseteq G/\Gamma$ we have $d_H(uA, uB) \leq Q^{O(1)}d_H(A, B)$. We also see that for any $u, v \in F$ and any closed $A \subseteq G$ we[2] have $d_H(uA, vA) \leq d_{\mathcal{X}}(u, v)$ due to the right invariance of $d_{\mathcal{X}}$. Recalling that $d_{\mathcal{X}}(x\Gamma, y\Gamma) \leq d_{\mathcal{X}}(x, y)$, we use the right invariance of $d_{\mathcal{X}}$ once again to see that $d_H(uA\Gamma, vA\Gamma) \leq d_H(uA, vA) \leq d_{\mathcal{X}}(u, v)$. We observe

[1] And it can be endowed with a $Q^{O(1)}\rho^{-O(1)}$-rational Mal'cev basis, at the same time, as we have been doing up to now.

[2] While G is not compact, the following calculations take place within $\overline{F}^2$, which is compact.

that the number of M-rational subgroups of G is at most $(2M+1)^{2m^2}$, so there are at most $Q^{O(1)}\rho^{-O(1)}$ subgroups of G that are $Q^{O(1)}n^{O(1)}$-rational. Now let $\{u_i\}_{i=1}^{Q^{O(1)}n^m}$ enumerate the points in F for which $\psi_{\mathcal{X}}(u_i)$ has coordinates of the form $a/(Q^{O(1)}n)$ for some $0 \leq a < nQ^{O(1)}$, and let $\{L_i\}_{i=1}^{Q^{O(1)}n^{O(1)}}$ enumerate the $Q^{O(1)}n^{O(1)}$-rational subnilmanifolds of G. Let $uK\Gamma/\Gamma \in \mathcal{A}$ be arbitrary. Lemma 3.3 gives us some L_i for which $K \leq L_i$ and $K\Gamma/\Gamma$ is $\frac{Q^{-O(1)}}{n}$-dense in $L_i\Gamma/\Gamma$. Lemma 2.7 tells us that $d_{\mathcal{X}}(u,u_j) \leq Q^{O(1)}|\psi_{\mathcal{X}}(u)-\psi_{\mathcal{X}}(u_j)|$, so we may pick a u_j for which $d_{\mathcal{X}}(u,u_j) \leq \frac{1}{n}$. We now see that

$$\begin{aligned} d_H(uK\Gamma/\Gamma, u_jL_i\Gamma/\Gamma) &\leq d_H(uK\Gamma/\Gamma, uL_i\Gamma/\Gamma) + d_H(uL_i\Gamma/\Gamma, u_jL_i\Gamma/\Gamma) \\ &\leq Q^{O(1)}d_H(K\Gamma/\Gamma, L_i\Gamma/\Gamma) + d_{\mathcal{X}}(u,u_j) \leq \frac{2}{n}. \end{aligned}$$

It follows that the number of balls of radius $\frac{2}{n}$ needed to cover $\mathcal{A}$ is $Q^{O(1)}n^{O(1)} = Q^{O(1)}\left(\frac{n}{2}\right)^{O(1)}$, from which the desired result follows. □

Lemma 3.5. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. If $H \leq G$ is a connected subgroup and $g \in G$, then there exists a rational subgroup $K \leq G$ for which $\overline{Hg\Gamma/\Gamma} = gK\Gamma/\Gamma$.

