A gated density obstruction for the unchanged Alweiss example

Produced by ChatGPT

September 15, 2026

Abstract

Let S be Alweiss's original set of allowed differences with the fixed parameters $a = 10^{-2}$ and $b = 10^{-10}$. We construct a set of integers of uniform density at least

$$\delta_* = \frac{9075692815599}{9084317000000} = 0.999050651314677\ldots$$

which contains no 10^{18}-term arithmetic progression with difference in S. Consequently S is not δ-density large for any $0 < \delta \le \delta_*$, including $\delta = 999/1000$. All smaller densities are attained exactly by thinning. The parameters and the sequence defining S are unchanged. The new device is a conditional, or gated, deletion: a small forbidden band for a root-scale observable is imposed only while a second observable lies in a narrow trigger band. Joint well-distribution makes the cost of this deletion a product. An elementary finite-window differentiation formula supplies the unusually small drift needed to keep the trigger active. All estimates, the uniform-density calculation, and the finite progression-length bound are proved below. No assertion for every $\delta < 1$ is made.

1 The original set and the precise conclusion

Throughout, $\mathbb{N} = \{1, 2, \ldots\}$ and $\|x\| = \operatorname{dist}(x, \mathbb{Z})$. We use the fixed numbers

$$a = \frac{1}{100}, \qquad b = \frac{1}{10^{10}}. \tag{1.1}$$

Let $(\alpha_j)_{j\ge1}$ be the real sequence in the original construction. The proof below uses only

$$\sum_{j\ge1} |\alpha_j| < \infty, \qquad 1, \alpha_1, \ldots, \alpha_m \text{ are } \mathbb{Q}\text{-linearly independent for every } m. \tag{1.2}$$

Alweiss chooses the sequence sufficiently rapidly decreasing to obtain the additional nil-Bohr recurrence property; no additional decrease is required here.

For clarity, we restate the allowed differences. For $d \in S$ there is a distinguished index $i = i(d)$ and real representatives σ_j of $d\alpha_j \pmod 1$ such that

$$\sigma_i = -a + \eta, \qquad |\eta| < 2b, \tag{1.3}$$

$$|\sigma_j| < 2b \qquad (j \ne i), \tag{1.4}$$

$$a - 2b < \sum_{j\ne i} |\sigma_j| < a + 2b. \tag{1.5}$$

These are exactly the conditions obtained by taking the inverse image of Definition 2.1 of [1], as in its Section 2.3. We may use weak inequalities in estimates. In particular,

$$\sum_j |\sigma_j| \le 2a + 4b, \qquad \sum_{j\neq i} \sigma_j^2 \le 2b(a+2b). \tag{1.6}$$

The second inequality follows by multiplying the bound on the largest tail coordinate by the bound on the sum of their absolute values. The lower bound in (1.5) is part of the original definition but is not needed for our obstruction.

Definition 1.1. For $A \subseteq \mathbb{N}$, its natural upper density is

$$\overline{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N}.$$

It has *uniform density* γ if

$$\lim_{N\to\infty} \sup_{M\ge 0} \left| \frac{|A \cap \{M+1,\dots,M+N\}|}{N} - \gamma \right| = 0,$$

where M, N are integers. A set $D \subseteq \mathbb{N}$ is *δ-density large* if every set A with $\overline{d}(A) = \delta$ contains arbitrarily long arithmetic progressions with common difference in D.

Theorem 1.2. *Assume (1.2), and let S be defined by (1.3)–(1.5). Set*

$$K = 10^{18}, \qquad \delta_* = \frac{9075692815599}{9084317000000}.$$

There is a set $A_* \subseteq \mathbb{N}$ of uniform density $\gamma \ge \delta_*$ containing no progression

$$n, n+d, \dots, n+(K-1)d \qquad (d \in S).$$

For every $0 < \delta \le \delta_*$ there is a subset of A_* of uniform density exactly δ. In particular, the unchanged original Alweiss example is not δ-density large for every δ in this range.

Remark 1.3. The displayed δ_* is a guaranteed lower bound for the density of the explicit set, not a claim that its density equals that fraction. It comes from a union bound. The progression length is deliberately very generous and is not asserted to be close to optimal.

