2-Large sets are sets of pointwise recurrence for nilsystems

Ryan Alweiss, Sohail Farhangi

October 27, 2026

Abstract

A set $S \subset \mathbb{N}$ is r-large if for any r-coloring of the natural numbers, there are arbitrarily long monochromatic arithmetic progressions whose common difference comes from S. The set D is large if it is r-large for all r. In 1999 Brown, Graham, and Landman conjectured that if S is 2-large, then it is also large.

We prove that every 2-large set is a set of pointwise recurrence for nilsystems. In other words, we show that if a set is not large because of a (generalized) polynomial obstacle, then it is also not 2-large.

1 Introduction

The subject of the present paper on arithmetic progressions is the following notion of *largeness*, introduced by Brown, Graham, and Landman [3].

Definition 1.1 (r-large)**.** For $r \in \mathbb{N}$ and $S \subseteq \mathbb{N}$, we say that S is r-large if for any r-coloring $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exists a color $1 \leq i \leq r$ so that C_i contains arbitrarily long arithmetic progressions with common difference in S.

We say that S is *large* if it is r-large for all r. Equivalently, S is large if and only if the restricted van der Waerden theorem with common difference in S is true. That $\mathbb{N}$ itself is large is the content of van der Waerden's theorem. Clearly if a set is large, then it is r-large for all $r \geq 2$, and an r-large set for $r \geq 2$ is clearly 2-large. Brown, Graham, and Landman conjectured that in fact all of these notions of largeness coincide.

Conjecture 1.2 ([3])**.** $D \subseteq \mathbb{N}$ is 2-large, then D is large.

Perhaps it is a little bit more intuitive to think about this conjecture in the contrapositive. If a set S is *not* large, then the restricted van der Waerden with common difference in S is false, and so some finite coloring blocks arbitrarily long arithmetic progressions. The question of Brown, Graham, and Landman is then whether two colors is sufficient to block such progressions.

One basic class of non-large sets are those coming from Bohr sets. For instance if S is the set of d so that $1/10 < \{d\alpha\} < 9/10$ or $1/10 < \{d\beta\} < 9/10$, then if we color x based on $\{x\alpha\}$ and

$\{x\beta\}$ up to ε error then even $x, x+d$ will not have the same color, so S will not be large. The first author [1] recently proved that such sets will not be 2-large.

A more general and very important class of non-large sets are those coming from generalized polynomials. For example, let S be the set of d so that $1/4 < \{d^2\alpha\} < 3/4$. Since $(x+2d)^2 - 2(x+d)^2 + x^2 = 2d^2$, if we color x by $x^2/2$ modulo α up to some ϵ error, then $x, x+d, x+2d$ cannot all have the same color in this coloring if $d \in S$. Equivalently, these sets come from nilsystems, which we will formally define in the next section.

Definition 1.3 (Pointwise recurrence for nilsystems)**.** A set $S \subset \mathbb{N}$ is a set of pointwise recurrence for nilsystems if for every d-step nil-system (X, T) there is a point $x \in X$ and a subsequence $S' \subset S$ so that for all $1 \leq d' \leq d$, $\lim_{n \in S'} T^{d'n} x = x$.

This has an equivalent combinatorial definition in terms of generalized polynomials. Let the nearest integer to x be $[x] := \lfloor x + 1/2 \rfloor$.

Definition 1.4 (Special Generalized Polynomials, after [7])**.** We define L so that $L(x) = x$ and $L(x_1, \ldots, x_\ell) = x_1[L(x_2, \ldots, x_\ell)]$.

Then the special generalized polynomials of degree at most d are the expressions

$$L(n^{j_1} a_1, \ldots, n^{j_\ell} a_\ell)$$

such that $j_i \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $1 \leq i \leq \ell$, and $\sum_{i=1}^{\ell} j_i \leq d$.

In particular, we must have $\ell \leq d$.

Definition 1.5 (Pointwise recurrence for nilsystems, combinatorial definition)**.** A set $S \subset \mathbb{N}$ is a set of pointwise recurrence for nilsystems if for every finite family of special generalized polynomials $P_1, \cdots, P_k$ with degree at most d and every ε, there exists $s \in S$ so that $||P_i(s)||_{\mathbb{R}/\mathbb{Z}} < \varepsilon$ for all $1 \leq i \leq k$.

Huang, Shao, and Ye [7] prove the equivalence of these definitions for a set of pointwise recurrence for nilsystems. One can check for instance that the set of α so that $1/4 < n^2\alpha < 3/4$ is not such a set, as then $||L(n^2\alpha)||_{\mathbb{R}/\mathbb{Z}} > 1/4$.

We also note that Huang, Shao, and Ye [7] also asked whether if S is a set of pointwise recurrence for nilsystems, then S is large. Recently, the first author [2] found an example of a non-large set which is nonetheless a set of pointwise recurrence for nilsystems.

We are now ready to state the main theorem of our paper.

Theorem 1.6. If $S \subseteq \mathbb{N}$ is $2-$large, then S is a set of pointwise recurrence for nilsystems.

The proof extends that of [1], but we must work with nilmanifolds rather than with only tori, which introduces significant complications.

1.1 Outline of the proof

We prove Theorem 1.6 by contradiction. Suppose that S is not a set of pointwise recurrence for nilsystems. After passing to a finite product nilsystem, we may reduce to the following situation: there is a nilmanifold G/Γ, an element $g \in G$, and some $\eta > 0$ such that

$$d(g^s\Gamma, \Gamma) \geq \eta$$

for every $s \in S$. Our goal is to construct a 2-coloring of $\mathbb{N}$ for which there is a uniform bound on the length of a monochromatic arithmetic progression whose common difference belongs to S. This will show that S is not 2-large.

The coloring is obtained by coloring a sufficiently fine partition of a nilmanifold and then pulling this coloring back along an orbit. There are, however, difficulties with carrying out this idea directly on G/Γ. For a fixed $s \in S$, the Green–Tao factorization theorem (cf. Theorem 2.22) decomposes a long segment of the polynomial sequence

$$p_s(n) = g^{sn}$$

as

$$p_s(n) = \epsilon_s(n)g'_s(n)\gamma_s(n),$$

where ϵ_s varies slowly, γ_s is periodic, and g'_s is quantitatively equidistributed on a rational subnilmanifold. In the abelian case of [1], the equidistribution of g'_s continues to control the translated segment

$$g^a p_s(n)\Gamma,$$

uniformly in the initial term a, although we need to ultimately pass to a larger subtorus to deal with the case where g^a is very close to the identity for a small integer a. On a nonabelian nilmanifold this uniformity is not guaranteed: multiplication by g^a can magnify a very small coordinate of ϵ_s and cancel the equidistributed component. The Heisenberg example discussed in Section 3 shows that the entire translated orbit segment may lie in a small ball even though g'_s equidistributes on a positive-dimensional subnilmanifold.

To overcome these obstructions, we must replace G/Γ by an auxiliary extended nilmanifold

$$\widetilde{G}/\widetilde{\Gamma}.$$

In addition to the original point $g^n\Gamma$, the extended system records finitely many reduction cocycles arising when elements of G are returned to a fixed Mal'cev fundamental domain. Equivalently, it records the lattice carries, or generalized-polynomial phases, that are invisible in the original quotient but that can contribute nontrivially after translation. The original nilrotation is a factor of this extended nilrotation. Most importantly, the extra coordinates prevent an equidistributed component from being hidden by an interaction between a smooth error and an unrecorded lattice

element.

We then construct a 2-coloring of a fine grid in a Mal'cev fundamental domain for $\widetilde{G}/\widetilde{\Gamma}$. For each $s \in S$, we apply the factorization theorem to the corresponding polynomial orbit in the extended nilmanifold. After restricting to a subprogression, the periodic factor is constant. There are then two possibilities. If the equidistributed factor is trivial, the smoothness of ϵ_s and the bounded period of γ_s imply that $g^s\Gamma$ is close to Γ. For a sufficiently long orbit segment, this contradicts the assumption

$$d(g^s\Gamma, \Gamma) \geq \eta.$$

Thus, for every $s \in S$, the factorization produces a genuinely positive-dimensional rational subnilmanifold on which the remaining factor is quantitatively equidistributed.

The initial term of an arithmetic progression is arbitrary, so we must deal simultaneously with all translates of all the rational subnilmanifolds that can arise in this way. There are infinitely many such translates. Theorem 5.1 is used to reduce this infinite family to a polynomially sized collection of model subnilmanifolds: at the scale of the grid, every possible translate is close in Hausdorff distance to one of these models. This polynomial bound is essential, since it allows us to apply the probabilistic method and take a union bound over all relevant geometric configurations.

Every positive-dimensional translated subnilmanifold passes through many well-separated cells of the grid. More precisely, Lemma 6.1 shows that its image in Mal'cev coordinates surjects onto at least one coordinate, and hence intersects a number of grid cells that grows linearly with the fineness of the partition. We color the grid cells independently and uniformly with two colors. Since each model subnilmanifold meets many separated cells, the probability that it fails to contain suitably robust cells of both colors is exponentially small. On the other hand, the number of model subnilmanifolds grows only polynomially. Choosing the grid sufficiently fine and applying the union bound therefore yields a deterministic coloring with the required property for every model subnilmanifold simultaneously.

Finally, we pull this coloring back to $\mathbb{N}$ along the extended nilrotation. Fix an initial term a and a difference $s \in S$. The equidistributed factor visits every sufficiently small ball in its positive-dimensional subnilmanifold. The smooth factor changes the orbit only by a small amount, while the periodic factor has already been made constant. The robustness built into the coloring therefore ensures that the orbit segment

$$a,\ a+s,\ a+2s,\ \ldots$$

visits cells of both colors before reaching the prescribed length. Consequently, it cannot be monochromatic.

We have thus constructed a 2-coloring of $\mathbb{N}$ containing no sufficiently long monochromatic arithmetic progression with common difference in S. Hence S is not 2-large. This proves the contrapositive of Theorem 1.6.

1.2 Organization and Acknowledgements

Organization:

Acknowledgements:

2 Nilmanifold Preliminaries

For our discussion about nilmanifolds, we use a mix of the presentations given in [5] and [6], and we refer the reader to those sources for additional details.

2.1 Basic properties

For the rest of this paper, we will use the following standard notation when talking about nilmanifolds. We use G to denote a finite dimensional nilpotent (real) Lie group with identity element id_G and let G^0 be the connected component containing the identity. Furthermore, we will have a standing assumption that G is generated by G^0 and finitely many commuting elements $\{\tau_i\}_{i=1}^k$. Moreover, we will assume that the group generated by $\{\tau_i\}_{i=1}^k$ is isomorphic to $\mathbb{Z}^k$ and has trivial intersection with G^0. These assumptions are standard in ergodic theory, but not in the general theory of Lie algebras and Lie groups. We will use Γ to denote a discrete cocompact subgroup of G, and we use $N := G/\Gamma$ to denote the set of left cosets of Γ in G. A **nilmanifold** is a space of the form $N = G/\Gamma$ with G and Γ satisfying the above conditions.

Now let G be a Lie group of (real) dimension m. As usual, let $[H, K]$ denote the subgroup generated by all commutators $[h, k] := hkh^{-1}k^{-1}$ for $h \in H, k \in K$.

A **filtration** $G_\bullet$ on G is a sequence of closed subgroups

$$G = G_0 = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_d \supseteq G_{d+1} = \{\mathrm{id}_G\}$$

such that $[G_i, G_j] \subseteq G_{i+j}$ for all integers $i, j > 0$. If G is connected, then we also require G_i to be connected for all i. The least integer d for which $G_{d+1} = \{\mathrm{id}_G\}$ is called the *degree* of the filtration.