Proof. Since the intersection of rational subgroups is again rational, we may define K to be the smallest rational subgroup of G containing $g^{-1}Hg$. It is clear that $\overline{Hg\Gamma} \subseteq gK\Gamma$, and hence $\overline{Hg\Gamma/\Gamma} \subseteq gK\Gamma/\Gamma$, so it only remains to show the reverse inclusion. It suffices to show that $\overline{g^{-1}Hg\Gamma/\Gamma} \supseteq K\Gamma/\Gamma$. Let $\Gamma_K := K\cap\Gamma$ and π denote the natural projection from K/Γ_K onto its horizontal torus $\mathbb{T}_K := K/([K,K]\cap\Gamma_K)$. We note that $\pi\left(\overline{g^{-1}Hg\Gamma_K/\Gamma_K}\right) = \overline{\pi(g^{-1}Hg\Gamma_K/\Gamma_K)}$. Leon Green's Theorem [7, 3] tells us that for $k \in K$ the orbit $\{k^n\Gamma_K\}_{n=1}^{\infty}$ is equidistributed in K/Γ_K if and only if $\{\pi(k^n\Gamma_K)\}_{n=1}^{\infty}$ is equidistributed in the horizontal torus. Since $\pi' : K \to K/([K,K]\Gamma_K)$ is a continuous homomorphism, we see that $\mathbb{T}_{H,g} := \pi\left(\overline{g^{-1}Hg\Gamma_K/\Gamma_K}\right) = \overline{\pi'(g^{-1}Hg)}$ is a closed connected subgroup of $\mathbb{T}_K$, hence it is a subtorus. If $\mathbb{T}_{H,g}$ is a proper subtorus, then we pick a nontrivial homomorphism $\eta : \mathbb{T}_K \to \mathbb{R}/\mathbb{Z}$ whose kernel contains $\mathbb{T}_{H,g}$, and we see that $\eta\circ\pi' : K \to \mathbb{R}/\mathbb{Z}$ is a nontrivial horizontal character of K whose kernel contains $g^{-1}Hg$. Since the kernel of $\eta\circ\pi'$ is a rational subgroup of K, and hence a rational subgroup of G, we have contradicted the minimality of K, so we must have that $\mathbb{T}_{H,g} = \mathbb{T}_K$. Since $\pi'(g^{-1}Hg)$ is a connected dense subgroup of $\mathbb{T}_K$, we may pick $h \in g^{-1}Hg$ for which $\{\pi'(h^n)\}_{n=1}^{\infty}$ is dense in $\mathbb{T}_K$. From Leon Green's Theorem we deduce that $\{h^n\Gamma_K/\Gamma_K\}_{n=1}^{\infty}$ is dense in K/Γ_K, hence $H\Gamma_K/\Gamma_K$ is dense in K/Γ_K as desired. □

Proof of Theorem 3.1. We see that for any $\gamma \in \Gamma$ that $\gamma H\gamma^{-1}$ is a connected rational subgroup of G. Indeed, since $H\cap\Gamma$ is cocompact in H, we see that $\gamma H\gamma^{-1}\cap\Gamma = \gamma(H\cap\Gamma)\gamma^{-1}$ is cocompact in $\gamma H\gamma^{-1}$. Using Lemma 3.5 we obtain for each $g_2 \in G$ a rational subgroup $K_{g_2} \leq G$ for which $\overline{Hg_2\Gamma/\Gamma} = g_2K_{g_2}\Gamma/\Gamma$. We now see that

$$\left\{\overline{g_1 H g_2 \Gamma/\Gamma}\right\}_{g_1,g_2\in G} = \{g_1 g_2 K_{g_2}\Gamma/\Gamma\}_{g_1,g_2\in G} = \left\{\{g_1 g_2\}\left([g_1 g_2]K_{g_2}[g_1 g_2]^{-1}\right)[g_1 g_2]\Gamma/\Gamma\right\}_{g_1,g_2\in G} \subseteq \mathcal{A}.$$

Using Lemma 3.4 we see that $N_H(\{g_1 H g_2\Gamma/\Gamma\}_{g_1,g_2\in G}, \frac{1}{n}) \leq N_H(\mathcal{A}, \frac{1}{n}) \leq Q^{O(1)} n^{O(1)}$. □

4 The proof

In this section, we will prove our main result. The proof is based on this is where you explain the proof, and say something here about what we are doing and why... like it's not clear to me what is happening from reading this

Lemma 4.1. Let G/Γ be a nilmanifold endowed with a Mal'cev basis $\mathcal{X}$. Let G' be a connected subgroup of G. For each $g_1, g_2 \in G$ there exists $1 \leq i_0 \leq m$ for which $\psi_{\mathcal{X}}\left(\overline{g_1 G' g_2 \Gamma/\Gamma}\right)$ surjects onto the i_0^{th} coordinate.