The construction in advance

We will use the cosine observable

$$F(n) = \sum_{j\ge 1} (\cos(2\pi n\alpha_j) - 1)$$

and a smoothed-ramp observable at a different scale,

$$G(n) = \sum_{j\ge 1} \psi(n\alpha_j/100).$$

The function ψ is defined in Section 3. For $P > 0$, write $x \bmod P$ for the representative in $[0, P)$. The explicit set is

$$A_* = \left\{ n \in \mathbb{N} : \begin{array}{l} F(n) \bmod 1000 \in [0.126, 1000), \\ G(n) \bmod 3000 \in [0.181, 3000), \\ \text{not both } F(n) \bmod 1.93 \in [0, 0.0886) \text{ and } G(n) \bmod 0.706035 \in [0, 0.011735) \end{array} \right\}. \tag{1.7}$$

Every terminating decimal in this note denotes the exact rational number with that decimal expansion.

The first two conditions are coarse barriers. They are imposed everywhere but are very inexpensive in density. The last condition is a fine barrier for G, imposed *only when F activates it*. Along a hypothetical allowed progression, the first coarse barrier makes the tail of F almost stationary. The distinguished cosine then enters the trigger band, and remains there along a short subprogression with index spacing 100. On that subprogression the root-scale observable G has precisely the original allowed step. Its coarse barrier makes its own tail slow enough for a linear segment of the ramp to cross the fine forbidden band. This is impossible inside A_*. Joint well-distribution makes the last deletion cost

$$\frac{0.0886}{1.93} \cdot \frac{0.011735}{0.706035},$$

rather than the sum of the two proportions.

2 A finite-window differentiation estimate

The following estimate is the analytic improvement needed for the gate. It uses the fact that the tail of a cosine sum has very small *frequencies*, not just small second differences. We include an elementary proof so that no theorem about entire functions is required.

Lemma 2.1 (Finite-window slope bound)**.** *Let $\omega, D > 0$ and*

$$f(t) = \sum_j c_j(\cos(\theta_j + \nu_j t) - \cos\theta_j), \qquad |\nu_j| \le \omega, \qquad \sum_j |c_j\nu_j| \le D, \tag{2.1}$$

where c_j, θ_j, ν_j are real. The sum may be finite or countably infinite. For an integer $N \ge 1$, put

$$R_N = \frac{(2N-1)\pi}{2\omega}.$$

If $\operatorname{osc}_{[t_0-R_N, t_0+R_N]} f \le M$, then

$$|f'(t_0)| \le \frac{\omega M}{2} + \frac{5D}{N}. \tag{2.2}$$

The same conclusion holds after adding any real constant to f.

Proof. The series for f converges locally uniformly, since the absolute value of its jth term is at most $|c_j\nu_j t|$. Its derivative series converges uniformly on $\mathbb{R}$ by the bound on $\sum_j |c_j\nu_j|$. Thus f is continuously differentiable and globally D-Lipschitz.

We first record the elementary triangle-wave Fourier identity

$$x = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\sin((2k+1)x) \qquad \left(-\frac{\pi}{2} \le x \le \frac{\pi}{2}\right). \tag{2.3}$$

One way to verify it is to extend x on $[-\pi/2, \pi/2]$ to the continuous 2π-periodic triangle wave. Its sine coefficients, computed on its linear pieces, vanish at even indices and equal $4(-1)^k/(\pi(2k+1)^2)$ at odd indices. The series is absolutely convergent, so it agrees everywhere with the continuous triangle wave. In particular, the endpoint identities hold.

Substituting $x = \nu\pi/(2\omega)$ in (2.3) gives

$$\nu = \frac{8\omega}{\pi^2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{(2k+1)\pi\nu}{2\omega}\right) \qquad (|\nu| \le \omega).$$

Using the difference of two cosines, this proves the differentiation formula

$$f'(t) = \frac{4\omega}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} (f(t+h_k) - f(t-h_k)), \qquad h_k = \frac{(2k+1)\pi}{2\omega}, \tag{2.4}$$

for a finite sum in (2.1). Constants cancel from the formula.

We need a tail bound which depends only on the global Lipschitz constant, not on the number or amplitudes of the summands. Put $c = \pi/(2\omega)$ and

$$g_k = f(t + (2k+1)c) - f(t - (2k+1)c), \qquad a_k = \frac{g_k}{(2k+1)^2}.$$

The Lipschitz bound implies

$$|g_k| \le 2Dc(2k+1), \qquad |g_{k+1} - g_k| \le 4Dc.$$

Consequently $a_k \to 0$ and

$$\begin{aligned} |a_{k+1} - a_k| &\le \frac{4Dc}{(2k+1)^2} + 2Dc(2k+3)\left(\frac{1}{(2k+1)^2} - \frac{1}{(2k+3)^2}\right) \\ &\le \frac{12Dc}{(2k+1)^2}. \end{aligned}$$

