An elementary two-color obstruction for special generalized polynomials

We give an elementary two-color obstruction for finite families of special generalized polynomials. The proof uses finite van der Waerden, finite differences, polynomial interpolation, compactness, and multiplication of upper-triangular matrices. It also gives simultaneous approximation at the reciprocal power of the progression length corresponding to each polynomial's assigned degree.

1 Statement and conventions

Write $[x]$ for a nearest integer, with any fixed convention at ties, and $\|x\| = \operatorname{dist}(x, \mathbb{Z})$. A special generalized polynomial is

$$P(t) = a_1 t^{w_1} \big[a_2 t^{w_2} \big[\cdots [a_\ell t^{w_\ell}] \cdots \big]\big], \qquad w_1, \dots, w_\ell \in \mathbb{N}. \tag{1}$$

For $\ell = 1$, this means $P(t) = a_1 t^{w_1}$, without a bracket. All the exponents are positive. The assigned degree of this expression is $\deg P = w_1 + \cdots + w_\ell$, regardless of any degeneracy in its coefficients.

A set $S \subseteq \mathbb{N}$ is **2-large** if every coloring $c : \mathbb{N} \to \{0, 1\}$ contains, for every positive integer N, a monochromatic progression $a, a + d, \dots, a + (N - 1)d$ with $d \in S$.

Theorem. *Given finitely many special generalized polynomials $P_1, \dots, P_r$ and $\varepsilon > 0$, the set*

$$\{d \geq 1 : \max_i \|P_i(d)\| \geq \varepsilon\}$$

is not 2-large.

In fact, for prescribed lists of positive exponents there are constants C, N_0, independent of the real coefficients, and a two-coloring of $\mathbb{N}$ for each choice of coefficients such that every monochromatic N-term progression, $N \geq N_0$, has difference d satisfying

$$\|P_i(d)\| \leq C N^{-\deg P_i} \qquad (1 \leq i \leq r). \tag{2}$$

The proof constructs one coloring that works simultaneously for all $N \geq N_0$.

2 Weighted polynomial matrices

Let $U_{\ell+1}(\mathbb{R})$ be the group of upper-triangular matrices with diagonal entries one. Give its adjacent entries the positive integer weights $w_1, \dots, w_\ell$, and give entry (i, j) weight

$$w_{ij} = w_i + \cdots + w_{j-1}.$$

Use a product of such groups, one for each special polynomial. Denote this product by G, its subgroup of integer matrices by Γ, the total number of off-diagonal coordinates by m, and the maximum weight by D. Coordinate bounds below refer to the off-diagonal entries in all blocks.

Let $\mathcal{H}$ consist of matrix polynomials $A(t)$ whose (i,j) entry has degree at most w_{ij}. It is closed under multiplication and inverse, since the weights add:

$$w_{ik} + w_{kj} = w_{ij}.$$

For $u > 0$, define the dilation $\delta_u : G \to G$ and the weighted leading-coefficient matrix $\tau(A)$ by

$$(\delta_u B)_{ij} = u^{w_{ij}} B_{ij}, \qquad \tau(A)_{ij} = [t^{w_{ij}}]A_{ij}(t).$$

Both matrices have diagonal entries one. Direct multiplication gives

$$\begin{gathered}\tau(AB) = \tau(A)\tau(B), \qquad \tau(A(b+qt)) = \delta_q \tau(A), \\ \delta_u(BC) = \delta_u(B)\delta_u(C).\end{gathered} \tag{3}$$

These identities are the reason for using positive, additive weights. A constant matrix has weighted leading-coefficient matrix equal to the identity.

3 An elementary shell coloring

For $A \in G$, set

$$F(A) = \min_{\gamma \in \Gamma} \sum_{i<j} (A\gamma)_{ij}^2, \tag{4}$$

where the sum includes every block. This function is invariant under right multiplication by Γ.