Proof. Lemma 3.5 lets us reduce to the case in which $g_2 = \mathrm{Id}_G$ and G' is a rational subgroup. Since $g_1 G'\Gamma/\Gamma = \{g_1\}([g_1]G'[g_1]^{-1})\Gamma/\Gamma$, we may further reduce to the case in which $\psi(g_1) \in [0,1)^m$. Now pick some $\gamma \in G' \cap \Gamma \setminus \{\mathrm{Id}_G\}$ and let i_0 denote the index of the first nonzero coordinate of $\psi(\gamma)$. For $0 \leq t \leq 1$ we define $\gamma^t = \exp(t\log(\gamma))$ and we observe that $\gamma^t \in G'$ for all t since G' is rational. Now consider the path $g_1\gamma^t\Gamma$ in $g_1 G'\Gamma/\Gamma$. The polynomials P_i appearing in Equation (2) have the property that $P_i(0,\cdots,0,s_1,\cdots,s_m) = P_i(t_1,\cdots,t_m,0,\cdots,0) = 0$, so this tells us that for $1 \leq j \leq i_0$ the jth coordinate of $\psi_{\mathcal{X}}(g_1\gamma^t)$ is the sum of the jth coordinates of $\psi(g_1)$ and $\psi(\gamma^t)$. We also see that for $1 \leq j < i_0$ the jth coordinate of $\psi(\gamma^t)$ is 0, and that the i_0th coordinate is $t\psi(\gamma)_{i_0}$. It follows that the i_0th coordinate of $\psi(g_1\gamma^t)$ for $0 \leq t \leq 1$ contains an interval of length 1. Since $\psi(g_1) \in [0,1)^m$, we see that for $0 \leq t \leq 1$ and $1 \leq j < i_0$ we must have $\psi([g_1\gamma^t])_j = 0$. It follows that $\psi(\{g_1\gamma^t\})_{i_0} = \psi(g_1\gamma^t)_{i_0} \pmod 1$, from which the desired result follows. □

Theorem 4.2. If $S \subseteq \mathbb{N}$ is 2−large, then S is a set of pointwise recurrence for basic nilsystems.

Proof. Since this proof is longer than any of the proofs appearing in Section 2, we will not use big O notation for the sake of clarity. Instead, let us pick $C = C(m,s,d)$ to be large enough that all of the results of Sections 2 and 3 hold with C in place of $O(1)$.

Let us fix an arbitrary nilmanifold G/Γ, a Q-rational Mal'cev basis $\mathcal{X}$ adapted to the lower central series filtration, an $\eta > 0$, and a $g \in G$. Let $S = \{n \in \mathbb{N} \mid d_{\mathcal{X}}(g^n\Gamma, \Gamma) \geq \eta\}$. Let us also fix large $M_0, N \in \mathbb{N}$ that will be specified later. By enlarging $C = C(m,s,d)$ if necessary, we will further assume that if $G' \subseteq G$ is a closed connected rational subgroup and $\mathcal{X}'$ is a Mal'cev basis adapted to $G'/(G' \cap \Gamma)$ for which every element of $\mathcal{X}'$ is a M-rational combination of elements of $\mathcal{X}$ for some $M \geq Q$, we have that $\mathcal{X}'$ is a M^C-rational Mal'cev basis.

We apply Theorem 2.21 with $A = (3C+1)(m+1)+3m^2(m+1)-(m+1)\log_Q(\frac{\delta^3}{m})$. Let $\{G_i'\}_{i=1}^{I_M}$ be an enumeration of the rational subgroups of G that possess a Mal'cev basis $\mathcal{X}_i'$ for G_i'/Γ_i' in which $\Gamma_i' = G_i' \cap \Gamma$ and each element of $\mathcal{X}_i'$ is a M-rational combination of elements of $\mathcal{X}$. We assume

without loss of generality that $M \geq Q \geq 3$ and observe that $I_M \leq (M^m(2M+1)^m)^m \leq M^{3m^2}$. We assume without loss of generality that $G'_1 = \mathrm{id}_G$ and $G_{I_M} = G$. Consider the fundamental domain $F = \psi_{\mathcal{X}}^{-1}([0,1)^m)$ for G/Γ, and for $1 \leq i \leq I_M$ let $F_i := (G_i\Gamma) \cap F$ be a fundamental domain for G_i/Γ_i.