Since G is nilpotent, G always possesses the lower central series filtration, namely, the filtration in which $G_i = \overline{[G, G_{i-1}]}$ for all $2 \leq i \leq c+1$, where c is the nilpotency class of G. However, G may also possess other filtrations, so the degree of a given filtration $G_\bullet$ may differ from the nilpotency class of G. When we are given a filtration $G_\bullet$ of G, we use m to denote the (real) dimension of G, and m_i to denote the dimension of G_i for $0 \leq i \leq d+1$.

Lemma 2.1 ([6, Corollary 10.26])**.** Every nilmanifold $N = G/\Gamma$ can be embdedded as a normal subnilmanifold of a nilmanifold $N' = G'/\Gamma'$ such that G' is connected and simply connected and such that every element of G is represented in G'.

A **nilsystem** is a pair (N, T_g) where N is a nilmanifold, $g \in G$, and $T_g : N \to N$ is given by $T_g(x\Gamma) = gx\Gamma$. It is well known that for nilsystems, transitivity, minimality, and unique ergodicity are equivalent. A

2.2 Lie algebras, logarithms, and exponentials

2.3 Mal'cev bases

Often, we will want to give real coordinates to a nilmanifold $N = G/\Gamma$. This can be accomplished by identifying a *Mal'cev basis*. We only define these in the case where G is connected and simply connected, as we will always be able to use Lemma 2.1 to reduce to this case.

Definition 2.2. [[5, Definition 2.1], Mal'cev basis] Let $N = G/\Gamma$ be a nilmanifold with G a m-dimensional connected and simply connected Lie group. Let $G_\bullet = \{G_i\}_{i=0}^{d+1}$ be a filtration on G and let $\mathfrak{g}$ denote the (real) Lie algebra of G. A basis $\mathcal{X} = \{X_1, \cdots, X_m\}$ for $\mathfrak{g}$ over $\mathbb{R}$ is called a Mal'cev basis for G/Γ adapted to $G_\bullet$ if the following four conditions are satisfied:

(i) For each $j = 0, \cdots, m-1$ the subspace $\mathfrak{h}_j := \text{Span}(X_{j+1}, \cdots, X_m)$ is a Lie algebra ideal in $\mathfrak{g}$, and hence $H_k := \exp \mathfrak{h}_j$ is a normal Lie subgroup of G.

(ii) For every $0 \leq i \leq d$ we have $G_i = H_{m-m_i}$ (recall that $m_i = \dim G_i$); I am a bit confused by this one ... oh ok

(iii) Each $g \in G$ can be written uniquely as $g = \exp(t_1 X_1)\exp(t_2 X_2) \cdots \exp(t_m X_m)$, for $t_i \in \mathbb{R}$;

(iv) Γ consists precisely of those elements which, when written in the above form, have all $t_i \in \mathbb{Z}$.

It is a well known result of Mal'cev [9] that any nilmanifold G/Γ can be equipped with a Mal'cev basis adapted to the lower central series filtration. More generally there is a Mal'cev basis adapted to any filtration $G_\bullet$ consisting of rational subgroups. We refer to the t_i as the **Mal'cev coordinates** of g, and we define the **Mal'cev coordinate map** $\psi = \psi_{\mathcal{X}} : G \to \mathbb{R}^m$ by

$$\psi(g) = (t_1, \cdots, t_m). \tag{1}$$

A nice property of Mal'cev coordinates is that they show the polynomial nature of the group operation of G. More specifically, there exist polynomials $\{P_i\}_{i=1}^m$ with $P_1 = 0$, taking integer values on integer inputs, such that for every $g, h \in G$ with $\psi(g) = (t_1, \cdots, t_m)$ and $\psi(h) = (s_1, \cdots, s_m)$ we have

$$\psi(gh) = (t_1 + s_1, t_2 + s_2 + P_2(t_1, s_1), \cdots, t_m + s_m + P_m(t_1, \cdots, t_{m-1}, s_1, \cdots, s_{m-1})). \tag{2}$$

Furthermore, there exist polynomials $\{\tilde{P}_i\}_{i=1}^m$ with $\tilde{P}_1 = 0$, taking integer values on integer inputs, such that for every $g \in G$ with $g = (t_1, \cdots, t_m)$ we have

$$\psi(g^{-1}) = (-t_1, -t_2 + \tilde{P}_2(t_1), \cdots, -t_m + \tilde{P}_m(t_1, \cdots, t_{m-1})). \tag{3}$$

Furthermore, if $G_\bullet$ is the lower central series filtration, and m_1 is the real dimension of $G_2 = [G, G]$, then $P_i = \tilde{P}_i = 0$ for $1 \leq i \leq m - m_1$. More generally, for $0 \leq j \leq d$ and $m - m_j < i \leq m - m_{j+1}$

we have $\deg_t(P_i), \deg_s(P_i), \deg_t(\tilde{P}_i), \deg_s(\tilde{P}_i) \leq j$, where $\deg_t(P)$ denotes the degree of P when viewed as a polynomial in the variables $\{t_i\}_{i=1}^m$ and the s_j are held constant. We will only need to make use of the simpler bounds that $\deg_t(P_i), \deg_s(P_i), \deg_t(\tilde{P}_i), \deg_s(\tilde{P}_i) \leq i$ for all $2 \leq i \leq m$. From this, we deduce that for any $g \in G$ with $\psi(g) = (t_1, \cdots, t_m)$, we have $\psi(g^n) = (Q_1(t_1; n) := nt_1, Q_2(t_1, t_2; n), \cdots, Q_m(t_1, \cdots, t_m; n))$ for some polynomials $\{Q_i\}_{i=1}^m$ satisfying $\deg_t(Q_i), \deg_n(Q_i) \leq i$. For the sake of notational convenience, we will write $P_i(t_1, \cdots, t_m, s_1, \cdots, s_m), \tilde{P}(t_1, \cdots, t_m)$ and $Q_i(t_1, \cdots, t_m; n)$ for all $1 \leq i \leq m$ with the understanding that P_i does not use t_j and s_j for $j > 1$, and similarly for $\tilde{P}_i$ and Q_i. We will also be using the fact that ψ is a diffeomorphism (cf. [4, Proposition 1.2.7]).

Definition 2.3 ([5, Definition 2.2], Metrics on G and G/Γ)**.** Let G/Γ be a nilmanifold with Mal'cev basis $\mathcal{X}$. We define $d = d_{\mathcal{X}} : G \times G \to \mathbb{R}_{\geq 0}$ to be maximal such that $d(x, y) \leq |\psi(xy^{-1})|$ for all $x, y \in G$, where $|\cdot|$ denotes the ℓ^∞ norm on $\mathbb{R}^m$. More explicitly, we have

$$d(x, y) = \inf \left\{ \sum_{i=0}^{n-1} \min \left(|\psi(x_{i-1}x_i^{-1})|, |\psi(x_i x_{i-1}^{-1})| \right) : x_0, \cdots, x_n \in G; x_0 = x; x_n = y \right\}.$$

This descends to a metric on G/Γ by setting

$$d(x\Gamma, y\Gamma) := \inf\{d(x', y') \ : \ x', y' \in G; x' \equiv x \pmod{\Gamma}; y' \equiv y \pmod{\Gamma}\}.$$

It turns out that this is indeed a metric on G/Γ (cf. [5, Lemma A.15]). We see that d is also right-invariant, hence

$$d(x\Gamma, y\Gamma) = \inf_{\gamma \in \Gamma} d(x, y\gamma) = \min_{\gamma \in \Gamma} d(x, y\gamma). \tag{4}$$

When the letter d is used for a metric, it will always denote the metric $d_{\mathcal{X}}$ relative to some basis $\mathcal{X}$ that is already under discussion. The symbol d' will be used for the metric defined using some other basis $\mathcal{X}'$. In the proof of Theorem 6.2 we will be considering many nilmanifolds and their respective Mal'cev bases, so then we will explicitly write $d_{\mathcal{X}}$ in each situation.

Remark 2.4 (Haar measure and Lebesgue measure). We see that for each $g \in G$, $\psi_{\mathcal{X}}(g)$ acts on $\mathbb{R}^m = \psi_{\mathcal{X}}(G)$ as an upper triangular matrix with 1s on the diagonal and polynomial functions in the entries above the diagonal. Since each $\psi_{\mathcal{X}}(g)$ has determinant 1, regardless of where it is acting on $\mathbb{R}^m$, we see that the m-dimensional Lebesgue measure μ on $\mathbb{R}^m$ is invariant under the action of $\psi_{\mathcal{X}}(g)$ for all $g \in G$. Since G is a nilpotent Lie group, it is unimodular, so the Haar measure μ of G normalized so that $\mu(\psi_{\mathcal{X}}^{-1}([0, 1)^m)) = 1$ is the pushforward of μ from $\mathbb{R}^m$, which is why we choose to abuse notation and write μ for both measures.

ryan bookmark

The **height** of a rational number $\frac{p}{q}$ with $\gcd(p, q) = 1$ is $\max(|p|, |q|)$, and by convention we set the height of 0 to be 1. The Lie algebra basis $\mathcal{X}$ is **Q-rational** if for every $1 \leq i, j \leq m$ we

have $[X_i, X_j] = \sum_{k=1}^m c_{i,j,k} X_k$ where every $c_{i,j,k}$ is a rational number of height at most Q. We will frequently use this rationality parameter Q of $\mathcal{X}$ in our inequalities, so we place a standing assumption that $Q \geq 2$ in any instance that we write $Q^{O(1)}$. If $\mathcal{X}$ is a Mal'cev basis of a nilmanifold G/Γ adapted to some filtration $G_\bullet$, then $\mathcal{X}$ is Q-rational for some $Q \in \mathbb{N}$.

Definition 2.5. Suppose that a nilmanifold G/Γ is given together with a Mal'cev basis $\mathcal{X} = \{X_1, \cdots, X_m\}$ and that $Q > 0$. Suppose that $G' \subseteq G$ is a closed connected subgroup. We say that G' is **Q-rational (relative to $\mathcal{X}$)** if the Lie algebra $\mathfrak{g}'$ of G' has a basis $\mathcal{X}' = \{X'_1, \cdots, X'_{m'}\}$ consisting of linear combinations $\sum_{i=1}^m a_i X_i$, where a_i are rational numbers with height at most Q for all i. If we can further assume that $\mathcal{X}'$ is a Q-rational Mal'cev basis for $G'/(\Gamma \cap G')$, then we call G' **strongly Q-rational (relative to $\mathcal{X}$)**. If G' is Q-rational for some $Q > 0$, then we say that G' is **rational**. We say that G' is **proper** if $G' \neq G$.

It is worth noting that G' is a rational subgroup of G if and only if $G' \cap \Gamma$ is cocompact in G' (cf. [5, Definition 1.10]). In particular, the quantitative rationality of G' with respect to G and Γ depends on the choice of Mal'cev basis $\mathcal{X}$, but the qualitative property of whether or not G' is rational is independent of $\mathcal{X}$.