For $N \ge 1$,

$$\sum_{k=N}^{\infty} \frac{1}{(2k+1)^2} \le \frac{1}{(2N+1)^2} + \int_N^{\infty} \frac{dx}{(2x+1)^2} \le \frac{1}{2N+1}.$$

Summation by parts, using the bound 1 on partial sums of $(-1)^k$, now gives

$$\left| \sum_{k=N}^{\infty} (-1)^k a_k \right| \le |a_N| + \sum_{k=N}^{\infty} |a_{k+1} - a_k| \le \frac{14Dc}{2N+1}.$$

After multiplication by $4\omega/\pi^2$, the corresponding tail of (2.4) is bounded by

$$\frac{28D}{\pi(2N+1)} < \frac{5D}{N}. \tag{2.5}$$

This also proves convergence of the series for any globally Lipschitz f. For the infinite sum in (2.1), apply the formula to finite partial sums, pass to the limit in any fixed truncation, and then use the common tail bound (2.5). This justifies (2.4) in that case as well.

For $0 \le k < N$, both $t_0 \pm h_k$ lie in the assumed window. Hence

$$|f(t_0 + h_k) - f(t_0 - h_k)| \le M.$$

The absolute value of the first N terms in (2.4) is therefore at most

$$\frac{4\omega M}{\pi^2} \sum_{k=0}^{N-1} \frac{1}{(2k+1)^2} \le \frac{4\omega M}{\pi^2} \cdot \frac{\pi^2}{8} = \frac{\omega M}{2}.$$

Here the identity for the sum of the reciprocal odd squares also follows by putting $x = \pi/2$ in (2.3). Adding (2.5) proves the lemma. □

Remark 2.2. The frequency bound is essential. A bound on $\Delta^2 f$ alone would give a much weaker drift estimate on a long bounded segment. In our application $\omega = 4\pi b$ is about 10^{-9}, while the total first-derivative bound D is of order 10^{-2}. The long window makes the second term in (2.2) negligible.

3 The ramp and a discrete drift estimate

Define the continuous one-periodic function

$$\chi(t) = \begin{cases} t, & 0 \le t \le 9/10, \\ 9(1-t), & 9/10 \le t \le 1, \end{cases}$$

and put

$$\varepsilon = \frac{1}{50}, \qquad \psi(t) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \chi(t-u)\,du - \frac{1}{20}. \tag{3.1}$$

The same ramp is useful in the earlier two-scale argument, but its use here is different: its fine forbidden band will be conditional on a separate cosine gate.

Lemma 3.1. *The function ψ is one-periodic and continuously differentiable, with*

$$\psi(0) = 0, \qquad \mathrm{Lip}(\psi) \le 9, \qquad \mathrm{osc}_{\mathbb{T}}\,\psi \le 0.9, \tag{3.2}$$

$$\psi(t) = t - 0.05 \qquad (0.02 \le t \le 0.88), \tag{3.3}$$

$$\mathrm{Lip}(\psi') \le 250. \tag{3.4}$$

In particular, for every $t, v \in \mathbb{R}$,

$$|\psi(t) - 2\psi(t+v) + \psi(t+2v)| \le 250v^2. \tag{3.5}$$

Proof. At zero the average in (3.1) is

$$\frac{1}{2\varepsilon}\left(\frac{\varepsilon^2}{2} + \frac{9\varepsilon^2}{2}\right) = \frac{5}{2}\varepsilon = \frac{1}{20}.$$

This proves $\psi(0) = 0$. Averaging preserves the Lipschitz bound 9 and cannot increase oscillation. On $[\varepsilon, 9/10 - \varepsilon]$ the whole averaging interval lies in the linear part of χ, so its average is t.

Differentiation gives

$$\psi'(t) = \frac{\chi(t+\varepsilon) - \chi(t-\varepsilon)}{2\varepsilon}.$$

This is continuous and piecewise linear. At points where its derivative exists, the numerator has derivative equal to the difference of two numbers in $\{1, -9\}$, hence of absolute value at most 10. Thus $|\psi''| \le 10/(2\varepsilon) = 250$ almost everywhere, and the piecewise-linear description proves the global Lipschitz bound. Integrating the difference of two first derivatives gives (3.5), including negative v. □