Every matrix can be reduced by integer column operations to a matrix whose off-diagonal entries lie in a fixed centered interval of length one. Therefore $0 \le F(A) \le m/4$. The minimum exists, and F is continuous. Indeed, on any compact set of matrices A, all minimizers satisfy a uniform bound on $A\gamma$, hence on $\gamma = A^{-1}(A\gamma)$. Only finitely many integer matrices can occur. Thus, locally, (4) is the minimum of finitely many continuous functions.

Here is the key elementary observation:

> *If $Q(t)$ is a nonconstant polynomial matrix, then $F(Q(t))$ is not constant on any nondegenerate real interval.*

On a compact parameter interval only finitely many γ's can minimize (4). For each one,

$$\sum_{i<j} (Q(t)\gamma)_{ij}^2$$

is a polynomial. If this polynomial were constant, every coordinate of $Q(t)\gamma$ would be constant: the highest nonzero even-degree coefficient of a sum of squares is a sum of positive squares. Hence Q would be constant. A finite collection of nonconstant polynomials cannot take one fixed value at every point of an interval, which proves the observation.

Fix $\rho = 1/4$. Consider all polynomial matrices $Q : [0,1] \to G$ with coordinate degrees at most D, coordinates bounded by one, and

$$\|Q(1) - Q(0)\|_\infty = \rho.$$

Their coefficient vectors form a compact set: boundedness on an interval bounds coefficients by interpolation. The observation and continuity imply that some $\eta > 0$ satisfies

$$\operatorname{osc}_{0\le t\le 1} F(Q(t)) \ge \eta \tag{5}$$

for every such curve.

Choose $0 < h < \eta/10$, and color matrices by

$$\kappa(A) = \lfloor F(A)/h \rfloor \pmod 2. \tag{6}$$

There is a number $T_0 \ge 1$ such that, for every curve in this compact family and every real $T \ge T_0$, the samples

$$Q(j/T), \qquad 0 \le j \le \lfloor T \rfloor, \tag{7}$$

receive both colors. To verify this, (5) makes the range of F contain points strictly inside bands of both parities, a distance at least $h/4$ from their endpoints. Bounded polynomial coefficients give uniform continuity in the parameter, and F is uniformly continuous on the bounded matrix box. A sufficiently fine mesh therefore samples the interior of both bands. This also handles points on color boundaries.

4 A short interpolation observation

Choose an integer $M > D + 1$, sufficiently large that the following holds:

> *If the coordinates of a polynomial matrix Q have degree at most D and $|Q_{ij}(n)| \le 1/2$ for every off-diagonal coordinate and all integers $0 \le n < M$, then*
>
> $$\|Q(t) - Q(0)\|_\infty < \rho \qquad (0 \le t \le T_0). \tag{8}$$

This follows from elementary interpolation. Rescale by $t = (M-1)u$, and interpolate at $D+1$ sample indices spaced approximately $(M-1)/D$ apart. The resulting rescaled polynomial and its derivative have bounds depending only on D. Thus the original polynomial moves by $O_D(T_0/(M-1))$ on $[0, T_0]$.

Fix a grid of side length

$$0 < \delta < 2^{-D-2}$$

in the centered fundamental coordinate box. Let R be its finite number of cells. Choose a finite van der Waerden number $W = W(R, M)$: every R-coloring of $\{0, \dots, W-1\}$ contains an M-term monochromatic progression.

Finally set

$$L = D! \operatorname{lcm}(1, \dots, W). \tag{9}$$

The order is: shell coloring, mesh threshold T_0, interpolation length M, auxiliary grid and van der Waerden number W, then dilation L. None depends on an arithmetic progression or its coefficients.

5 Polynomial rounding inside one small box

We need an exact finite-difference lemma.

