Two-syndetic largeness implies nil-Bohr recurrence

A standalone proof prepared for Sohail Farhangi

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Abstract

We prove that if a set of positive integers supplies common differences for arbitrarily long arithmetic progressions in every two-syndetic set, then it meets every nil-Bohr neighborhood of zero. More precisely, every nil-Bohr neighborhood contains all the common differences of sufficiently long progressions lying in one suitably constructed two-syndetic set. The argument combines a continuous shell coloring on a nilmanifold with a binary encoding that meets every pair of consecutive integers. The shell estimate is proved using polynomial Mal'cev coordinates, exact integer-valued corrections, compactness, and a first-exit argument. A slow root of the nilrotation makes the encoding stable under a one-step error, while a selected subprogression recovers the original common difference exactly. The standard structural inputs from nilmanifold theory are stated explicitly, with references; all the encoding and shell lemmas are proved here.

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1 Definitions and the result

Throughout, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and every arithmetic progression has positive integer common difference. A progression of length K is a set of the form $\{a, a+d, \ldots, a+(K-1)d\}$, where $a, d \in \mathbb{N}$.

Definition 1.1. A set $A \subseteq \mathbb{N}$ is *two-syndetic* if

$$A \cap \{n, n+1\} \neq \varnothing \qquad (n \in \mathbb{N}). \tag{1.1}$$

A set $S \subseteq \mathbb{N}$ is *two-syndetic-large* if, for every two-syndetic $A \subseteq \mathbb{N}$ and every $K \geq 2$, there are $a \in \mathbb{N}$ and $d \in S$ such that

$$\{a, a+d, \ldots, a+(K-1)d\} \subseteq A. \tag{1.2}$$

A finite-step nilmanifold is a compact quotient $X = H/\Lambda$, where H is a finite-dimensional nilpotent Lie group and Λ is a discrete cocompact subgroup. A nilrotation is a left translation $T(x) = gx$ on such a quotient. The pair (X, T) is called a nilsystem.

Definition 1.2. A set $B \subseteq \mathbb{N}$ is a *nil-Bohr neighborhood of zero* if it contains a set

$$\{n \in \mathbb{N} : T^n x \in V\}, \tag{1.3}$$

where (X, T) is a finite-step nilsystem, $x \in X$, and V is an open neighborhood of x. A set $S \subseteq \mathbb{N}$ is a *set of nil-Bohr recurrence* if it meets every such B.

Thus nil-Bohr recurrence is precisely pointwise recurrence for nilsystems: for every (X, T), x, and neighborhood V of x, there is $d \in S$ with $T^d x \in V$.

Theorem 1.3. *Every two-syndetic-large set is a set of nil-Bohr recurrence. In fact, for every nil-Bohr neighborhood $B \subseteq \mathbb{N}$, there exist a two-syndetic set $A \subseteq \mathbb{N}$ and an integer $K \geq 2$ such that*

$$\{a, a+d, \ldots, a+(K-1)d\} \subseteq A \quad \Longrightarrow \quad d \in B \qquad (a, d \in \mathbb{N}). \tag{1.4}$$

We will prove the finite witness assertion and then apply the definition of two-syndetic largeness. The length K and the set A may depend on B.

Structural inputs. We use the classical finite van der Waerden theorem, the standard Mal'cev coordinate theorem, and the multiple-return realization of nil-Bohr neighborhoods due to Huang, Shao, and Ye. The latter realization and the connected ambient model are stated in the next section. The coordinate facts used later are also stated explicitly. All subsequent lemmas, including the robust shell estimate, are proved in this document.

2 A nilmanifold model that detects the desired difference

The structural result we need is the following form of the nil-Bohr multiple-return theorem.

Theorem 2.1 (Huang–Shao–Ye)**.** *Given a nil-Bohr neighborhood $B \subseteq \mathbb{N}$, there exist a minimal nilsystem (Z, T), a nonempty open set $U \subseteq Z$, and an integer $\ell \geq 1$ such that*

$$\{d \in \mathbb{N} : U \cap T^{-d}U \cap \cdots \cap T^{-\ell d}U \neq \varnothing\} \subseteq B. \tag{2.1}$$

This is Theorem A(1) of [2], restricted from integer times to positive times. We also use the standard ambient realization from [2, Section 3.1.2]: for the selected nilrotation, we may identify Z with a closed invariant subset of

$$Y = G/\Gamma, \tag{2.2}$$

where G is connected, simply connected, and nilpotent, Γ is a lattice, and T is the restriction of left translation by some $\tau \in G$. This allows the original nilsystem to have more than one connected component. The embedding is equivariant for the selected translation.