Remark 2.6 (Asymptotic Notation). We use $a \ll b$ to mean that there exists a constant $C > 0$ for which $a \leq Cb$. Similarly, we use $a = O(b)$ to mean that there is a constant C for which $C^{-1}b \leq a \leq Cb$. In the discussion that follows, the constants C that are implied by our usage of $\ll$ and $O(1)$ will be allowed to depend on the global structure of the nilmanifold G/Γ, but not the local structure. For example, the constant C may depend on the dimension m, the nilpotence degree s, and the degree d of the filtration $G_\bullet$, but not on the Mal'cev basis $\mathcal{X}$ or on the points $g, g_1, g_2, x, y \in G$. For a given result, we will always assume without loss of generality that $C = C(m, s, d)$ is increasing as a function of m, s, and d, i.e., $C(m, s, d) \leq C(m', s', d')$ if $m \leq m'$, $s \leq s'$, or $d \leq d'$. The constants generated by the Mal'cev basis $\mathcal{X}$ and the polynomials $P_i, \tilde{P}_i$, and Q_i will be absorbed into the $Q^{O(1)}$ terms. We will also differ from the conventions of [5] in that we will write $a \leq Q^{O(1)}b$ instead of $a \ll Q^{O(1)}b$, as we will always assume that $Q \geq 2$, so the constant from $\ll$ can be absorbed into the $Q^{O(1)}$ term.

Lemma 2.7 ([5, Lemma A.4])**.** Suppose that $Q \geq 2$. Suppose that $\mathcal{X}, \mathcal{X}'$ are two Q-rational Mal'cev bases for G/Γ. Suppose that each X'_i is given by a Q-rational combination of the X_i and vice versa. Then for all $x, y \in G$ with $|\psi'(x)|, |\psi'(y)| \leq Q$, we have the bound

$$d(x, y) \ll Q^{O(1)} |\psi'(x) - \psi'(y)|,$$

and for all $x, y \in G$ with $d(x, \mathrm{id}_G), d(y, \mathrm{id}_G) \leq Q$, we have the bound

$$|\psi'(x) - \psi'(y)| \ll Q^{O(1)} d(x, y).$$

Lemma 2.8 (Approximate left-invariance of d, [5, Lemma A.5])**.** Suppose $Q \geq 2$ and that $\mathcal{X}$ is a Q-rational basis for G/Γ adapted to some filtration $G_\bullet$. suppose that $g, x, y \in G$ are elements with $|\psi(x)|, |\psi(y)|, |\psi(g)| \leq Q$. Then we have the bound

$$d(gx, gy) \ll Q^{O(1)} d(x, y).$$

Lemma 2.9 (Comparison lemma [5, Lemma A.6])**.** Suppose that $G' \subseteq G$ is a closed subgroup and that $\mathcal{X}, \mathcal{X}'$ are bases for $\mathfrak{g}, \mathfrak{g}'$ respectively which have the nesting property (A.1). Let $Q \geq 2$, and suppose that each X_i' is a Q-rational combination of the X_i. Then we have the bounds

$$d'(x, y) \ll Q^{O(1)} d(x, y)$$

uniformly for all $x, y \in G'$ with $|\psi(x)|, |\psi(y)| \leq Q$ and

$$d(x, y) \ll Q^{O(1)} d'(x, y)$$

uniformly for all $x, y \in G'$ with $|\psi'(x)|, |\psi'(y)| \leq Q$.

Lemma 2.10 ([5, Proposition A.10])**.** Suppose that $\mathcal{X} = \{X_1, \cdots, X_m\}$ is a Q-rational Mal'cev basis for G/Γ adapted to a filtration $G_\bullet$. Suppose that $G' \subseteq G$ is a Q-rational subgroup of G and, furthermore, that $G'_\bullet$ is a filtration of G' in which each of the groups G'_i is Q-rationa (with respect to the basis $\mathcal{X}$). Write $\Gamma' := \Gamma \cap G'$. Then G'/Γ' has a Mal'cev basis $\mathcal{X}' = \{X_1, \cdots, X_m\}$ adapted to $G'_\bullet$ in which each X_i' is a $Q^{O(1)}$-rational combination of the X_i.

Remark 2.11. We will only be interested in applying Lemma 2.10 in the case that the filtration $G'_\bullet$ is given by $G_\bullet \cap G'$, i.e., when $G'_i = G_i \cap G'$. In general, the intersection of a Q_1-rational subgroup with a Q_2-rational subgroup is a $(Q_1 Q_2)^{O(1)}$-rational subgroup, hence each G'_i is a $Q^{O(1)}$-rational subgroup. Consequently, we will assume throughout this paper that a rational subgroup G' of G is endowed with the filtration $G'_\bullet := G_\bullet \cap G'$, and a Mal'cev basis $\mathcal{X}'$ adapted to $G'_\bullet$.

Lemma 2.12 ([5, Lemma A.14])**.** Let G/Γ be a nilmanifold endowed with a Mal'cev basis $\mathcal{X}$. For any $g \in G$, we may write $g = \{g\}[g]$ in a unique way, where $\psi_{\mathcal{X}}(\{g\}) \in [0, 1)^m$ and $[g] \in \Gamma$.

Lemma 2.13 (Nilmanifolds are bounded [5, Lemma A.16])**.** Let $Q \geq 2$, and suppose that $\mathcal{X}$ is a Q-rational Mal'cev basis for a nilmanifold G/Γ with respect to some filtration $G_\bullet$. Then $d(x\Gamma, y\Gamma) \ll Q^{O(1)}$ uniformly in $x, y \in G$.

Lemma 2.14. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For $C > 0$ let $R_C := \psi_{\mathcal{X}}^{-1}([-Q^C, Q^C])$. There exists $0 < C_0 \leq Q^{O(1)}$ such that for any $x, y \in G$, there exist $\gamma_1, \gamma_2 \in \Gamma$ with $d_{\mathcal{X}}(x\Gamma, y\Gamma) = d_{\mathcal{X}}(x\gamma_1, y\gamma_2)$ and $x\gamma_1, y\gamma_2 \in R_{C_0}$. Furthermore, we may assume without loss of generality that $x\gamma_1 \in \psi_{\mathcal{X}}^{-1}([0, 1)^m)$.

Proof. We first pick $\gamma_1 \in \Gamma$ for which $x\gamma_1 \in \psi_{\mathcal{X}}^{-1}([0, 1)^m)$. Let $\gamma_2 \in \Gamma$ be such that $d_{\mathcal{X}}(x\Gamma, y\Gamma) = d_{\mathcal{X}}(x\gamma_1, y\gamma_2)$. Using Lemma 2.13 we see that

$$d_{\mathcal{X}}(y\gamma_2, \mathrm{Id}_G) \leq d_{\mathcal{X}}(x\gamma_1, y\gamma_2) + d_{\mathcal{X}}(x\gamma_1, \mathrm{Id}_G) \leq d_{\mathcal{X}}(x\Gamma, y\Gamma) + |\psi_{\mathcal{X}}(x\gamma_1)| \leq Q^{O(1)}. \tag{5}$$

The desired result now follows from Lemma 2.7. □

When the nilmanifold G/Γ is endowed with a Mal'cev basis $\mathcal{X}$, we use d_∞ to denote the pull back of the ℓ^∞ metric on $\mathbb{R}^m$ to G via $\psi_{\mathcal{X}}$. More concretely, we have $d_\infty(x,y) = |\psi_{\mathcal{X}}(x) - \psi_{\mathcal{X}}(y)|$ for all $x, y \in G$.

Lemma 2.15 (Comparison of metrics on nilmanifolds [5, Lemma A.17])**.** Let $Q \geq 2$. Suppose that $G' \subseteq G$ is a closed subgroup and that $\mathcal{X}, \mathcal{X}'$ are Q-rational Mal'cev bases for G/Γ and G'/Γ' respectively such that each X_i' is a Q-rational combination of the X_i. Let d and d' be the metrics induced on G/Γ and G'/Γ' respectively. Then for any $x, y \in G'$, we have

$$d'(x\Gamma', y\Gamma') \ll Q^{O(1)} d(x\Gamma, y\Gamma) \text{ and } d(x\Gamma, y\Gamma) \ll Q^{O(1)} d'(x\Gamma', y\Gamma').$$

2.4 Equidistribution in nilmanifolds

We want to not only discuss equidistribution in nilmanifolds G/Γ, but also in translated subnilmanifolds of the form $xG'y\Gamma/\Gamma$, so we introduce the definitions of equidistribution in more generality than what is present in [5]. We will see later that we can always reduce considerations about the translated subnilmanifold $xG'y\Gamma/\Gamma$ to considerations about another translated subnilmanifold $x'G''\Gamma/\Gamma$. After making such a reduction, we will always assume that $x'G''\Gamma/\Gamma$ is endowed with the pushforward of the normalized Haar measure of $G''\Gamma/\Gamma$ with respect to left-translation by x'.

Definition 2.16. Let (X, d) be a compact metric space endowed with a Borel probability measure μ.

(i) Given a length $N \in \mathbb{N}$ and an error tolerance $\delta \in (0, \frac{1}{2})$, a finite sequence $(p(n))_{n=1}^N$ in X is said to be **δ-equidistributed** if we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} F(p(n)) - \int_X F d\mu \right| \leq \delta ||F||_{\text{Lip}}$$

for all Lipschitz functions $F : X \to \mathbb{C}$, where

$$||F||_{\text{Lip}} := ||F||_\infty + \sup_{x,y \in X, x \neq y} \frac{|F(x) - F(y)|}{d(x,y)}. \tag{6}$$

(ii) The finite sequence $(p(n))_{n=1}^N$ is **totally δ-equidistributed** if we have

$$\left| \frac{1}{|P|} \sum_{n \in P} F(p(n)) - \int_X F d\mu \right| \leq \delta ||F||_{\text{Lip}}$$

for all Lipschitz functions $F : X \to \mathbb{C}$ and all arithmetic progressions $P \subseteq [1, N]$ of length at least δN. In other words, $(p(n))_{n=1}^N$ is totally δ-equidistributed if for any arithmetic progression $P \subseteq [1, N]$ of length at least δN, the sequence $(p(n))_{n \in P}$ is δ-equidistributed.

Lemma 2.17. Let G/Γ be a nilmanifold of dimension m endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For $g \in G$ let $B_r(g\Gamma)$ denote the ball of radius r in G/Γ with respect to the $d_{\mathcal{X}}$ metric.

(i) For $0 < r < r_0 \asymp Q^{-O(1)}$ and $g \in G$, we have

$$Q^{-O(1)} r^m \leq \mu(B_r(g\Gamma)) \leq Q^{O(1)} r^m. \tag{7}$$

(ii) If $\delta < r_0$ and $(p(n)\Gamma)_{n=1}^N$ is $Q^{-O(1)}\delta^{m+1}$-equidistributed, then $(p(n)\Gamma)_{n=1}^N$ visit every ball of radius δ in G/Γ.

Proof of (i). Firstly, we will show that the injectivity radius of the quotient map q given by $q(x) = x\Gamma$ is $Q^{-O(1)}$. Indeed, suppose that $0 < r < \mathrm{diam}(G/\Gamma), x, y \in \psi^{-1}([0,1)^m)$, and $\gamma_1 \neq \gamma_2 \in \Gamma$ are such that $d(x, y\gamma_1), d(x, y\gamma_2) < r$. We see that $d(y^{-1}, \mathrm{Id}_G) = d(y, \mathrm{Id}_G) \leq 1$, so Lemma 2.7 tells us that $|\psi(y^{-1})| < Q^{O(1)}$. Since $d(y\gamma_1, y\gamma_2) < 2r$, by Lemma 2.8 we have $d(\gamma_1, \gamma_2) < 2rQ^{O(1)}$. Similarly, we see that $d(y\gamma_i, y) \leq d(y\gamma_i, x) + d(x, y) \leq r + Q^{O(1)}|\psi(x) - \psi(y)| \leq 2Q^{O(1)}$, hence we may use Lemma 2.8 again to see that $d(\gamma_i, \mathrm{Id}_G) < Q^{O(1)}$ for $i = 1, 2$. We may now use Lemma 2.7 to see that $d(\gamma_1, \gamma_2) \geq Q^{-O(1)}|\psi(\gamma_1) - \psi(\gamma_2)| \geq Q^{-O(1)}$, hence $r \geq Q^{-O(1)}$ as claimed.