We partition $[0,1)^m$ into boxes of side length M^{-1}. Let $B_r(x\Gamma'_i)$ denote the ball of radius r in the $d_{\mathcal{X}'_i}$ metric centered at $x\Gamma'_i$. Using Theorem 3.1 and Lemma 3.5, for each $1 \leq i \leq I_M$, let $\{g_{i,j}\}_{j=1}^{Q^C M^C}$ be such that for any $g'_1, g'_2 \in G$ there exists j for which we have $d_H(g'_1 G'_i g'_2 \Gamma/\Gamma, g_{i,j} G'_i \Gamma/\Gamma) < \frac{1}{M}$.

We will now use the probabilistic method to show that a random coloring almost surely gives the desired result. We see that there are $2^{M^{-m}}$ ways to 2-color the M^{-1}-cells of $[0,1)^m$. Lemma 4.1 tells us that for each $2 \leq i \leq I_M$ and $1 \leq j \leq Q^C M^C \leq M^{2C}$ there exists $1 \leq k_{i,j} \leq m$ for which $\psi_{\mathcal{X}}(g_{i,j} G'_i \Gamma/\Gamma)$ surjects onto the $k_{i,j}$ coordinate. Let $\{C_{i,j,k}\}_{k=1}^{M}$ be an enumeration of cells intersecting $\psi_{\mathcal{X}}(g_{i,j} G'_i \Gamma/\Gamma)$ for which we also have that the $k_{i,j}$ coordinate of $C_{i,j,k}$ is between $\frac{k-1}{M}$ and $\frac{k}{M}$. Now fix some i and j. We consider the restriction of our random 2-coloring to the cells $\{C_{i,j,3k-1}\}_{k=1}^{\frac{M}{3}}$. Hoeffding's inequality tells us that with probability at most $2\exp(-\frac{M}{24})$, we have that the number of red cells in the coloring is either bigger than $\frac{M}{4}$ or smaller than $\frac{M}{12}$. We consider all of these colorings to be bad colorings. Of the remaining colorings, we still consider them to be bad if for every $1 \leq k \leq \frac{M}{3}$, every cell $C_{i,j,k}$ has a neighboring cell in the ambient grid that has a different color, and this happens with probability $(1 - 2^{-3^m+1})^{\frac{M}{3}}$. It follows that for each i, j, the probability of a bad coloring is bounded above by $2\exp(-\frac{M}{24}) + (1 - 2\exp(-\frac{M}{24}))(1 - 2^{-3^m+1})^{\frac{M}{3}}$. The union bound over all possible pairs (i, j) shows us that the probability of a bad coloring is at most

$$M^{3m^2+2Cm}\left(2\exp\left(-\frac{M}{24}\right) + \left(1 - 2\exp\left(-\frac{M}{24}\right)\right)\left(1 - 2^{-3^m+1}\right)^{\frac{M}{3}}\right). \tag{14}$$

We pick M_0 to be so large that for all $M \geq M_0$ the quantity in Equation (14) is smaller than 1. It follows that there will exist a good coloring χ' of the $[0,1)^m$. Let $g_0 := \exp(\frac{1}{M!}\log(g))$, and observe that $g = g_0^{M!}$ Now we produce a 2-coloring χ of $\mathbb{N}$ by $\chi(n) = \chi'(g_0^n)$. It remains to show that the coloring χ does not have an N-term S-A.P.