The exponential map of G is a diffeomorphism, so

$$\tau^t = \exp(t \log \tau), \qquad t \in \mathbb{R}, \tag{2.3}$$

defines a one-parameter subgroup acting continuously on Y. Fix a smooth Riemannian metric on Y, and write its distance as d_Y. Put

$$Z_* = \{\tau^t z : 0 \le t \le 1,\ z \in Z\}. \tag{2.4}$$

This set is compact. Moreover, if $x \in Z$, then $\tau^s x \in Z_*$ for every $s \in \mathbb{R}$: write $s = m + t$ with $m \in \mathbb{Z}$, $0 \le t < 1$, and use $\tau^m Z = Z$.

Lemma 2.2 (Uniform detection on the flow extension)**.** *There is a number $\nu > 0$ such that, for every $d \in \mathbb{N}$ and every $y \in Z_*$,*

$$\max_{1 \le j \le \ell} d_Y(\tau^{jd} y, y) < \nu \quad \Longrightarrow \quad d \in B. \tag{2.5}$$

Proof. Minimality gives a finite open cover of Z by sets $T^{-r_1}U, \dots, T^{-r_s}U$, where the r_i are integers. Choose a Lebesgue number in the following form: there is $\lambda > 0$ such that, for every $z \in Z$, the relative ball $B_Y(z, \lambda) \cap Z$ lies in one member of this cover. If

$$\max_{1 \le j \le \ell} d_Y(\tau^{jd} z, z) < \lambda,$$

then $z, \tau^d z, \dots, \tau^{\ell d} z$ all lie in the same $T^{-r_i}U$. Consequently, $T^{r_i} z$ belongs to $U \cap T^{-d}U \cap \cdots \cap T^{-\ell d}U$. Equation (2.1) gives $d \in B$.

The maps $y \mapsto \tau^{-t} y$, for $0 \le t \le 1$, form an equicontinuous family because $(t, y) \mapsto \tau^{-t} y$ is continuous on the compact space $[0, 1] \times Y$. Choose $\nu > 0$ such that

$$d_Y(y, y') < \nu \quad \Longrightarrow \quad d_Y(\tau^{-t} y, \tau^{-t} y') < \lambda \qquad (0 \le t \le 1).$$

For $y \in Z_*$, write $y = \tau^t z$ with $z \in Z$ and $0 \le t \le 1$. Since all the powers of τ commute, hypothesis (2.5) implies

$$d_Y(\tau^{jd} z, z) = d_Y(\tau^{-t} \tau^{jd} y, \tau^{-t} y) < \lambda \qquad (1 \le j \le \ell).$$

The preceding paragraph now yields $d \in B$. □

This lemma lets us detect membership in B using a cluster of iterates about any point of Z_*. The point at which the cluster occurs will be supplied by the shell argument.

3 Polynomial coordinates and exact lattice corrections

For the next three sections, fix a positive-dimensional $Y = G/\Gamma$ as in (2.2). The zero-dimensional case is a single point and requires no argument.

3.1 The coordinate facts

A lattice-adapted strong Mal'cev coordinate map

$$\psi : G \longrightarrow \mathbb{R}^m$$

is a smooth global coordinate system with $\psi(1_G) = 0$ and $\psi(\Gamma) = \mathbb{Z}^m$. The coordinates can be ordered so that, writing $x = \psi(g)$ and $y = \psi(h)$,

$$\psi_i(gh) = x_i + y_i + R_i(x_{<i}, y_{<i}), \qquad 1 \le i \le m, \tag{3.1}$$

where R_i is a polynomial in the coordinates with index less than i, and $R_1 = 0$. The change between exponential coordinates and ψ is polynomial. These are the standard Mal'cev facts recorded in [1, Definition 2.1 and Lemmas A.2–A.3].

In particular, there is an integer $E \ge 1$ such that

$$\deg_t \psi_i(\exp(tX)) \le E \tag{3.2}$$

for every X in the Lie algebra and every i. Indeed, exponential coordinates of $\exp(tX)$ are linear in t, and a fixed polynomial coordinate change has a fixed finite degree.