Now let r_0' denote the injectivity radius of q. We see that for $r \in (0, r_0')$ and $x \in \psi^{-1}([0,1)^m)$ we have that q is a diffeomorphism between $B_r(x)$ and $B_r(x\Gamma)$, hence $\mu(B_r(x)) = \mu(B_r(x\Gamma))$. Using Lemma 2.7, we set $r_0 = Q^{-O(1)} r_0'$ and we see that for $r \in (0, r_0)$ and $x \in \psi^{-1}([0,1)^m)$ we have

$$B_{Q^{-O(1)}r}(\psi(x)) \subseteq \psi(B_r(x)) \subseteq B_{Q^{O(1)}r}(\psi(x)). \tag{8}$$

As discussed in Remark 2.4, ψ pushesforward the Haar measure of G to the Lebesgue measure of $\mathbb{R}^m$, so $Q^{-O(1)} r^m \leq \mu(B_r(x)) \leq Q^{O(1)} r^m$, from which the desired result follows. □

Proof of (ii). For $g \in G$ let $F_g : G/\Gamma \to \mathbb{C}$ be given by

$$F_g(x\Gamma) = \begin{cases} \delta - d_{\mathcal{X}}(x\Gamma, g\Gamma) & \text{if } d_{\mathcal{X}}(x\Gamma, g\Gamma) \leq \delta \\ 0 & \text{else} \end{cases}. \tag{9}$$

We see that for all $g \in G$ we have

$$\int_{G/\Gamma} F_g d\mu \geq \frac{\delta}{2} \mu(B_{\frac{\delta}{2}}(g\Gamma)) \geq Q^{-A} \left(\frac{\delta}{2}\right)^{m+1}, \text{ and } ||F_g||_{\mathrm{Lip}} = 1 + \delta, \tag{10}$$

where the constant $A > 0$ is given to us by part (i). The desired result now follows from the observation that

$$\int_{G/\Gamma} F_g d\mu \geq Q^{-A-m-2} \left(\frac{\delta}{6}\right)^{m+1} ||F_g||_{\mathrm{Lip}}. \tag{11}$$

□

Lemma 2.18 (Balls and equidistribution in two-sided translates)**.** Let G/Γ be an m-dimensional nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$ adapted to a filtration $G_\bullet$ of degree d, and let $R \geq Q \geq 2$. Let $H \leq G$ be a closed connected R-rational subgroup, let $x, y \in G$, and suppose

that $d_{\mathcal{X}}(x, \mathrm{id}_G) \leq R$ and that y is R-rational. Set $Z := xHy\Gamma/\Gamma \subseteq G/\Gamma$ and let $k := \dim H$. Then there exists a constant $C = C(m, d) > 0$ with the following properties:

(i) There exist an element $u \in \psi_{\mathcal{X}}^{-1}([0,1)^m)$ and a closed connected R^C-rational subgroup $K \leq G$ such that $Z = uK\Gamma/\Gamma$. Writing $\Gamma_K := K \cap \Gamma$, the nilmanifold K/Γ_K admits an R^C-rational Mal'cev basis $\mathcal{X}_K$ such that the map $\Phi : K/\Gamma_K \longrightarrow Z$, given by $\Phi(h\Gamma_K) := uh\Gamma$, is a homeomorphism and satisfies

$$R^{-C} d_{\mathcal{X}_K}(h_1\Gamma_K, h_2\Gamma_K) \leq d_{\mathcal{X}}\big(uh_1\Gamma, uh_2\Gamma\big) \leq R^C d_{\mathcal{X}_K}(h_1\Gamma_K, h_2\Gamma_K) \tag{12}$$

for all $h_1, h_2 \in K$.

(ii) We endow Z with the probability measure $\mu_Z := \Phi_* \mu_{K/\Gamma_K}$, where μ_{K/Γ_K} is normalized Haar measure on K/Γ_K. For $z \in Z$, let $B_r^Z(z) := B_r(z) \cap Z$, where $B_r(z)$ is the ball of radius r in G/Γ with respect to $d_{\mathcal{X}}$. If $0 < r < R^{-C}$, then

$$R^{-C} r^k \leq \mu_Z\big(B_r^Z(z)\big) \leq R^C r^k. \tag{13}$$

(iii) Let $0 < \delta < R^{-C}$, and let $(z_n)_{n=1}^N$ be a finite sequence in Z. Suppose that $(z_n)_{n=1}^N$ is

$$R^{-C}\delta^{k+1}$$

-equidistributed in Z, where equidistribution is defined using the measure μ_Z and the restriction of $d_{\mathcal{X}}$ to Z. Then $(z_n)_{n=1}^N$ visits every ball $B_\delta^Z(z)$, $z \in Z$.

Proof. Throughout the proof, the value of $C = C(m, d)$ may be increased a bounded number of times. We first rewrite the two-sided translate as a bounded left translate of a rational subnilmanifold. Write $y = \{y\}[y]$, where $\psi_{\mathcal{X}}(\{y\}) \in [0,1)^m$ and $[y] \in \Gamma$. Since $y\Gamma = \{y\}\Gamma$, we have $xHy\Gamma/\Gamma = xH\{y\}\Gamma/\Gamma$. Thus, replacing y by $\{y\}$, we may assume that $\psi_{\mathcal{X}}(y) \in [0,1)^m$. Since y is R-rational and $[y] \in \Gamma$ is rational, [5, Lemma A.11] implies that $\{y\}$ and $\{y\}^{-1}$ are $R^{O(1)}$-rational and have Mal'cev coordinates bounded by $R^{O(1)}$. Consequently, [5, Lemma A.13] implies that $H_0 := y^{-1}Hy$ is an $R^{O(1)}$-rational subgroup of G. We have $Hy = yH_0$, and hence $Z = xyH_0\Gamma/\Gamma$. Write

$$xy = u\lambda, \qquad u := \{xy\} \in \psi_{\mathcal{X}}^{-1}([0,1)^m), \qquad \lambda := [xy] \in \Gamma.$$

Since $d_{\mathcal{X}}(x, \mathrm{id}_G) \leq R$, Lemma 2.7, together with the polynomial form of multiplication and inversion in Mal'cev coordinates, gives

$$|\psi_{\mathcal{X}}(x)|, |\psi_{\mathcal{X}}(xy)|, |\psi_{\mathcal{X}}(\lambda)| \leq R^{O(1)}.$$

The last estimate can also be read directly from the iterative reduction procedure in [5, Lemma A.14].

Define $K := \lambda H_0 \lambda^{-1}$. Since $\lambda \in \Gamma$, it is rational, and its Mal'cev coordinates are bounded by $R^{O(1)}$. Another application of [5, Lemma A.13] shows that K is $R^{O(1)}$-rational. Moreover,

$$Z = u\lambda H_0 \Gamma/\Gamma = u(\lambda H_0 \lambda^{-1})\lambda\Gamma/\Gamma = uK\Gamma/\Gamma.$$

Since K is rational, $K\Gamma/\Gamma$ is a subnilmanifold of G/Γ, hence Z is closed.

Equip K with the filtration $K_\bullet := K \cap G_\bullet$. By Lemma 2.10, after increasing C, the nilmanifold K/Γ_K admits an R^C-rational Mal'cev basis $\mathcal{X}_K$, adapted to $K_\bullet$, whose elements are R^C-rational combinations of the elements of $\mathcal{X}$.

We next compare metrics. First, Lemma 2.15 gives

$$R^{-C} d_{\mathcal{X}_K}(h_1\Gamma_K, h_2\Gamma_K) \le d_{\mathcal{X}}(h_1\Gamma, h_2\Gamma) \le R^C d_{\mathcal{X}_K}(h_1\Gamma_K, h_2\Gamma_K). \tag{14}$$

We also claim that left multiplication by u is bi-Lipschitz on G/Γ, with constants bounded by R^C. To prove the upper bound, let $a, b \in G$. By Lemma 2.14, we may choose representatives $a_0 \in a\Gamma$ and $b_0 \in b\Gamma$ such that

$$d_{\mathcal{X}}(a\Gamma, b\Gamma) = d_{\mathcal{X}}(a_0, b_0) \text{ and } |\psi_{\mathcal{X}}(a_0)|, |\psi_{\mathcal{X}}(b_0)| \le R^C.$$

Since u lies in the fixed fundamental domain, both $|\psi_{\mathcal{X}}(u)|$ and $|\psi_{\mathcal{X}}(u^{-1})|$ are at most R^C. Lemma 2.8 therefore gives

$$d_{\mathcal{X}}(ua\Gamma, ub\Gamma) \le d_{\mathcal{X}}(ua_0, ub_0) \le R^C d_{\mathcal{X}}(a_0, b_0) = R^C d_{\mathcal{X}}(a\Gamma, b\Gamma).$$

Applying the same estimate to left multiplication by u^{-1} gives the reverse inequality

$$d_{\mathcal{X}}(a\Gamma, b\Gamma) \le R^C d_{\mathcal{X}}(ua\Gamma, ub\Gamma).$$

Combining this with (14) proves (12).

We now prove (ii). Let $z = \Phi(w)$ with $w \in K/\Gamma_K$. Equation (12) gives the inclusions

$$B^{K/\Gamma_K}_{R^{-C}r}(w) \subseteq \Phi^{-1}\big(B^Z_r(z)\big) \subseteq B^{K/\Gamma_K}_{R^C r}(w), \tag{15}$$

where the balls on the left and right are taken with respect to $d_{\mathcal{X}_K}$.

Apply Lemma 2.17(i) to K/Γ_K. Since $\dim K = k$ and $\mathcal{X}_K$ is R^C-rational, after increasing C and requiring $0 < r < R^{-C}$, Equation (15) yields

$$R^{-C} r^k \le \mu_Z\big(B^Z_r(z)\big) \le R^C r^k.$$

This proves (ii).

Finally, suppose that $0 < \delta < R^{-C}$ and that $(z_n)_{n=1}^N$ is $R^{-C}\delta^{k+1}$-equidistributed in Z. Fix

$z \in Z$ and define

$$F_z(w) := \max\big(0, \delta - d_{\mathcal{X}}(w,z)\big), \qquad w \in Z.$$

Then

$$\|F_z\|_{\mathrm{Lip}} \leq 1 + \delta \leq 2,$$

while part (ii) gives

$$\int_Z F_z \, d\mu_Z \geq \frac{\delta}{2} \mu_Z\big(B^Z_{\delta/2}(z)\big) \geq R^{-C}\delta^{k+1},$$

after increasing C once more.

The assumed equidistribution therefore implies

$$\frac{1}{N}\sum_{n=1}^{N} F_z(z_n) > 0.$$

Hence $F_z(z_n) > 0$ for at least one n, which means $d_{\mathcal{X}}(z_n, z) < \delta$. Since $z \in Z$ was arbitrary, the sequence visits every ball $B^Z_\delta(z)$. □

Definition 2.19. Suppose that G/Γ is a nilmanifold with a Mal'cev basis $\mathcal{X}$. Suppose that $\eta : G \to \mathbb{R}/\mathbb{Z}$ is a **horizontal character**, that is to say a homomorphism from G to $\mathbb{R}\mathbb{Z}$ which annihilates Γ. Then, when written in coordinates relative to $\mathcal{X}$, properties (iii) and (iv) of Definition 2.2 imples that $\eta(g) = k \cdot \psi_{\mathcal{X}}(g)$ for some unique $k \in \mathbb{Z}^m$. We write $|\eta| := |k|$.