Polynomial rounding lemma. *Suppose $A \in \mathcal{H}$. Assume that the first M canonical representatives, those of $A(0), \dots, A(M-1)$ modulo right multiplication by Γ, lie in one grid cell. Then there exist $Q, C \in \mathcal{H}$ such that*

$$A(t)C(t) = Q(t), \qquad C(j) \in \Gamma \quad (j \in \mathbb{Z}), \tag{10}$$

and $Q(j)$ equals the given canonical representative for $0 \le j < M$.

Proof. Process entries in increasing matrix gap $j - i$. Assume the entries $C_{kj}(t)$, $i < k < j$, have already been constructed as integer-valued polynomials. Put

$$f_{ij}(t) = A_{ij}(t) + \sum_{i<k<j} A_{ik}(t)C_{kj}(t).$$

Its degree is at most $w = w_{ij}$. At the initial integer samples, the already constructed corrections agree with the lattice corrections for the prescribed representatives. Therefore the desired reduced coordinate $r_{ij}(n)$ differs from $f_{ij}(n)$ by an integer. Therefore $\Delta^{w+1} r_{ij}(n)$ is an integer wherever this difference is defined. All $r_{ij}(n)$ lie in an interval of length δ, so

$$|\Delta^{w+1} r_{ij}(n)| \le 2^{w+1}\delta < 1.$$

Thus it vanishes. The samples $r_{ij}(n)$ agree with a polynomial $Q_{ij}(t)$ of degree at most w. Set $C_{ij} = Q_{ij} - f_{ij}$. It takes integer values at at least $w + 1$ consecutive integers. Its Newton expansion in $\binom{t}{k}$ consequently has integer coefficients, so $C_{ij}(n) \in \mathbb{Z}$ for every integer n. This completes the induction. □

In particular, since C_{ij} is integer-valued of degree at most w_{ij},

$$w_{ij}!\,\tau(C)_{ij} \in \mathbb{Z}. \tag{11}$$

6 Control of the weighted leading coefficients

Let $N \ge 2W + 2$, let $p \in \mathcal{H}$, and suppose

$$p(0), \dots, p(N-1)$$

are monochromatic under κ. Apply the auxiliary R-coloring by canonical grid cells to the first W values. It gives integers $b \ge 0$, $q \ge 1$, with $b, q \le W$, such that

$$p(b), p(b+q), \dots, p(b+(M-1)q)$$

have canonical representatives in one cell. Apply the lemma to $A(t) = p(b + qt)$, obtaining

$$p(b+qt)C(t) = Q(t). \tag{12}$$

For every integer j, $\kappa(Q(j)) = \kappa(p(b+qj))$, by right-lattice invariance. Put

$$J = \left\lfloor \frac{N-1-b}{q} \right\rfloor.$$

We claim that

$$\|Q(t) - Q(0)\|_\infty < \rho \qquad (0 \le t \le J). \tag{13}$$

Otherwise, let $T \le J$ be the first time the displacement equals ρ. By (8), $T > T_0$. The rescaled curve $s \mapsto Q(Ts)$, $0 \le s \le 1$, stays in the coordinate box of radius $3/4$, since $Q(0)$ is a canonical representative. Its endpoint displacement is ρ. Therefore (7) says that the integer samples $Q(j)$, $0 \le j \le \lfloor T \rfloor$, receive both colors. Their colors agree with those of points in the original monochromatic progression, a contradiction. This proves (13).

Each coordinate of Q is thus bounded by $3/4$ on $[0, J]$. Polynomial interpolation, now on this long real interval, bounds its leading coefficient:

$$|\tau(Q)_{ij}| \le C_D J^{-w_{ij}}. \tag{14}$$

Taking weighted leading matrices in (12), and using (3), gives

$$\delta_q \tau(p)\, \tau(C) = \tau(Q).$$

Apply $\delta_{L/q}$. Since L/q is a multiple of $D!$, (11) shows that

$$\gamma := \delta_{L/q} \tau(C) \in \Gamma.$$

Consequently

$$\delta_L \tau(p)\, \gamma = E, \qquad E := \delta_{L/q} \tau(Q), \tag{15}$$

and

$$|E_{ij}| \le C_D \left(\frac{L}{qJ}\right)^{w_{ij}} \le C_D \left(\frac{L}{N-1-2W}\right)^{w_{ij}}. \tag{16}$$

Here we used $qJ \ge N-1-b-q \ge N-1-2W$. This is the complete two-color argument for polynomial matrices.