Choose positive integers $D_1, \dots, D_m$ recursively as follows. Take $D_1 \ge E$. For $i > 1$, take $D_i \ge E$ large enough that every monomial of R_i has weighted degree at most D_i, when each variable x_j or y_j has weight D_j. This is possible because R_i involves only earlier coordinates. Put $D = \max_i D_i$, and let $\mathcal{P}$ consist of all maps $A : \mathbb{R} \to G$ such that

$$\deg_t \psi_i(A(t)) \le D_i \qquad (1 \le i \le m). \tag{3.3}$$

Lemma 3.1. *The class $\mathcal{P}$ is closed under multiplication, inversion, and affine changes of parameter. It contains every curve $t \mapsto g^t v$, where $g, v \in G$.*

Proof. For multiplication, substitute the coordinate polynomials of A and B into (3.1). The defining choices of the D_i give the required degree bound in every coordinate. For inversion, solve $AA^{-1} = 1_G$ recursively:

$$\psi_i(A^{-1}) = -\psi_i(A) - R_i\big(\psi_{<i}(A), \psi_{<i}(A^{-1})\big).$$

Induction gives the same degree bounds. Affine changes of the real parameter do not increase the degree of a polynomial. Finally, (3.2) puts g^t in $\mathcal{P}$, and constant curves belong to $\mathcal{P}$. Closure under multiplication therefore puts $g^t v$ in $\mathcal{P}$. □

The integers D_i are only coordinate degree bounds. No grading of the Lie group is required.

3.2 Centered representatives and finite differences

Let $I = [-1/2, 1/2)$. Every right coset $g\Gamma$ has a unique representative $\mathrm{rep}(g)$ with coordinates in I^m. To see this, choose the integer coordinates z_i of a correction $\gamma \in \Gamma$ recursively. If the earlier coordinates have been chosen, set

$$z_i = -\left\lfloor x_i + R_i(x_{<i}, z_{<i}) + \frac{1}{2} \right\rfloor, \qquad x = \psi(g).$$

Then the ith coordinate of $g\gamma$ lies in I. This also gives uniqueness, since there is a unique integer producing a value in I at each stage. Consequently, $\mathrm{rep}(g\gamma) = \mathrm{rep}(g)$ whenever $\gamma \in \Gamma$.

For a scalar sequence f, put $\Delta f(n) = f(n+1) - f(n)$. We will use

$$\Delta^r f(n) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(n+j). \tag{3.4}$$

A polynomial of degree at most r has vanishing $(r+1)$st differences. Conversely, if a finite sequence has vanishing $(r+1)$st differences wherever they are defined, interpolation at its first $r+1$ nodes agrees with the whole sequence: the difference equation determines each successive value from the preceding $r+1$ values.

We also use the Newton interpolation identity

$$f(t) = \sum_{j=0}^{r} \Delta^j f(0) \binom{t}{j} \qquad (\deg f \le r). \tag{3.5}$$

If $f(0), \dots, f(r)$ are integers, this formula shows that $f(n)$ is an integer for every $n \in \mathbb{Z}$. Here $\binom{n}{j}$ is integral also for negative n, since $\binom{-u}{j} = (-1)^j \binom{u+j-1}{j}$.

Lemma 3.2 (Polynomial lattice correction)**.** *Set $M = D+2$, and let $\xi > 0$ satisfy $2^{D+1}\xi < 1$. Suppose $A \in \mathcal{P}$ and the coordinate vectors of*

$$\mathrm{rep}(A(0)), \dots, \mathrm{rep}(A(M-1))$$

all lie in one box whose coordinate intervals have length at most ξ. Then there are $Q, C \in \mathcal{P}$ such that

$$A(t)C(t) = Q(t) \quad (t \in \mathbb{R}), \qquad C(n) \in \Gamma \quad (n \in \mathbb{Z}), \tag{3.6}$$

and

$$Q(n) = \mathrm{rep}(A(n)) \qquad (0 \le n < M). \tag{3.7}$$

Proof. For $0 \le n < M$, write $A(n)\lambda(n) = \mathrm{rep}(A(n))$ with $\lambda(n) \in \Gamma$. Let $a_i(t) = \psi_i(A(t))$, $r_i(n) = \psi_i(\mathrm{rep}(A(n)))$, and $z_i(n) = \psi_i(\lambda(n)) \in \mathbb{Z}$. We construct the coordinate polynomials $c_i(t) = \psi_i(C(t))$ and $q_i(t) = \psi_i(Q(t))$ by induction on i. The induction will preserve agreement with $z_i(n)$ and $r_i(n)$ at every one of the M initial nodes.