Lemma 3.2 (Discrete drift)**.** *Suppose a real sequence $z_0, \dots, z_L$ has oscillation at most M and*

$$|z_{t+2} - 2z_{t+1} + z_t| \le B \qquad (0 \le t \le L-2).$$

For integers $q \ge 1$ and $t \ge 0$ with $t + q \le L$,

$$|z_{t+1} - z_t| \le \frac{M}{q} + \frac{B(q-1)}{2}. \tag{3.6}$$

Proof. Summing first differences and then second differences gives the exact identity

$$z_{t+q} - z_t = q(z_{t+1} - z_t) + \sum_{r=0}^{q-2}(q-1-r)(z_{t+r+2} - 2z_{t+r+1} + z_{t+r}).$$

The left side has absolute value at most M, and the sum of the nonnegative coefficients on the right is $q(q-1)/2$. Division by q proves the result. □

4 The observables and exact parameters

Set

$$F(n) = \sum_{j\geq 1} (\cos(2\pi n\alpha_j) - 1), \qquad G(n) = \sum_{j\geq 1} \psi(n\alpha_j/100). \tag{4.1}$$

Both series converge absolutely for each integer n: their absolute values term by term are bounded by $2\pi|n\alpha_j|$ and $9|n\alpha_j|/100$, respectively.

For convenient reference, the parameters in (1.7) are

$$\begin{aligned} P_0 &= 1000, & h_0 &= 0.126, \\ P_1 &= 3000, & h_1 &= 0.181, \\ p_0 &= 1.93, & w_0 &= 0.0886, \\ p_1 &= 0.706035, & h_f &= 0.011735. \end{aligned} \tag{4.2}$$

We also use

$$N_0 = 50{,}000{,}000, \qquad e_0 = 0.00000064, \qquad q_1 = 3{,}500{,}000, \qquad e_1 = 0.001734. \tag{4.3}$$

Here N_0 is the truncation parameter in Lemma 2.1, not a progression cutoff. The gate has an interior core

$$[0.0128, 0.0758] \subset [0, w_0), \tag{4.4}$$

of width 0.063, leaving a margin 0.0128 on each side.

5 The cosine gate remains active on a short subprogression

Suppose, toward a contradiction, that

$$n, n+d, \ldots, n+(K-1)d \in A_*, \qquad d \in S, \qquad K = 10^{18}. \tag{5.1}$$

Fix the distinguished index i and representatives σ_j from (1.3)–(1.5). For integer t put

$$y_t = F(n+td), \qquad u_t = \cos(2\pi(n\alpha_i + t\sigma_i)) - 1, \qquad z_t = y_t - u_t.$$

When using real t below, extend the tail by the absolutely convergent difference series

$$z(t) = z_0 + \sum_{j\neq i} \left[\cos(2\pi(n\alpha_j + t\sigma_j)) - \cos(2\pi n\alpha_j)\right]. \tag{5.2}$$

At integer t, this agrees with z_t by periodicity. It is globally D_0-Lipschitz, where

$$D_0 = 2\pi(a + 2b). \tag{5.3}$$

5.1 Coarse confinement

By (1.6),

$$|y_{t+1} - y_t| \leq 2\pi(2a + 4b) < 0.126 = h_0. \tag{5.4}$$

Every y_t is in the union of retained intervals

$$\bigcup_{m\in\mathbb{Z}} [mP_0 + h_0, (m+1)P_0).$$

Two different components of this union are separated by a gap of length at least h_0. Thus consecutive values cannot pass between components. All K samples belong to one retained interval, and their oscillation is at most $P_0 - h_0$.

Since $\operatorname{osc} u_t \le 2$, the integer samples of z have oscillation at most $P_0 - h_0 + 2$. Every point of $[0, K-1]$ is within $1/2$ of an integer in that interval. The Lipschitz bound therefore gives

$$\operatorname{osc}_{[0,K-1]} z \le P_0 - h_0 + 2 + D_0 < 1002. \tag{5.5}$$

5.2 Frequency control gives a small tail slope

The frequencies in (5.2) have absolute value at most

$$\omega = 4\pi b,$$

and their absolute sum is at most D_0. Lemma 2.1, with $M = 1002$ and $N = N_0$, has sampling radius

$$R_{N_0} = \frac{(2N_0 - 1)\pi}{2\omega} = \frac{99{,}999{,}999}{8b} < 1.25 \times 10^{17}. \tag{5.6}$$

Set $t_* = K/2$. Every radius-R_{N_0} window centered in $[t_*, t_* + 20{,}000]$ is contained in $[0, K-1]$. Consequently

$$|z'(t)| \le 2\pi b(1002) + \frac{5D_0}{N_0} < e_0 \quad (t_* \le t \le t_* + 20{,}000). \tag{5.7}$$

For a rational check of the final inequality, use $\pi < 22/7$. Its left side is then less than 0.000000636115, whereas $e_0 = 0.00000064$.