Theorem 2.20 ([5, Theorem 1.16])**.** Let $m, d \geq 0$, $0 < \delta < \frac{1}{2}$, and $N \geq 1$. Let G/Γ be an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d and a $\frac{1}{\delta}$-rational Mal'cev basis $\mathcal{X}$ adapted to this filtration. Suppose that $g \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. Then at least one of the following statements is true:

(i) $(g(n)\Gamma)_{n=1}^N$ is δ-equidistributed in G/Γ.

(ii) There exists a non-trivial horizontal character $\eta : G \to \mathbb{R}/\mathbb{Z}$ with $|\eta| \ll \delta^{-O_{m,d}(1)}$ such that $||\eta \circ g(n) - \eta \circ g(n-1)||_{\mathbb{R}/\mathbb{Z}} \ll \delta^{-O_{m,d}(1)}/N$ for all $n \in [1, N]$.

Definition 2.21 (cf. [5, Definition 1.18])**.** Let G/Γ be a nilmanifold with a Mal'cev basis $\mathcal{X}$. Let $(\epsilon(n))_{n\in\mathbb{Z}}$ be a sequence in G, and let $M, N \geq 1$. We say that $(\epsilon(n))_{n\in\mathbb{Z}}$ is (M,N)**-smooth** if we have $d(\epsilon(n), \mathrm{id}_G) \leq M$ and $d(\epsilon(n), \epsilon(n-1)) \leq M/N$ for all $1 \leq n \leq N$.

Theorem 2.22 ([5, Theorem 1.19])**.** Let $m, d \geq 0$, and let $M_0, N \geq 1$ and $A > 0$ be real numbers. Suppose that G/Γ is an m-dimensional nilmanifold together with a filtration $G_\bullet$ of degree d. Suppose that $\mathcal{X}$ is an M_0-rational Mal'cev basis adapted to $G_\bullet$ and that $p \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$. Then there is an integer M with $M_0 \leq M \ll M_0^{O_{A,m,d}(1)}$, a rational subgroup $G' \subseteq G$, a Mal'cev basis $\mathcal{X}'$ for G'/Γ' in which each element is an M-rational combination of the elements of $\mathcal{X}$, and a decomposition $p = \epsilon g' \gamma$ into polynomial sequences $\epsilon, g', \gamma \in \mathrm{poly}(\mathbb{Z}, G_\bullet)$ with the following properties:

(i) $\epsilon : \mathbb{Z} \to G$ is (M,N)-smooth;

(ii) $g' : \mathbb{Z} \to G'$ takes values in G', and the finite sequence $(g'(n)\Gamma')_{n=1}^N$ is totally M^{-A}-equidistributed in G'/Γ', using the metric $d_{\mathcal{X}'}$ on G'/Γ';

(iii) $\gamma : \mathbb{Z} \to G$ is M-rational, and $(\gamma(n)\Gamma)_{n\in\mathbb{Z}}$ is periodic with period at most M.

We mention that [8] has quantitative improvements of the equidistribution results of [5], but we do not state them here since we do not need them for our purposes.

Lemma 2.23. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$ adapted to the filtration $G_\bullet$. Let $H \leq G$ be a Q-rational subgroup, let $\gamma \in \psi_{\mathcal{X}}^{-1}([0,1)^m)$ be Q-rational, and note that $K := \gamma^{-1}H\gamma$ is a $Q^{O(1)}$-rational subgroup of G. Let $\mathcal{X}_H$ and $\mathcal{X}_K$ be $Q^{O(1)}$-rational Mal'cev bases for $H\Gamma/\Gamma$ and $K\Gamma/\Gamma$ adapted to the filtrations $H_\bullet := H \cap G_\bullet$ and $K_\bullet := K \cap G_\bullet$ respectively, for which we also have that each element of $\mathcal{X}_H$ and $\mathcal{X}_K$ is a $Q^{O(1)}$-rational linear combination of elements of $\mathcal{X}$.

(i) Given a nontrivial horizontal character η of $K\Gamma/\Gamma$, there exists a $0 < q < Q^{O(1)}$ and a nontrivial horizontal character χ of $H\Gamma/\Gamma$ given by $\chi(h) := q\eta(\gamma^{-1}h\gamma)$. Furthermore, we have that $|\chi| \leq Q^{O(1)}|\eta|$.

(ii) There exists a constant $C = C(m,d) > 0$ such that for every sequence $p \in \mathrm{poly}(\mathbb{Z}, H_\bullet)$ for which $(p(n)\Gamma)_{n=1}^N$ is δ-totally equidistributed in $H\Gamma/\Gamma$ with respect to $d_{\mathcal{X}_H}$, the sequence $(\gamma^{-1}p(n)\gamma\Gamma)_{n=1}^N$ is $Q^C\delta^{\frac{1}{C}}$-totally equidistributed in $K\Gamma/\Gamma$ with respect to $d_{\mathcal{X}_K}$.

Proof. □

Our next result is a stronger variant of [5] that is an immediate consequence of Theorem 2.22 and Lemma 2.23. While we will not make use of the next result directly, we record it because it is of independent interest.

Corollary 2.24.

2.5 Recurrence in nilmanifolds

3 A translation obstruction and the need for an extension

A natural first attempt at proving our main theorem is to imitate the argument from [1] for tori. Suppose that, after applying the factorization theorem to the polynomial sequence $p(n) = g^n$, we obtain

$$p(n) = \epsilon(n)g'(n)\gamma(n),$$

where ϵ is smooth, $(g'(n)\Gamma')_{n=1}^N$ is quantitatively equidistributed in a positive-dimensional subnilmanifold G'/Γ', and $(\gamma(n)\Gamma)_{n\in\mathbb{Z}}$ is periodic. After restricting n to a suitable arithmetic progression, one may assume that $\gamma(n)\Gamma$ is constant.

If an arithmetic progression has initial term a, however, the points that must be colored are

$$g^a p(n)\Gamma = g^a \epsilon(n) g'(n)\gamma(n)\Gamma.$$

In the torus case, the metric is translation invariant, so the smooth factor remains small after multiplication by g^a, uniformly in a. It is therefore natural to hope that the equidistribution of $g'(n)$ forces the translated sequence $g^a p(n)\Gamma$ to visit many cells of any sufficiently fine partition. The following example shows that this conclusion is false on a general nilmanifold.

Example 3.1 (Equidistribution can disappear after translation)**.** Let H be the Heisenberg group, written in coordinates as

$$(x, y, z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus,

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

Let

$$\Lambda := \{(x, y, z) : x, y, z \in \mathbb{Z}\},$$

and endow H/Λ with the metric associated to the standard Mal'cev basis.

Fix $N \geq 2$ and a positive integer a, and set

$$\beta := \frac{1}{aN}, \qquad c := -\frac{1}{N}, \qquad g := (1, \beta, c).$$

The formula for powers in the Heisenberg group gives

$$p(n) := g^n = \left(n, n\beta, nc + \binom{n}{2}\beta\right).$$

Let

$$H' := \{(0, 0, t) : t \in \mathbb{R}\}, \qquad \Lambda' := H' \cap \Lambda,$$

and define

$$\epsilon(n) := \left(0, n\beta, \binom{n}{2}\beta\right),$$

$$g'(n) := \left(0, 0, -\frac{n}{N}\right), \qquad \gamma(n) := (n, 0, 0).$$

A direct calculation gives

$$p(n) = \epsilon(n) g'(n) \gamma(n).$$

All three sequences are polynomial relative to the lower central series filtration. Moreover, $\gamma(n) \in \Lambda$ for every n, so $(\gamma(n)\Lambda)_{n\in\mathbb{Z}}$ is constant.

The sequence $(g'(n)\Lambda')_{n=1}^N$ consists of the N equally spaced points of the circle H'/Λ'. Conse-

quently, for every Lipschitz function $F : H'/\Lambda' \to \mathbb{C}$,

$$\left| \frac{1}{N} \sum_{n=1}^{N} F(g'(n)\Lambda') - \int_{H'/\Lambda'} F \, d\mu \right| \leq \frac{1}{N} \|F\|_{\mathrm{Lip}}.$$

In particular, $(g'(n)\Lambda')_{n=1}^{N}$ is δ-equidistributed whenever $N \geq \delta^{-1}$.

The factor ϵ is extremely small on $[1, N]$. Indeed,

$$\max_{1 \leq n \leq N} d\big(\epsilon(n), \mathrm{id}_H\big) \leq \frac{N}{2a},$$

and

$$\max_{1 \leq n \leq N} d\big(\epsilon(n), \epsilon(n-1)\big) \leq \frac{1}{a}.$$

In particular, if $a \geq N$, then ϵ is $(1, N)$-smooth in the sense of Definition 2.21. By increasing a, we may therefore make ϵ as small and as slowly varying as desired.

Nevertheless, the translated orbit segment $(g^a p(n)\Lambda)_{n=1}^{N}$ may be contained in an arbitrarily small ball. For each $k \in \mathbb{Z}$, the element

$$x_k := g^k(-k, 0, 0) = \left(0, k\beta, kc + \binom{k}{2}\beta\right)$$

represents the coset $g^k\Lambda$. For $1 \leq n \leq N$, we have

$$\begin{aligned} x_{a+n}x_a^{-1} &= \left(0, n\beta, nc + \left(\binom{a+n}{2} - \binom{a}{2}\right)\beta\right) \\ &= \left(0, \frac{n}{aN}, \frac{1}{aN}\binom{n}{2}\right). \end{aligned}$$

It follows that

$$d(g^{a+n}\Lambda, g^a\Lambda) \leq \max\left(\frac{n}{aN}, \frac{1}{aN}\binom{n}{2}\right) \leq \frac{N}{2a}.$$

Therefore, if $a > N/(2\delta)$, then

$$\{g^a p(n)\Lambda : 1 \leq n \leq N\} = \{g^{a+n}\Lambda : 1 \leq n \leq N\} \subseteq B_\delta(g^a\Lambda),$$

even though $(g'(n)\Lambda')_{n=1}^{N}$ is δ-equidistributed on the positive-dimensional nilmanifold H'/Λ' once N is sufficiently large.

The mechanism behind Example 3.1 is the noncommutative term xy' in the Heisenberg group law. The y-coordinate

$$\frac{n}{aN}$$

of $\epsilon(n)$ is very small, but left multiplication by an element whose x-coordinate is a creates the

central term

$$a \cdot \frac{n}{aN} = \frac{n}{N}.$$

This term cancels the equidistributed central coordinate $-n/N$ of $g'(n)$. Thus, a smooth error that is negligible before translation can become macroscopically important after translation.

Equivalently, the conclusion

$$(g'(n)\Gamma')_{n=1}^{N} \text{ is } \delta\text{-equidistributed on a positive-dimensional subnilmanifold}$$

does not imply that

$$(g^a p(n)\Gamma)_{n=1}^{N}$$

escapes every δ-ball, uniformly in the initial term a. This is precisely the translation invariance that is available in the torus case and that fails in the general nilmanifold case.

The example may also be understood in terms of reduction to a fundamental domain. Recall that every $h \in H$ has a unique decomposition

$$h = \{h\}[h],$$

where $\{h\}$ lies in the chosen fundamental domain and $[h] \in \Lambda$. When g^a is reduced to the fundamental domain, its large integer x-coordinate a is absorbed into the lattice component $[g^a]$. The point $g^a\Lambda$ in the original nilmanifold does not separately record this lattice carry. Nevertheless, the carry interacts with the small y-coordinate of $\epsilon(n)$ and produces the central term that cancels the equidistributed part of the factorization.