Suppose the preceding coordinates have been constructed, and put

$$f_i(t) = a_i(t) + R_i\big(a_{<i}(t), c_{<i}(t)\big).$$

The degree of f_i is at most D_i. At every initial node, the triangular group law and the induction hypothesis give

$$r_i(n) - f_i(n) = z_i(n) \in \mathbb{Z} \qquad (0 \le n < M). \tag{3.8}$$

Since $\Delta^{D_i+1} f_i = 0$, each defined value of $\Delta^{D_i+1} r_i$ is an integer. All the numbers $r_i(n)$ lie in an interval of length at most ξ. Subtracting an endpoint of that interval does not change a positive-order finite difference, so (3.4) gives

$$|\Delta^{D_i+1} r_i(n)| \le 2^{D_i+1}\xi < 1.$$

These integer differences are therefore zero.

The finite interpolation observation gives a polynomial q_i of degree at most D_i agreeing with all M values of r_i. Define $c_i = q_i - f_i$. Equation (3.8) shows that $c_i(n) = z_i(n)$ at all the initial nodes, including $0, \dots, D_i$. Formula (3.5) then makes c_i integer-valued at every integer. This proves the induction step, including agreement at all M initial nodes.

After all coordinates have been constructed, define $C = \psi^{-1}(c_1, \dots, c_m)$ and $Q = \psi^{-1}(q_1, \dots, q_m)$. Their coordinate degree bounds put them in $\mathcal{P}$. The identities defining c_i give $AC = Q$, and $\psi(C(n)) \in \mathbb{Z}^m$ gives $C(n) \in \Gamma$. □

The correction is lattice-valued at *every* integer, even though only its first M values were used to construct it. The coefficients of the multiplication polynomials R_i need not be integers; the integrality in the proof comes from (3.8).

4 A continuous shell function and uniform oscillation

4.1 The size function

Define a function on the quotient by

$$F(g\Gamma) = \min_{\gamma\in\Gamma} \|\psi(g\gamma)\|_2^2. \tag{4.1}$$

Lemma 4.1. *The minimum in (4.1) is attained. The function $F : Y \to \mathbb{R}$ is continuous and satisfies $0 \le F \le m/4$. On any compact set of representatives $g \in G$, only finitely many lattice elements can occur as minimizers.*

Proof. The centered representative supplies a candidate of value at most $m/4$. If g lies in a fixed compact set and $\|\psi(g\gamma)\|_2^2 \le m/4$, then $g\gamma$ lies in a fixed compact coordinate box. Hence $\gamma = g^{-1}(g\gamma)$ lies in a compact subset of G. Its intersection with the discrete subgroup Γ is finite. The minimum is therefore attained, and a common finite set of candidates suffices on any compact set of representatives. Locally the minimum is consequently a minimum of finitely many continuous functions. Right multiplication by an element of Γ merely permutes the candidates, so the continuous function descends to Y. □

Lemma 4.2. *If $Q : \mathbb{R} \to G$ has polynomial Mal'cev coordinates and is nonconstant, then $t \mapsto F(Q(t)\Gamma)$ is nonconstant on every nondegenerate real interval.*

Proof. Consider a compact nondegenerate interval. By Lemma 4.1, only finitely many lattice elements $\gamma_1, \dots, \gamma_s$ need be considered there. For each of them,

$$f_j(t) = \|\psi(Q(t)\gamma_j)\|_2^2$$

is a polynomial. Right multiplication is an invertible polynomial map in Mal'cev coordinates, so $\psi(Q(t)\gamma_j)$ is a nonconstant polynomial vector. The sum of squares of a nonconstant real polynomial vector is nonconstant: the coefficient of its highest even power is a positive sum of squares.

If $F(Q(t)\Gamma)$ were identically c on the interval, then every point of the interval would be a zero of at least one of the finitely many nonzero polynomials $f_j - c$. Each such polynomial has only finitely many zeros. Their union cannot contain an interval, giving the required contradiction. □

4.2 Excursions of a fixed size

Fix $\rho = 1/4$. Let $\mathcal{K}$ be the family of maps $Q \in \mathcal{P}$ restricted to $[0, 1]$ whose coordinate vectors $q = \psi \circ Q$ satisfy

$$\begin{aligned} &q(0) \in [-1/2, 1/2]^m, \qquad \sup_{0\le t\le 1} \|q(t) - q(0)\|_\infty \le \rho, \\ &\|q(1) - q(0)\|_\infty = \rho. \end{aligned} \tag{4.2}$$

We regard this as a family in the finite-dimensional space of coordinate polynomial coefficients. All its coordinate polynomials are uniformly bounded by $3/4$ on $[0, 1]$. Interpolation at a

fixed set of nodes bounds their coefficients, and the conditions in (4.2) are closed. Thus $\mathcal{K}$ is compact. It is nonempty, for example by taking one coordinate to be ρt and the others to be zero.