5.3 Entering the interior of a trigger band

Consider the 100 values $y_{t_*}, \dots, y_{t_*+99}$. Put $\theta = n\alpha_i + t_*\sigma_i$. The ideal cosine values

$$\cos(2\pi(\theta - r/100)) - 1 \qquad (0 \le r \le 99)$$

have oscillation at least $2\cos(\pi/100)$. Indeed, the translated $1/100$ grid has a point within $1/200$ of zero and another within $1/200$ of $1/2$.

The actual distinguished phase differs at index r by $r\eta$. Its cosine differs from the ideal value by at most $2\pi \cdot 99 \cdot 2b$. The tail has oscillation at most $99e_0$ on this block. Therefore

$$\max_{0\le r\le 99} y_{t_*+r} - \min_{0\le r\le 99} y_{t_*+r} \ge 2\cos(\pi/100) - 4\pi \cdot 99 \cdot 2b - 99e_0 > 1.998. \tag{5.8}$$

The elementary inequality $\cos x \ge 1 - x^2/2$, together with $\pi < 22/7$, already gives this numerical bound. Furthermore,

$$|y_{t_*+r+1} - y_{t_*+r}| \le 2\pi(a + 2b) + e_0 < 0.063. \tag{5.9}$$

We use a simple fact about real walks. If a finite real sequence has successive increments of absolute value less than c, and its range has length greater than $p + c$, it meets one of the intervals $[mp + u, mp + u + c]$, for any fixed u. To see this, choose an entire such interval inside the range, and follow the walk between indices attaining the minimum and maximum. The first crossing of the left endpoint cannot jump over the right endpoint. The same reasoning applies if the minimum occurs after the maximum, by reversing that portion of the walk.

Apply this with

$$c = 0.063, \qquad p = p_0 = 1.93, \qquad u = 0.0128.$$

Since $p + c = 1.993 < 1.998$, there are $0 \le r_0 \le 99$ and $m \in \mathbb{Z}$ such that, with $r = t_* + r_0$,

$$y_r \in [mp_0 + 0.0128, mp_0 + 0.0758]. \tag{5.10}$$

5.4 Keeping the gate active for 200 samples

For $0 \le q \le 199$ we have

$$100q\sigma_i = -q + 100q\eta.$$

The integer term does not change the cosine. Using (5.7),

$$\begin{aligned} |y_{r+100q} - y_r| &\le 2\pi \cdot 100q \cdot 2b + 100qe_0 \\ &\le 19900(e_0 + 4\pi b) < 0.0128. \end{aligned} \tag{5.11}$$

All times involved lie in $[t_*, t_* + 20{,}000]$. The last bound is less than 0.012762, even using $\pi < 22/7$.

Combining (5.10) and (5.11) yields

$$F(n + (r + 100q)d) \bmod p_0 \in (0, w_0) \qquad (0 \le q \le 199). \tag{5.12}$$

Thus the fine condition on G in (1.7) is compulsory at every one of these 200 points.

6 The root-scale ramp contradicts the active gate

For $0 \le q \le q_1 + 199$, put

$$Y_q = G(n + (r + 100q)d).$$

Every one of these indices is in the original progression: the largest is at most

$$t_* + 99 + 100(q_1 + 199) < K - 1.$$

By (4.1), the jth phase of Y_q is

$$\frac{(n + rd)\alpha_j}{100} + qd\alpha_j.$$

This is the reason for the root scale: the step in q is exactly $d\alpha_j$, with representatives σ_j satisfying the original allowed-step conditions.

Write

$$U_q = \psi(\vartheta_i + q\sigma_i), \qquad Z_q = Y_q - U_q, \qquad \vartheta_j = \frac{(n + rd)\alpha_j}{100}.$$

The global coarse restriction on G, and (1.6), give

$$|Y_{q+1} - Y_q| \le 9(2a + 4b) < 0.181 = h_1.$$

As before, all the Y_q lie in one coarse retained interval, now of length $P_1 - h_1$. Since $\operatorname{osc} U_q \le 0.9$,

$$\operatorname{osc}_{0 \le q \le q_1 + 199} Z_q \le P_1 - h_1 + 0.9 = 3000.719. \tag{6.1}$$

Lemma 3.1 and (1.6) also give

$$|Z_{q+2} - 2Z_{q+1} + Z_q| \le 250 \sum_{j \ne i} \sigma_j^2 \le 250 \cdot 2b(a + 2b) = 5.0000001 \times 10^{-10}. \tag{6.2}$$

All summations and differences are justified by absolute convergence, or directly by the summable majorants in these inequalities.