This explains why it is not enough to work only in the nilmanifold that defines the nil-Bohr set under consideration. The original nilmanifold records the reduced position $\{g^n\}\Gamma$, but it forgets the lattice reduction data $[g^n]$. In an abelian group, this loss of information is harmless. In a nonabelian group, the forgotten lattice data can interact with later coordinates through commutator terms.

To obtain an argument that is uniform in the initial term of an arithmetic progression, we must therefore pass to an extended nilmanifold that records the relevant reduction cocycles, or, equivalently, the generalized-polynomial phases generated by those cocycles. In the extended space, the term responsible for the cancellation in Example 3.1 appears as an additional coordinate rather than being hidden inside the choice of a lattice representative.

The purpose of the extension is not to make every left translation an isometry. Rather, it is to retain enough information that a quantitatively equidistributed component cannot be annihilated by the interaction between a smooth factor and an unrecorded lattice carry. This is the additional structure that is absent from the torus argument and that is needed in the general nilmanifold case.

4 A translation-stable nilmanifold extension

Example 3.1 shows that the nilmanifold defining a nil-Bohr set need not retain enough information to control translated orbit segments. In particular, a component that is equidistributed before translation can be cancelled by the interaction of a smooth factor with lattice-reduction data coming from the initial point of the progression. We now construct an auxiliary nilsystem in which a bad generalized-polynomial difference cannot undergo such a complete collapse. The construction uses the realization theorem of Huang, Shao, and Ye for generalized polynomials.

For generalized polynomials $P_1, \dots, P_k$ and $\varepsilon > 0$, write

$$\mathcal{R}(P_1, \dots, P_k; \varepsilon) := \left\{ n \in \mathbb{Z} : \|P_i(n)\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon \text{ for every } 1 \le i \le k \right\}. \tag{16}$$

We first record the precise consequence of [7] that we use.

Theorem 4.1 (Huang–Shao–Ye)**.** Let $P_1, \dots, P_k$ be generalized polynomials of degree at most d. There exist a minimal d-step nilsystem (Y, T) such that for every $\epsilon > 0$ there exists a nonempty open set $U \subseteq Y$ for which

$$\left\{ n \in \mathbb{Z} : U \cap T^{-n}U \cap \cdots \cap T^{-dn}U \neq \varnothing \right\} \subseteq \mathcal{R}(P_1, \dots, P_k; \varepsilon). \tag{17}$$

Proof. This is the specialization of [7, Theorem C] to the set in Equation (16). While the statement of [7, Theorem C] implies that (Y, T) also depends on the choice of ϵ, it is clear from the proof that this is not the case. □

The next theorem is the translation-stable form of this result that will be used below.

Theorem 4.2 (Translation-stable realization)**.** Let $P_1, \dots, P_k$ be special generalized polynomials of degree at most D. There exist a D-step nilsystem (Y, T) such that for any metric d_Y compatible with the topology of Y and any $\varepsilon > 0$, there exists $\rho > 0$ for which the following holds for every $n \in \mathbb{Z}$ and every $y \in Y$:

$$\operatorname{diam}_{d_Y} \left\{ y, T^n y, \dots, T^{Dn} y \right\} < \rho \quad \Longrightarrow \quad \|P_i(n)\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon \text{ for every } 1 \le i \le k. \tag{18}$$

Equivalently, if

$$\max_{1 \le i \le k} \|P_i(n)\|_{\mathbb{R}/\mathbb{Z}} \ge \varepsilon, \tag{19}$$

then, for every $a \in \mathbb{Z}$ and every $y \in Y$,

$$\operatorname{diam}_{d_Y} \left\{ T^a y, T^{a+n} y, \dots, T^{a+Dn} y \right\} \ge \rho. \tag{20}$$

Proof. Apply Theorem 4.1 to obtain a minimal D-step nilsystem (Y, T) and a nonempty open set $U \subseteq Y$ satisfying Equation (17). Write $Y = \widetilde{G}/\widetilde{\Gamma}$ and $T = T_{\widetilde{g}}$, and fix any metric d_Y compatible with the topology of Y.

Since (Y, T) is minimal and U is nonempty and open, the collection

$$\{T^r U : r \in \mathbb{Z}\}$$

is an open cover of Y. By compactness, there exist integers $r_1, \ldots, r_M$ such that

$$Y = \bigcup_{j=1}^{M} T^{r_j} U. \tag{21}$$

Let $\rho > 0$ be a Lebesgue number for this finite open cover. Thus every subset of Y having d_Y-diameter smaller than ρ is contained in one of the sets $T^{r_j} U$.

Suppose that $n \in \mathbb{Z}$ and $y \in Y$ satisfy

$$\operatorname{diam}_{d_Y}\{y, T^n y, \ldots, T^{Dn} y\} < \rho.$$

By the choice of ρ, there is some $1 \leq j \leq M$ such that

$$y, T^n y, \ldots, T^{Dn} y \in T^{r_j} U.$$

Set $z := T^{-r_j} y$. Since powers of T commute, for every $0 \leq \ell \leq d$ we have

$$T^{\ell n} z = T^{-r_j} T^{\ell n} y \in U.$$

Consequently,

$$z \in U \cap T^{-n} U \cap \cdots \cap T^{-dn} U,$$

so Equation (17) implies that

$$n \in \mathcal{R}(P_1, \ldots, P_k; \varepsilon).$$

This proves Equation (18).

Taking the contrapositive gives the same conclusion with $a = 0$ in Equation (20). The general case follows by applying that conclusion to the point $T^a y$ in place of y. □

Remark 4.3 (The reduction coordinates). The nilsystem supplied by Theorem 4.1 is not merely an abstract compact dynamical system. The construction in [7, Chapters 3 and 6] uses nilmanifolds arising from unitriangular matrix groups. Its calculations introduce integer lattice corrections in the matrix coordinates, and these corrections realize the nested nearest-integer operations occurring in generalized polynomials. Thus the auxiliary coordinates retain the reduction data needed to evaluate the chosen polynomials $P_1, \ldots, P_k$, even when that data is invisible in the original quotient.

5 Polynomial covering number

Given a compact metric space (X, d), we define the Hausdorff distance d_H on the collection of closed subsets of X by

$$d_H(C_1, C_2) = \max\left(\sup_{c_1 \in C_1} \inf_{c_2 \in C_2} d(c_1, c_2), \sup_{c_2 \in C_2} \inf_{c_1 \in C_1} d(c_2, c_1)\right). \tag{22}$$

Given a collection $\mathcal{C}$ of closed subsets of X, we let $N_H(\mathcal{C}, \epsilon)$ denote the minimum numbers of ϵ-balls in the d_H metric needed to cover $\mathcal{C}$. The goal of this section is to prove the following result.

Theorem 5.1. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$, and $H \leq G$ a connected subgroup. The set $\{g_1 H g_2 \Gamma/\Gamma\}_{g_1, g_2 \in G}$ can be covered by at most $Q^{O(1)} n^{O(1)}$ balls of radius $1/n$ in the Hausdorff metric induced on the closed subsets of G/Γ by $d_{\mathcal{X}}$.

We first require some auxillary results. The next lemma is essentially a reformulation of Theorem 2.20. We recall from Remark 2.11 that once we fix a filtration and a Q-rational Mal'cev basis $\mathcal{X}$ of the ambient Nilmanifold G/Γ, all subnilmanifolds G'/Γ' with G' a Q-rational subgroup of G are endowed with a canonical filtration $G'_\bullet$ consisting of $Q^{O(1)}$-rational subgroups. Furthermore, we can pick a (non-canonical) $Q^{O(1)}$-rational Mal'cev basis $\mathcal{X}'$ adapted to G'/Γ' in which each element of $\mathcal{X}'$ is a $Q^{O(1)}$-rational combination of elements of $\mathcal{X}$. While the Mal'cev basis $\mathcal{X}'$ is not uniquely determined, it does not matter which one we pick.

Lemma 5.2. Let G/Γ be a nilmanifold endowed with a Mal'cev basis $\mathcal{X}$. Let $L \leq G$ be a closed connected Q-rational subgroup, let $\Lambda = L \cap \Gamma$, let $\mathcal{X}_L$ be a $Q^{O(1)}$-rational Mal'cev basis for L/Λ adapted to $L \cap G_\bullet$ and for which each element of $\mathcal{X}_L$ is a $Q^{O(1)}$-rational combination of elements of $\mathcal{X}$, and let $K \leq L$ be a closed connected rational subgroup. For every $0 < \rho \leq \frac{1}{2}$, one of the following alternatives holds:

(i) $K\Lambda/\Lambda$ is ρ-dense in L/Λ with respect to $d_{\mathcal{X}_L}$.

(ii) There is a proper connected rational subgroup $L_1 < L$ such that $K \leq L_1$ and L_1 is a $Q^{O(1)}\rho^{-O(1)}$-rational subgroup of G.

Proof. Assume that $K\Lambda/\Lambda$ is not ρ-dense in L/Λ with respect to $d_{\mathcal{X}_L}$. Let $\ell \in L$ be such that $B_r(\ell\Lambda) \cap K\Lambda/\Lambda = \emptyset$. Lemma 2.17(ii) tells us that any sequence which fails to visit $B_r(\ell\Lambda)$ is not $Q^{-O(1)}\rho^{m_L+1}$-equidistributed. In particular, any orbit that is contained in $K\Lambda/\Lambda$ will fail to be $Q^{-O(1)}\rho^{m_L+1}$-equidistributed. Now pick some $b \in K$ for which $\{b^n K\Lambda/\Lambda\}_{n=1}^{\infty}$ is dense in $K\Lambda/\Lambda$. We apply Theorem 2.20 with $\delta = Q^{-O(1)}\rho^{m_L+1}$ and some $N \in \mathbb{N}$ to obtain a horizontal character $\eta_N : L \to \mathbb{R}/\mathbb{Z}$ with $|\eta_N| \ll Q^{O(1)}\rho^{-(m_L+1)O(1)}$. Since there are only finitely many horizontal characters of height at most $Q^{O(1)}\rho^{-(m_L+1)O(1)}$, there is some such η for which $\eta_N = \eta$ for infinitely many N. For each such N, we see that $|\eta(b)| = |\eta_N(b)| \ll Q^{-O(1)}\rho^{-(m_L+1)O(1)}/N$, hence $\eta(b) = 0$. Since η annihilates Λ, and $b^n\Lambda$ is dense in $K\Lambda/\Lambda$, we see that η annihilates K. It follows that $L_1 := (\ker(\eta))^\circ$ contains K. We see that L_1 is a $|\eta|$-rational subgroup of L with respect to $\mathcal{X}_L$, hence it is a $Q^{O(1)} - \rho^{-O(1)}$-rational subgroup of G with respect to $\mathcal{X}$. □

Lemma 5.3. Suppose that G/Γ is a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. For every connected rational subgroup $K \leq G$ and every $0 < \rho \leq \frac{1}{2}$ there is a connected $Q^{O(1)}\rho^{-O(1)}$-rational subgroup $L \leq G$ for which $K \leq L$ and $d_H(K\Gamma/\Gamma, L\Gamma/\Gamma) \leq \rho$.