7 Recovering the nested brackets

For (1), define $g(t) \in U_{\ell+1}(\mathbb{R})$ to have zero entries above the first superdiagonal, and superdiagonal

$$-a_1 t^{w_1},\ -a_2 t^{w_2}, \dots, -a_{\ell-1} t^{w_{\ell-1}},\ +a_\ell t^{w_\ell}. \tag{17}$$

For $\ell = 1$, its only off-diagonal entry is $+a_1 t^{w_1}$. For the finite family, take the product of these band matrices.

They are weighted homogeneous:

$$g(t) = \delta_t g(1), \qquad \tau\big(g(x+ut)\big) = g(u). \tag{18}$$

Now color the positive integers by

$$\boxed{c(n) = \kappa(g(n/L)).} \tag{19}$$

Suppose $a, a+d, \dots, a+(N-1)d$ is monochromatic. Apply (15)–(16) to

$$p(t) = g\big((a+dt)/L\big).$$

By (18), $\delta_L \tau(p) = g(d)$. We obtain an integer matrix $\gamma \in \Gamma$ such that

$$g(d)\gamma = E, \qquad |E_{ij}| \le C_D \left(\frac{L}{N-1-2W}\right)^{w_{ij}}. \tag{20}$$

Choose $N_0 > 2W+1$ large enough that every off-diagonal entry on the right has absolute value less than $1/4$ whenever $N \ge N_0$.

To read one special polynomial, write $x_i = a_i d^{w_i}$, and consider the last column of its block. The last nontrivial row of (20) says

$$x_\ell + \gamma_{\ell,\ell+1} = E_{\ell,\ell+1},$$

so $\gamma_{\ell,\ell+1} = -[x_\ell]$. The next row says

$$\gamma_{\ell-1,\ell+1} - x_{\ell-1}\gamma_{\ell,\ell+1} = E_{\ell-1,\ell+1},$$

so $\gamma_{\ell-1,\ell+1} = -[x_{\ell-1}[x_\ell]]$. Continue upward. Every rounding is unique because the residual has absolute value below $1/4$. The top row becomes

$$P(d) + \gamma_{1,\ell+1} = E_{1,\ell+1}.$$

Thus

$$\|P(d)\| \le |E_{1,\ell+1}| \le C_D \left(\frac{L}{N-1-2W}\right)^{\deg P}. \tag{21}$$

For $\ell = 1$, the same conclusion follows directly from the single entry. All blocks are handled by the same binary coloring. Enlarging the constants in (21) gives (2); choosing N sufficiently large gives $\|P_i(d)\| < \varepsilon$ for every i. Hence no such monochromatic progression has its difference in the proposed bad set. This proves the theorem. □

8 What makes the proof work

The auxiliary grid uses finitely many colors only to find a short subprogression inside an already monochromatic binary progression. Finite differences then replace all its lattice carries by integer-valued polynomials. The shell coloring prevents the resulting ordinary polynomial lift from escaping a bounded box. Finally, weighted leading coefficients remove the starting point, and the fixed dilation clears the factorial denominators of the integer-valued polynomials.

The positive exponent convention is essential to this argument. The proof above is for special generalized polynomials (1); it does not silently extend the statement to arbitrary floor expressions or to arbitrary generalized-polynomial syntax.

Provenance. This exposition is an elementary reworking of the Alweiss–Farhangi argument, developed with Codex assistance. Its architecture was informed by the earlier manuscript. The proof above does not invoke that manuscript's results, or any nilrecurrence, nilsequence equidistribution, orbit-closure, or factorization theorem.