Apply Lemma 3.2 with step length $q_1 = 3{,}500{,}000$. For every $0 \le q \le 199$ the required endpoint is available, and

$$|Z_{q+1} - Z_q| \le \frac{3000.719}{3{,}500{,}000} + \frac{(5.0000001 \times 10^{-10})(3{,}499{,}999)}{2} < 0.001733 < e_1 = 0.001734. \tag{6.3}$$

The bound also covers $q = 199$ because $q_1 \ge 1$ and the sequence extends to $q_1 + 199$.

Now choose $r_1 \in \{0, \dots, 99\}$ so that a representative of $\vartheta_i - r_1/100$ lies in $[0.865, 0.875)$. Such a choice exists because these phases form a translated $1/100$ grid. For $0 \le j \le 84$, representatives of the actual phases at $q = r_1 + j$ lie between

$$0.025 - 366b \quad \text{and} \quad 0.875 + 366b.$$

This interval lies strictly inside $[0.02, 0.88]$. Thus the exact linear formula (3.3) applies to all 85 samples, giving

$$U_{r_1+j+1} - U_{r_1+j} = \sigma_i \qquad (0 \le j < 84).$$

Together with (6.3), this implies that the corresponding Y values decrease strictly and satisfy

$$|Y_{r_1+j+1} - Y_{r_1+j}| \le a + 2b + e_1 = 0.0117340002 < 0.011735 = h_f. \tag{6.4}$$

Their total decrease is at least

$$Y_{r_1} - Y_{r_1+84} \ge 84(a - 2b - e_1) = 0.6943439832 > 0.6943 = p_1 - h_f. \tag{6.5}$$

All these indices are at most 183, and hence lie in the 200-sample region where (5.12) holds. Because the original progression is in A_*, the conditional fine restriction therefore gives

$$Y_q \bmod p_1 \in [h_f, p_1) \qquad (0 \le q \le 199).$$

In particular, each of the 85 samples in (6.4) belongs to a retained interval of length $p_1 - h_f = 0.6943$. The successive increments are smaller than the separating gap h_f, so these 85 samples must all lie in one such interval. This contradicts (6.5).

We have proved that the explicit set (1.7) contains no K-term progression with difference in S. It remains to prove its density.

7 Joint well-distribution of the two infinite sums

We give the full equidistribution argument. In particular, it is not legitimate to assume that the tails of the original infinite sums are uniformly small in n. Only tails of fixed *differences* have the necessary uniform bound.

Lemma 7.1 (Infinite-sum well-distribution)**.** *Let (β_j) satisfy (1.2) with β in place of α. Let $\varphi_1, \dots, \varphi_s : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be Lipschitz, with $\varphi_\ell(0) = 0$. Suppose every nonzero real linear combination of the φ_ℓ is nonconstant. Define*

$$H_\ell(n) = \sum_{j \ge 1} \varphi_\ell(n\beta_j).$$

For every choice $L_1, \dots, L_s > 0$, the sequence

$$(H_1(n) \bmod L_1, \dots, H_s(n) \bmod L_s)$$

is well-distributed in $\prod_{\ell=1}^{s} \mathbb{R}/L_\ell\mathbb{Z}$, uniformly over translated intervals of integers. In particular, inverse images of finite unions of rectangles have uniform density equal to their normalized Lebesgue measure whenever their boundaries have measure zero.

Proof. It suffices to prove the uniform Weyl criterion. Fix a nonzero integer vector $(k_1, \dots, k_s)$ and put

$$\varphi = 2\pi \sum_{\ell=1}^{s} \frac{k_\ell}{L_\ell} \varphi_\ell, \qquad H(n) = \sum_{j \geq 1} \varphi(n\beta_j), \qquad u(n) = \exp(iH(n)).$$

The function φ is nonconstant and Lipschitz, with $\varphi(0) = 0$.