Proof. Let $L_0 = G$. If $K\Gamma/\Gamma$ is $Q^{-O(1)}\rho$-dense in G/Γ, then we are done. Otherwise, we induct on i as follows. We apply Lemma 5.2 to obtain a connected rational $Q^{O(1)}\rho^{-O(1)}$-rational subgroup L_{i+1} of L_i for which $K \leq L_{i+1}$. Since the dimension of L_{i+1} is at most the dimension of L_i minus 1, we see that we can only iterate this process at most $m-1$ times before we obtain some L_i for which $K(\Gamma \cap L_i)/(\Gamma \cap L_i)$ is $Q^{-O(1)}\rho$-dense in $L_i/(\Gamma \cap L_i)$ with respect to the metric $d_{\mathcal{X}_{L_i}}$. We see that L_i is a $Q^{O(1)}$-rational subgroup of G[1] so Lemma 2.15 tells us that $d_{\mathcal{X}_{L_i}}$ on $L_i/(L_i \cap \Gamma)$ is $Q^{O(1)}$-comparable to $d_{\mathcal{X}}$ on G/Γ, from which the desired result follows. □

Lemma 5.4. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. Let $F = \psi_{\mathcal{X}}^{-1}([0,1)^m)$ be a fundamental domain for G/Γ. Consider

$$\mathcal{A} := \{uK\Gamma/\Gamma \ : \ u \in F, K \leq G \text{ a connected rational subgroup}\}. \tag{23}$$

We have $N_H(\mathcal{A}, \frac{1}{n}) \leq Q^{O(1)}n^{O(1)}$.

Proof. Lemma 2.8 tells us that for any $u \in F$ and any closed $A, B \subseteq G/\Gamma$ we have $d_H(uA, uB) \leq Q^{O(1)}d_H(A, B)$. We also see that for any $u, v \in F$ and any closed $A \subseteq G$ we[2] have $d_H(uA, vA) \leq d_{\mathcal{X}}(u, v)$ due to the right invariance of $d_{\mathcal{X}}$. Recalling that $d_{\mathcal{X}}(x\Gamma, y\Gamma) \leq d_{\mathcal{X}}(x, y)$, we use the right invariance of $d_{\mathcal{X}}$ once again to see that $d_H(uA\Gamma, vA\Gamma) \leq d_H(uA, vA) \leq d_{\mathcal{X}}(u, v)$. We observe that the number of M-rational subgroups of G is at most $(2M+1)^{2m^2}$, so there are at most $Q^{O(1)}\rho^{-O(1)}$ subgroups of G that are $Q^{O(1)}n^{O(1)}$-rational. Now let $\{u_i\}_{i=1}^{Q^{O(1)}n^m}$ enumerate the points in F for which $\psi_{\mathcal{X}}(u_i)$ has coordinates of the form $a/(Q^{O(1)}n)$ for some $0 \leq a < nQ^{O(1)}$, and let $\{L_i\}_{i=1}^{Q^{O(1)}n^{O(1)}}$ enumerate the $Q^{O(1)}n^{O(1)}$-rational subnilmanifolds of G. Let $uK\Gamma/\Gamma \in \mathcal{A}$ be arbitrary. Lemma 5.3 gives us some L_i for which $K \leq L_i$ and $K\Gamma/\Gamma$ is $\frac{Q^{-O(1)}}{n}$-dense in $L_i\Gamma/\Gamma$. Lemma 2.7 tells us that $d_{\mathcal{X}}(u, u_j) \leq Q^{O(1)}|\psi_{\mathcal{X}}(u) - \psi_{\mathcal{X}}(u_j)|$, so we may pick a u_j for which $d_{\mathcal{X}}(u, u_j) \leq \frac{1}{n}$. We now see that

$$\begin{aligned} d_H(uK\Gamma/\Gamma, u_jL_i\Gamma/\Gamma) &\leq d_H(uK\Gamma/\Gamma, uL_i\Gamma/\Gamma) + d_H(uL_i\Gamma/\Gamma, u_jL_i\Gamma/\Gamma) \\ &\leq Q^{O(1)}d_H(K\Gamma/\Gamma, L_i\Gamma/\Gamma) + d_{\mathcal{X}}(u, u_j) \leq \frac{2}{n}. \end{aligned}$$

It follows that the number of balls of radius $\frac{2}{n}$ needed to cover $\mathcal{A}$ is $Q^{O(1)}n^{O(1)} = Q^{O(1)}\left(\frac{n}{2}\right)^{O(1)}$, from which the desired result follows. □

[1] And it can be endowed with a $Q^{O(1)}\rho^{-O(1)}$-rational Mal'cev basis, at the same time, as we have been doing up to now.

[2] While G is not compact, the following calculations take place within $\overline{F}^2$, which is compact.

Lemma 5.5. Let G/Γ be a nilmanifold endowed with a Q-rational Mal'cev basis $\mathcal{X}$. If $H \leq G$ is a connected subgroup and $g \in G$, then there exists a rational subgroup $K \leq G$ for which $\overline{Hg\Gamma/\Gamma} = gK\Gamma/\Gamma$.

Proof. updated proof needed. □

Proof of Theorem 5.1. We see that for any $\gamma \in \Gamma$ that $\gamma H \gamma^{-1}$ is a connected rational subgroup of G. Indeed, since $H \cap \Gamma$ is cocompact in H, we see that $\gamma H \gamma^{-1} \cap \Gamma = \gamma(H \cap \Gamma)\gamma^{-1}$ is cocompact in $\gamma H \gamma^{-1}$. Using Lemma 5.5 we obtain for each $g_2 \in G$ a rational subgroup $K_{g_2} \leq G$ for which $\overline{Hg_2\Gamma/\Gamma} = g_2 K_{g_2}\Gamma/\Gamma$. We now see that

$$\left\{\overline{g_1 H g_2 \Gamma/\Gamma}\right\}_{g_1,g_2\in G} = \{g_1 g_2 K_{g_2}\Gamma/\Gamma\}_{g_1,g_2\in G} = \left\{\{g_1g_2\}\left([g_1g_2]K_{g_2}[g_1g_2]^{-1}\right)[g_1g_2]\Gamma/\Gamma\right\}_{g_1,g_2\in G} \subseteq \mathcal{A}.$$

Using Lemma 5.4 we see that $N_H(\{g_1 H g_2\Gamma/\Gamma\}_{g_1,g_2\in G}, \frac{1}{n}) \leq N_H(\mathcal{A}, \frac{1}{n}) \leq Q^{O(1)} n^{O(1)}$. □

6 The proof

In this section, we will prove our main result.

Lemma 6.1. Let G/Γ be a nilmanifold endowed with a Mal'cev basis $\mathcal{X}$. Let G' be a connected subgroup of G. For each $g_1, g_2 \in G$ there exists $1 \leq i_0 \leq m$ for which $\psi_{\mathcal{X}}\left(\overline{g_1 G' g_2 \Gamma/\Gamma}\right)$ surjects onto the i_0^{th} coordinate.

Proof. Lemma 5.5 lets us reduce to the case in which $g_2 = \mathrm{Id}_G$ and G' is a rational subgroup. Since $g_1 G'\Gamma/\Gamma = \{g_1\}([g_1]G'[g_1]^{-1})\Gamma/\Gamma$, we may further reduce to the case in which $\psi(g_1) \in [0,1)^m$. Now pick some $\gamma \in G' \cap \Gamma \setminus \{\mathrm{Id}_G\}$ and let i_0 denote the index of the first nonzero coordinate of $\psi(\gamma)$. For $0 \leq t \leq 1$ we define $\gamma^t = \exp(t\log(\gamma))$ and we observe that $\gamma^t \in G'$ for all t since G' is rational. Now consider the path $g_1\gamma^t\Gamma$ in $g_1 G'\Gamma/\Gamma$. The polynomials P_i appearing in Equation (2) have the property that $P_i(0,\cdots,0,s_1,\cdots,s_m) = P_i(t_1,\cdots,t_m,0,\cdots,0) = 0$, so this tells us that for $1 \leq j \leq i_0$ the jth coordinate of $\psi_{\mathcal{X}}(g_1\gamma^t)$ is the sum of the jth coordinates of $\psi(g_1)$ and $\psi(\gamma^t)$. We also see that for $1 \leq j < i_0$ the jth coordinate of $\psi(\gamma^t)$ is 0, and that the i_0th coordinate is $t\psi(\gamma)_{i_0}$. It follows that the i_0th coordinate of $\psi(g_1\gamma^t)$ for $0 \leq t \leq 1$ contains an interval of length 1. Since $\psi(g_1) \in [0,1)^m$, we see that for $0 \leq t \leq 1$ and $1 \leq j < i_0$ we must have $\psi([g_1\gamma^t])_j = 0$. It follows that $\psi(\{g_1\gamma^t\})_{i_0} = \psi(g_1\gamma^t)_{i_0} \pmod 1$, from which the desired result follows. □

Theorem 6.2. If $S \subseteq \mathbb{N}$ is 2−large, then S is a set of pointwise recurrence for basic nilsystems.

Proof. Let us fix an arbitrary generalized polynomials $P_1,\cdots,P_v$ of degree at most D and let (Y,T) be the D-step nilsystem given to use by Theorem 4.2. Let $Y = G/\Gamma$ and $T(x) = gx$ for some $g \in G$. Let $\eta > 0$ be arbitrary, and consider the nilbohr set $A := \{n \mid \|P_i(n)\|_{\mathbb{R}/\mathbb{Z}} < \eta \ \forall\ 1 \leq i \leq v\}$. We want to show that $S \cap A \neq \emptyset$. We endow G/Γ with a Q-rational Mal'cev basis $\mathcal{X}$ adapted to the

lower central series filtration, and let $\rho > 0$ be given by Theorem 4.2 applied to $\{P_i\}_{i=1}^v$ and $\epsilon = \eta$. Let us assume for the sake of contradiction that $S \cap A = \emptyset$, so we may then assume without loss of generality that $S = \{n \in \mathbb{N} \mid \|P_i(n)\|_{\mathbb{R}/\mathbb{Z}} \geq \eta \text{ for some } 1 \leq i \leq v\}$.

Since this proof is longer than any of the proofs appearing in Section 2, we will not use big O notation for the sake of clarity. Instead, let us pick $C = C(m, s, d)$ to be large enough that all of the results of Sections **??** and 3 hold with C in place of $O(1)$. By enlarging $C = C(m, s, d)$ if necessary, we will further assume that if $G' \subseteq G$ is a closed connected rational subgroup and $\mathcal{X}'$ is a Mal'cev basis adapted to $G'/(G' \cap \Gamma)$ for which every element of $\mathcal{X}'$ is a M-rational combination of elements of $\mathcal{X}$ for some $M \geq Q$, we have that $\mathcal{X}'$ is a M^C-rational Mal'cev basis. Let $A = 3C + 1$, then let $B = B(A, m, d) > 0$ be such that whenever we apply Theorem 2.22, we have $M_0 \leq M \leq M_0^B$.

We partition $[0, 1)^m$ into boxes of side length M^{-1}, and we refer to these boxes as M^{-1}-cells.[3] Let $B_r(x\Gamma_i')$ denote the ball of radius r in the $d_{\mathcal{X}_i'}$ metric centered at $x\Gamma_i'$. Using Theorem 5.1 and Lemma 5.5, for each $1 \leq i \leq I_M$, let $\{g_{i,j}\}_{j=1}^{Q^C M^C}$ be such that for any $g_1', g_2' \in G$ there exists j for which we have $d_H(g_1' G_i' g_2' \Gamma/\Gamma, g_{i,j} G_i' \Gamma/\Gamma) < \frac{1}{M}$.