Every curve in $\mathcal{K}$ is nonconstant. Lemma 4.2, compactness, and continuity therefore give

$$\eta := \min_{Q\in\mathcal{K}} \operatorname{osc}_{0\le t\le 1} F(Q(t)\Gamma) > 0. \tag{4.3}$$

Here the oscillation is the maximum minus the minimum. To justify the continuity used in this minimum, convergence of polynomial coefficients gives uniform convergence of the curves on $[0,1]$, and F is uniformly continuous on the compact coordinate box containing them. Oscillation is a continuous functional in the uniform norm.

Set

$$h = \frac{\eta}{20}, \qquad \beta = \frac{h}{8}, \qquad \mathcal{I}_e = \bigcup_{\substack{j\in\mathbb{Z}\\ j\equiv e \pmod 2}} [jh,(j+1)h] \quad (e=0,1). \tag{4.4}$$

The corresponding coloring of Y is

$$\kappa(y) = \left\lfloor \frac{F(y)}{h} \right\rfloor \pmod 2. \tag{4.5}$$

In particular, $\kappa(y) = e$ implies $F(y) \in \mathcal{I}_e$.

Lemma 4.3 (Robust sampling of an excursion)**.** *There exists a real number $T_0 \ge D+1$ such that, for every $Q \in \mathcal{K}$, every $T \ge T_0$, and either $e \in \{0,1\}$, there is an integer n with $0 \le n \le \lfloor T \rfloor$ for which*

$$\operatorname{dist}\big(F(Q(n/T)\Gamma), \mathcal{I}_e\big) > \beta. \tag{4.6}$$

Proof. The functions $t \mapsto F(Q(t)\Gamma)$, with $Q \in \mathcal{K}$, form an equicontinuous family. Choose $T_0 \ge D+1$ so large that

$$|t-t'| \le T_0^{-1} \quad \Longrightarrow \quad |F(Q(t)\Gamma) - F(Q(t')\Gamma)| < h/8$$

for every $Q \in \mathcal{K}$ and $t, t' \in [0,1]$. The range of $F \circ Q$ is an interval of length at least $\eta = 20h$. It contains a midpoint $(j+1/2)h$ of a shell whose parity is $1-e$. Choose t_* at which this value is attained. That midpoint is at distance $h/2$ from $\mathcal{I}_e$.

For $T \ge T_0$, take $n = \lfloor Tt_* \rfloor$. Then $0 \le n \le \lfloor T \rfloor$ and $|t_* - n/T| < 1/T$, except that equality of the sample and t_* is also possible. The corresponding values of F differ by less than $h/8$. The distance of the sampled value from $\mathcal{I}_e$ is therefore greater than $h/2 - h/8 = 3h/8$, which exceeds β. This works also when T is not an integer. □

5 The robust slow-motion lemma

We now turn band confinement of a long sampled orbit into a cluster at a prescribed divisible spacing. The construction is uniform over all polynomial curves in the class $\mathcal{P}$.

Lemma 5.1 (Robust slow motion)**.** *For the fixed nilmanifold, coordinate system, and metric, there are an integer $W \ge 1$ and a constant $C > 0$ with the following property. Put*

$$L_0 = \operatorname{lcm}(1,2,\dots,W). \tag{5.1}$$

Let $p \in \mathcal{P}$, let $k \ge 4W+2$ be an integer, and suppose that, for some $e \in \{0,1\}$,

$$\operatorname{dist}\big(F(p(n)\Gamma), \mathcal{I}_e\big) < \beta \qquad (0 \le n < k). \tag{5.2}$$

Then there is an integer b with $0 \le b < W$ such that, for every positive multiple L of L_0 and every integer $j \ge 1$ with $jL \le k/2$,

$$d_Y\left(p(b+jL)\Gamma, p(b)\Gamma\right) \le C\frac{jL}{k}. \tag{5.3}$$

The constants F, h, β, W, C, L_0 are fixed before p, k, and L are selected.