Uniform limits of fixed correlations. For every fixed integer h,

$$\sup_{n \in \mathbb{Z}} \left| \sum_{j > m} \big(\varphi((n+h)\beta_j) - \varphi(n\beta_j)\big) \right| \leq \mathrm{Lip}(\varphi)|h| \sum_{j > m} |\beta_j| \longrightarrow 0. \tag{7.1}$$

Define the continuous function

$$B(v) = \int_0^1 \exp\big(i(\varphi(x+v) - \varphi(x))\big)\, dx.$$

For finite m, torus equidistribution and rational independence give, uniformly in $M \geq 0$,

$$\frac{1}{N} \sum_{n=M+1}^{M+N} \exp\left(i \sum_{j=1}^{m} [\varphi((n+h)\beta_j) - \varphi(n\beta_j)] \right) \longrightarrow \prod_{j=1}^{m} B(h\beta_j).$$

For completeness, uniformity follows by approximating the continuous integrand by a trigonometric polynomial. For every nonzero integer frequency vector, its average is a geometric sum with ratio different from one; its bound after division by N is independent of M. The constant Fourier coefficient is the integral, and then the approximation error is sent to zero.

Using (7.1) and $|e^{ix} - e^{iy}| \leq |x - y|$ shows that the products have a limit $C(h)$ and that

$$\lim_{N \to \infty} \sup_{M \geq 0} \left| \frac{1}{N} \sum_{n=M+1}^{M+N} u(n+h)\overline{u(n)} - C(h) \right| = 0. \tag{7.2}$$

Because $|B(v)| \leq 1$, for every fixed m we also have

$$|C(h)| \leq \prod_{j=1}^{m} |B(h\beta_j)|. \tag{7.3}$$

The averaged absolute correlation is strictly contracting. We claim

$$q := \int_0^1 |B(v)|\, dv < 1. \tag{7.4}$$

If $q = 1$, continuity and $|B| \leq 1$ imply $|B(v)| = 1$ for every v. Put $f(x) = e^{i\varphi(x)}$. Equality in Cauchy–Schwarz then says that for each v the translate $f(x+v)$ is a scalar multiple of $f(x)$. Continuity makes the identity pointwise. The scalars form a continuous character of $\mathbb{R}/\mathbb{Z}$, so

$$f(x) = f(0)e^{2\pi i m x}$$

for some $m \in \mathbb{Z}$. To justify the usual character description directly, lift its values continuously along $[0, 1]$ and use the homomorphism law; the lifted phase is additive and hence linear, and periodicity makes its slope an integer multiple of 2π.

Now $\varphi(x) - \varphi(0) - 2\pi m x$ is continuous and takes values in $2\pi\mathbb{Z}$, and it is zero at $x = 0$. It is therefore identically zero. Since φ is periodic, $m = 0$, and φ is constant, a contradiction. This proves (7.4).

A second application of finite-dimensional torus equidistribution, now in the variable h, gives from (7.3)

$$\limsup_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} |C(h)| \le q^m.$$

Since this holds for every m, letting $m \to \infty$ yields

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} |C(h)| = 0. \tag{7.5}$$

Uniform van der Corput. For completeness, the required inequality follows by replacing a sum of $u(n)$ by the average of its first H shifts, with an $O(H)$ boundary error, and then applying Cauchy–Schwarz. Expanding the square of the shifted average gives a diagonal contribution $1/H$ and correlation terms of weights $2(H-h)/H^2$. By (7.2), after $N \to \infty$ this gives

$$\limsup_{N\to\infty} \sup_{M\ge 0} \left| \frac{1}{N} \sum_{n=M+1}^{M+N} u(n) \right|^2 \le \frac{1}{H} + \frac{2}{H} \sum_{h=1}^{H-1} |C(h)|.$$

The uniform correlation limits justify taking the supremum over M here. Sending $H \to \infty$ and using (7.5) proves that every nontrivial character has uniformly vanishing averages.

Finite Fourier approximation gives the assertion for continuous functions on the target torus. Continuous upper and lower approximations to indicators of rectangles, followed by finite unions, give the stated counting assertion. Their boundary sets have Lebesgue measure zero, so the errors of these approximations tend to zero. This proves the lemma. □

7.1 Application to the present observables

Take

$$\beta_j = \alpha_j/100, \qquad \varphi_0(x) = \cos(200\pi x) - 1, \qquad \varphi_1(x) = \psi(x).$$

The sequence (β_j) is summable and retains rational independence with 1. The associated infinite sums are exactly $F(n)$ and $G(n)$.

The two real functions have no nonzero constant linear combination. Indeed, on $(0.02, 0.88)$ the derivative of such a combination would be

$$-200\pi c_0 \sin(200\pi x) + c_1.$$

If this vanishes throughout the interval, first $c_0 = 0$ and then $c_1 = 0$. Thus Lemma 7.1 applies to (F, G) with arbitrary target periods.