Now fix some $s_0 \in S$. Let $1 \leq i \leq I_M$ be such that when we apply Theorem 2.21 to $p(n) = g_0^{s_0 n}$, we obtain $p(n) = \epsilon g' \gamma$ with g' taking values in G'_i. We claim that for $N > M(M!)\eta^{-1}$, we cannot have $i = 1$. Indeed, if we assume for the sake of contradiction that $i = 1$, then $G = g' = \mathrm{Id}_G$, hence $g^{s_0} = g_0^{s_0 M!} = p(M!) = \epsilon(M!)g'(M!)\gamma(M!) = \epsilon(M!)\gamma(0)$, and similarly $\mathrm{Id}_G = g^0 = \epsilon(0)\gamma(0)$. To obtain our desired contradiction, we observe that

$$\begin{aligned}\eta &\leq d_{\mathcal{X}}(g^{s_0}\Gamma, \Gamma) = d_{\mathcal{X}}(\epsilon(M!)\gamma(0)\Gamma, \epsilon(0)\gamma(0)\Gamma) \leq d_{\mathcal{X}}(\epsilon(M!)\gamma(0), \epsilon(0)\gamma(0)) \\ &= d_{\mathcal{X}}(\epsilon(M!), \epsilon(0)) \leq M(M!)/N < \eta.\end{aligned}$$

We now take $N'' = M^{A+1}Q^C M^{m+2} + M(M!)\epsilon^{-1}$. Let $a \in \mathbb{N}$ be arbitrary, and let j be such that $d_H(g^a G'_i \Gamma/\Gamma, g_{i,j} G'_i \Gamma/\Gamma) < \frac{1}{M}$. Let $M_1 \leq M$ be the period of γ, $s'_0 = M_1 s_0$ and $N' =$

N''/M and note that $\{g'(M_1n)\}_{n=1}^{N'}$ is $M^{-(3C+1)(m+1)}\delta^{m+1}$-equidistributed in G_i'/Γ_i' by our choice of A. Lemma 2.17(iii) tells us that $\{g'(n)\Gamma_i'\}_{n=1}^{N'}$ visits every ball in G_i'/Γ_i' of radius $M^{-2C-1}\rho\delta$. Let $q_i : G_i' \to G_i'/\Gamma_i'$ denote the quotient map $x \mapsto x\Gamma_i'$. We see that the image of a ball of radius r centered at $x \in F_i \subseteq G_i'$ under q_i contains the ball of radius r centered at $x\Gamma_i'$, so $\{\{g'(M_1n)\}\}_{n=1}^{N'}$ visits every ball of radius $M^{-2C-1}\delta$ in F_i. Lemma 2.9 tells us that when we view F_i as a subset of G endowed with the $d_{\mathcal{X}}$ metric instead of the $d_{\mathcal{X}_i'}$ metric, we have that $\{\{g'(M_1n)\}\}_{n=1}^{N'}$ visits every ball of radius $M^{-C-1}\delta$. We see that for $x \in F$ and $r(0, \mathrm{Diam}(G/\Gamma))$, $\psi_{\mathcal{X}}^{-1}(B_{Q^Cr}(\psi_{\mathcal{X}}(x)) \supseteq B_r(x)$, so $\{\psi_{\mathcal{X}}(\{g'(M_1n)\})\}_{n=1}^{N'}$ visits every balls of radius $M^{-1}\delta$ (in the ℓ^∞ norm) in $[0,1)^m$ centered at a point on $\psi_{\mathcal{X}_i'}(F_i)$. In particular, we have that $\{\psi_{\mathcal{X}}(\{g'(M_1n)\})\}_{n=1}^{N'}$ visits every balls of radius $M^{-1}\rho\delta$ centered at a point in $B_{i,j} \cap \psi_{\mathcal{X}}(F_i)$ for all $1 \leq j \leq n$. We let N be such that $MN''/N < r_1$, as we then have that $d_{\mathcal{X}}(\epsilon(n), \mathrm{Id}_G) < r_1$ for $1 \leq n \leq N''$. It follows that $d_{\mathcal{X}}(\epsilon(M_1n)g'(M_1n), g'(M_1n)) < M^{-1}\rho\delta$, hence for each $1 \leq j \leq \delta^{-1}$, there is a point in the orbit $\{\psi_{\mathcal{X}}(\{g_0^{d_0n})\}\}_{n=1}^{N}$ that is the same connected component of $\mathrm{Grid}(\rho, \delta^{-1}, \theta)$ as $B_{i,j}$

□

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