We will now use the probabilistic method to show that a random 2-coloring almost surely gives the desired result. Let us use the colors red and blue. We see that there are $2^{M^{-m}}$ ways to 2-color the M^{-1}-cells of $[0, 1)^m$. Lemma 6.1 tells us that for each $2 \leq i \leq I_M$ and $1 \leq j \leq Q^C M^C \leq M^{2C}$ there exists $1 \leq k_{i,j} \leq m$ for which $\psi_{\mathcal{X}}(g_{i,j} G_i' \Gamma/\Gamma)$ surjects onto the $k_{i,j}$ coordinate. Let $\{C_{i,j,k}\}_{k=1}^M$ be an enumeration of cells intersecting $\psi_{\mathcal{X}}(g_{i,j} G_i' \Gamma/\Gamma)$ for which we also have that the $k_{i,j}$ coordinate of $C_{i,j,k}$ is between $\frac{k-1}{M}$ and $\frac{k}{M}$. Now fix some i and j. We consider the restriction of our random 2-coloring to the cells $\{C_{i,j,3k-1}\}_{k=1}^{\frac{M}{3}}$. Hoeffding's inequality tells us that with probability at most $2\exp(-\frac{M}{24})$, we have that the number of red cells in the coloring is either bigger than $\frac{M}{4}$ or smaller than $\frac{M}{12}$. We consider all of these colorings to be bad colorings. Of the remaining colorings, we still consider them to be bad if either every red M^{-1}-cell has a neighboring blue M^{-1}-cell in the ambient grid, or every blue cell has a neighboring red M^{-1}-cell in the ambient grid. This happens with probability at most $2(1 - 2^{-3^m+1})^{\frac{M}{12}}$. It follows that for each i, j, the probability of a bad coloring is bounded above by $2\exp(-\frac{M}{24}) + 2(1 - 2\exp(-\frac{M}{24}))(1 - 2^{-3^m+1})^{\frac{M}{12}}$. The union bound over all possible pairs (i, j) shows us that the probability of a bad coloring is at most

$$M^{(3m^2+2C)B} \left(2\exp\left(-\frac{M}{24}\right) + 2\left(1 - 2\exp\left(-\frac{M}{24}\right)\right)\left(1 - 2^{-3^m+1}\right)^{\frac{M}{12}} \right). \tag{24}$$

We pick $M_0 > \rho^{-1} + Q$ to be so large that the quantity in (24) is smaller than 1 when $M = M_0$. We now pick a good coloring χ of M_0^{-1}-cells of $[0, 1)^m$, i.e., one of the random 2-colorings of the M_0^{-1}-cells of $[0, 1)^m$ that is not a bad coloring.

Let $R = M_0^B$ and let $g_0 := \exp\left(\frac{1}{R!}\log(g)\right)$, then observe that $g = g_0^{R!}$ Now we produce a 2-coloring χ of $\mathbb{N}$ by $\chi(n) = \chi'(g_0^n)$. It remains to show that the coloring χ does not have a monochromatic N-term S-A.P.

To this end, fix some $s_0 \in S$ and $a \in \mathbb{N}$. We apply Theorem 2.22 to G/Γ with $A = (3C+1)(m+1)$

[3] We mention that for our argument it is not important what convention we use for the boundaries of the M^{-1}-cells.

and M_0 as above and $p(n) := g_0^{a+s_0 n}$. Let $\{G_i'\}_{i=1}^{I_R}$ be an enumeration of the rational subgroups of G that possess a Mal'cev basis $\mathcal{X}_i'$ for G_i'/Γ_i' in which $\Gamma_i' = G_i' \cap \Gamma$ and each element of $\mathcal{X}_i'$ is a R-rational combination of elements of $\mathcal{X}$. Since $R \geq 3$, we observe that $I_R \leq (R^m(2R+1)^m)^m \leq R^{3m^2}$. We assume without loss of generality that $G_1' = \mathrm{id}_G$ and $G_{I_R} = G$. Consider the fundamental domain $F = \psi_{\mathcal{X}}^{-1}([0,1)^m)$ for G/Γ, and for $1 \leq i \leq I_R$ let $F_i := (G_i'\Gamma) \cap F$ be a fundamental domain for G_i/Γ_i.

Let $1 \leq i \leq I_R$ be such that when we apply Theorem 2.22 to $p(n) = g_0^{a+s_0 n}$, we obtain $p(n) = \epsilon g' \gamma$ with g' taking values in G_i'. We claim that for $N > DR(R!)\rho^{-1}$, we cannot have $i = 1$. Indeed, if we assume for the sake of contradiction that $i = 1$, then $G_1 = \{\mathrm{Id}_G\}$ and $g'(n) \equiv \mathrm{Id}_G$, hence $g_0^a g^{s_0 n} = g_0^{a+s_0 n \cdot R!} = p(n \cdot R!) = \epsilon(n \cdot R!) g'(n \cdot R!)\gamma(n \cdot R!) = \epsilon(n \cdot R!)\gamma(0)$ for all $0 \leq n \leq D$. To obtain our desired contradiction, we observe that since $s_0 \in S$, there exists $0 \leq n_1, n_2 \leq D$ for which we have

$$\begin{aligned}\rho &\leq d_{\mathcal{X}}(g^{s_0 n_1} g_0^a \Gamma, g^{s_0 n_2} g_0^a \Gamma) = d_{\mathcal{X}}(\epsilon(n_1 \cdot R!)\gamma(0)\Gamma, \epsilon(n_2 \cdot R!)\gamma(0)\Gamma) \\ &\leq d_{\mathcal{X}}(\epsilon(n_1 \cdot R!)\gamma(0), \epsilon(n_2 \cdot R!)\gamma(0)) = d_{\mathcal{X}}(\epsilon(n_1 \cdot R!), \epsilon(n_2 \cdot R!)) \leq DR(R!)/N < \rho.\end{aligned}$$

Now that we know $i > 1$, take $N'' = 2DR(R!)\rho^{-1}$, and let j be such that $d_H(g^a G_i'\Gamma/\Gamma, g_{i,j} G_i'\Gamma/\Gamma) < \frac{1}{R}$. Recall that $M \leq R$ is a period of γ, $s_0' = Ms_0$ and $N' = N''/M$, then note that $(g'(Mn))_{n=1}^{N'}$ is $M^{-(3C+1)(m+1)}$-equidistributed in G_i'/Γ_i' by our choice of A, so it is also $M_0^{-(3C+1)(m+1)}$-equidistributed in G_i'/Γ_i'. Lemma 2.17(ii) tells us that $(g'(n)\Gamma_i')_{n=1}^{N'}$ visits every ball in G_i'/Γ_i' of radius M_0^{-2C-1} in the $d_{\mathcal{X}_i'}$ metric. Lemma 2.15 tells us that when we view G_i'/Γ_i' as a subnilmanifold of G/Γ with the metric induced by $d_{\mathcal{X}}$, the sequence $(g(n)\Gamma_i')_{n=1}^{N'}$ visits every ball of radius M_0^{-C-1}. Let $q_i : G_i' \to (G_i'\Gamma)/\Gamma \cong G_i'/\Gamma_i'$ denote the quotient map $x \mapsto x\Gamma$. We see that the image of a ball of radius r centered at $x \in F_i \subseteq G_i'\Gamma$ under q_i contains the ball of radius r centered at $x\Gamma$, so $(\{g'(Mn)\})_{n=1}^{N'}$ visits every ball of radius M_0^{-C-1} in F_i, where we use the $d_{\mathcal{X}}$ metric on both spaces. Lemma 2.9 tells us that when we view F_i as a subset of G endowed with the $d_{\mathcal{X}}$ metric instead of the $d_{\mathcal{X}_i'}$ metric, we have that $\{\{g'(Mn)\}\}_{n=1}^{N'}$ visits every ball of radius M_0^{-C-1}. We see that for $x \in F$ and $r \in (0, \mathrm{Diam}(G/\Gamma))$, $\psi_{\mathcal{X}}^{-1}(B_{Q^C r}(\psi_{\mathcal{X}}(x))) \supseteq B_r(x)$, so $(\psi_{\mathcal{X}}(\{g'(Mn)\}))_{n=1}^{N'}$ visits every balls of radius M_0^{-1} (in the ℓ^∞ norm) in $[0,1)^m$ centered at a point on $\psi_{\mathcal{X}_i'}(F_i)$.

We let N be such that $RN''/N < \frac{1}{6M_0}$, as we then have that $d_{\mathcal{X}}(\epsilon(n), \epsilon(0)) < \frac{1}{6M_0}$ for $1 \leq n \leq N''$. It follows that $d_{\mathcal{X}}(\epsilon(Mn)g'(Mn), \epsilon(0)g'(Mn)) < \frac{1}{6M_0}$. Now let $1 \leq j \leq Q^C R^C$ be such that $d_H(\epsilon(0)G_i'\gamma(0)\Gamma/\Gamma, g_{i,j}G_i'\Gamma/\Gamma) < \frac{1}{6M_0}$. Since χ is a good coloring, let $1 \leq k_0, k_1 \leq \frac{M_0}{3}$ be such that $C_{i,j,3k_0-1}$ and $C_{i,j,3k_1-1}$ have distinct colors, but each have the same color as all of their neighboring M_0^{-1}-cells. Let $x_1 \in \psi_{\mathcal{X}}(\epsilon(0)G_i'\Gamma)$ be a point that is within $\frac{1}{6M_0}$ center of $C_{i,j,3k_1-1}$, and define x_2 similarly. Recall that $(\psi_{\mathcal{X}}(\{g'(Mn)\}))_{n=1}^{N'}$ visits $B_{M_0^{-1}}(x_1)$ and $B_{M_0^{-1}}(x_2)$, so let $1 \leq n_1, n_2 \leq N'$ be such that $\psi_{\mathcal{X}}(\{g'(Mn_1)\})$ and $\psi_{\mathcal{X}}(\{g'(Mn_2)\})$ are within $\frac{7}{6M_0}$ of the centers of $C_{i,j,3k_0-1}$ and $C_{i,j,3k_1-1}$ respectively. □

References

[1] R. Alweiss. 2-large sets are sets of Bohr recurrence. *arXiv preprint arXiv:2512.01997*, 2025.

[2] R. Alweiss. New Obstacles to Multiple Recurrence. *arXiv preprint arXiv:2511.21680*, 2025.

[3] T. C. Brown, R. L. Graham, and B. M. Landman. On the set of common differences in van der waerden's theorem on arithmetic progressions. *Canadian Mathematical Bulletin*, 42(1):25–36, 1999.

[4] L. J. Corwin and F. P. Greenleaf. *Representations of nilpotent Lie groups and their applications. Part 1: Basic theory and examples*, volume 18 of *Camb. Stud. Adv. Math.* Cambridge etc.: Cambridge University Press, 1990.

[5] B. Green and T. Tao. The quantitative behaviour of polynomial orbits on nilmanifolds. *Annals of Mathematics*, pages 465–540, 2012.

[6] B. Host and B. Kra. *Nilpotent structures in ergodic theory*, volume 236 of *Math. Surv. Monogr.* Providence, RI: American Mathematical Society (AMS), 2018.

[7] W. Huang, S. Shao, and X. Ye. Nil Bohr-sets and almost automorphy of higher order. *Memoirs of the American Mathematical Society*, 241(1143):v+83, 2016.

[8] J. Leng. Efficient Equidistribution of Nilsequences. Preprint, arXiv:2312.10772 [math.NT] (2023), 2023.

[9] A. I. Mal'cev. On a class of homogeneous spaces. *Izv. Akad. Nauk SSSR, Ser. Mat.*, 13:9–32, 1949.