All conditions on F in (1.7) have common period $L_0 = 193000$, and all conditions on G have common period $L_1 = 141207000$. For example,

$$\frac{L_0}{1000} = 193, \quad \frac{L_0}{1.93} = 100000, \qquad \frac{L_1}{3000} = 47069, \quad \frac{L_1}{0.706035} = 200000000.$$

Thus (1.7) is the inverse image of a finite union of rectangles, with boundary measure zero, on

$$(\mathbb{R}/L_0\mathbb{Z}) \times (\mathbb{R}/L_1\mathbb{Z}).$$

It has a uniform density γ.

The excluded points are in the union of three events: the coarse F band, the coarse G band, and the conjunction of the trigger F band and the fine G band. Joint well-distribution makes the density of the last event the product of its two marginal proportions. Therefore

$$\begin{aligned} 1-\gamma &\le \frac{0.126}{1000}+\frac{0.181}{3000}+\frac{0.0886}{1.93}\frac{0.011735}{0.706035} \\ &= \frac{8624184401}{9084317000000} = 0.000949348685322\ldots. \end{aligned} \tag{7.6}$$

It follows that

$$\gamma \ge 1-\frac{8624184401}{9084317000000} = \frac{9075692815599}{9084317000000} = \delta_*. \tag{7.7}$$

We have not subtracted overlaps in (7.6); doing so is unnecessary for the stated bound.

8 Thinning to an exact prescribed density

Lemma 8.1. *If $B=\{b_1<b_2<\cdots\}$ has uniform density $\gamma>0$, then for every $0<\delta\le\gamma$ it has a subset of uniform density exactly δ.*

Proof. Put $\theta=\delta/\gamma$ and retain b_j precisely when

$$\lfloor \theta j\rfloor > \lfloor \theta(j-1)\rfloor.$$

Call the resulting set A. With $B(N)=|B\cap[1,N]|$, we have

$$|A\cap[1,N]| = \lfloor \theta B(N)\rfloor.$$

Consequently, uniformly in nonnegative integers M,

$$\begin{aligned} |A\cap\{M+1,\ldots,M+N\}| &= \lfloor \theta B(M+N)\rfloor - \lfloor \theta B(M)\rfloor \\ &= \theta\,|B\cap\{M+1,\ldots,M+N\}| + O(1). \end{aligned}$$

The absolute error is less than 2. Division by N and uniform density of B give uniform density $\theta\gamma=\delta$ for A. □

Apply the lemma to $B=A_*$ and (7.7). A subset cannot acquire a progression that was absent from A_*. This completes the proof of Theorem 1.2.

9 What has changed, and what has not

The construction is not a re-selection of the original parameters. The set of allowed differences is still defined using $a=1/100$, $b=10^{-10}$ and the original real numbers α_j. The use of $n\alpha_j/100$ occurs only in a *test observable*. After passing to a subprogression of index spacing 100, that observable has increment $d\alpha_j$, so its step is again the original allowed step. At no point do we assume that $100d$ itself belongs to S.

The other new point is that the fine deletion is conditional. Deleting its band everywhere would cost approximately 1.66%. Here it is imposed only on a trigger of proportion about 4.59%, so the joint cost is below 0.077%. The two globally imposed coarse bands together cost about 0.01864%. The exact union-bound calculation is (7.6).

The differentiation estimate is also stronger than the second-difference estimate used for the ramp. Its proof uses only a triangle-wave Fourier series, summation by parts, and the very small tail frequencies. The large value $K = 10^{18}$ provides an explicit window large enough for that estimate. The actual contradiction uses only 200 gated samples and then 85 consecutive ramp samples within them.

Finally, the theorem does not establish failure of δ-density largeness for every $\delta < 1$. The interval

$$\frac{9075692815599}{9084317000000} < \delta < 1$$

is not resolved here. An iteration of the gate would require new uniform persistence estimates; the present proof does not supply them automatically. The result and constants above are supported by the complete argument, rather than by an assumed density-amplification principle.

Provenance

The original example and the fact that its suitably chosen integer realization meets every finite-step nil-Bohr neighborhood are from [1]. The obstruction, the root-scale gate, and the quantitative estimates in this note are a further argument for that unchanged example. This note is self-contained as a proof of the stated density obstruction; it does not use the earlier two-color or fixed-density nilrecurrence manuscripts as black boxes.

References

[1] R. Alweiss, *New obstacles to multiple recurrence*, arXiv:2511.21680v3, December 24, 2025. The fixed parameters and the group-level set are in Definition 2.1; the integer realization is in Section 2.3. https://arxiv.org/abs/2511.21680.