Proof. **Step 1: choose the auxiliary box size.** Let T_0 be as in Lemma 4.3, and retain $M = D+2$. Let $\ell_0, \dots, \ell_D$ be the Lagrange interpolation polynomials for the nodes $0, 1, \dots, D$. The finite constant

$$H = \max_{0 \le t \le T_0} \sum_{r=0}^{D} |\ell_r(t)|$$

has the following useful consequence: if f has degree at most D and $|f(r)| \le \xi$ at these nodes, then $|f(t)| \le H\xi$ for $0 \le t \le T_0$. Choose $\xi > 0$ so small that

$$2^{D+1}\xi < 1, \qquad H\xi < \rho/2. \tag{5.4}$$

Partition I^m into finitely many boxes whose coordinate intervals have length at most ξ.

Step 2: select an initial progression of representatives. The finite van der Waerden theorem gives an integer W such that every coloring of $\{0, \dots, W-1\}$ by these boxes has a monochromatic progression of length M. Apply it to the box containing $\psi(\mathrm{rep}(p(n)))$. There are integers b, q such that

$$0 \le b < W, \qquad 1 \le q \le W, \qquad b + (M-1)q < W, \tag{5.5}$$

and the representatives at $b, b+q, \dots, b+(M-1)q$ all occupy one box. Set $A(t) = p(b+qt)$. Lemma 3.1 gives $A \in \mathcal{P}$, and Lemma 3.2 gives $Q, C_p \in \mathcal{P}$ with

$$p(b+qt)C_p(t) = Q(t), \qquad C_p(n) \in \Gamma \quad (n \in \mathbb{Z}). \tag{5.6}$$

Write $q_i(t) = \psi_i(Q(t))$ and $q(t) = (q_1(t), \dots, q_m(t))$. For all $0 \le n < M$, these vectors are the selected representatives. Hence $q(0) \in I^m$ and

$$\|q(n) - q(0)\|_\infty \le \xi \qquad (0 \le n < M).$$

Interpolation and (5.4) imply

$$\sup_{0 \le t \le T_0} \|q(t) - q(0)\|_\infty < \rho/2. \tag{5.7}$$

Step 3: keep the polynomial lift bounded. Put

$$J = \left\lfloor \frac{k-1-b}{q} \right\rfloor. \tag{5.8}$$

Since $k > W$, we have $J \ge M-1$. For each integer $0 \le n \le J$, the correction identity gives

$$Q(n)\Gamma = p(b+qn)\Gamma.$$

Thus (5.2) holds also for every such $Q(n)$.

Suppose there is a first real time $T \in [0, J]$ at which

$$\|q(T) - q(0)\|_\infty = \rho.$$

Equation (5.7) gives $T > T_0$. The rescaled curve $\widetilde{Q}(u) = Q(Tu)$ belongs to $\mathcal{K}$: before this first exit its coordinate displacement is at most ρ, and at $u = 1$ it equals ρ. Lemma 4.3 supplies an integer $0 \le n \le \lfloor T \rfloor$ for which

$$\operatorname{dist}\big(F(\widetilde{Q}(n/T)\Gamma), \mathcal{I}_e\big) > \beta.$$

But $\widetilde{Q}(n/T) = Q(n)$ and $n \le J$, contradicting band confinement. There is no first exit, so

$$\sup_{0 \le t \le J} \|q(t) - q(0)\|_\infty \le \rho. \tag{5.9}$$

Step 4: estimate the speed. There is a finite constant B_D such that every real polynomial f of degree at most D obeys

$$\sup_{[0,1]} |f'| \le B_D \sup_{[0,1]} |f|. \tag{5.10}$$

For completeness, interpolate f at $0, 1/D, \dots, 1$, differentiate the interpolation identity, and take B_D to be the maximum over $[0,1]$ of the sum of the absolute values of the derivatives of these Lagrange polynomials. Apply (5.10) to $f(u) = q_i(Ju) - q_i(0)$. Equation (5.9) and integration from 0 to t/J give

$$\|q(t) - q(0)\|_\infty \le B_D \rho \frac{t}{J} \qquad (0 \le t \le J). \tag{5.11}$$

All these coordinate vectors lie in $[-3/4, 3/4]^m$. The smooth map $v \mapsto \psi^{-1}(v)\Gamma$ has bounded derivative on this compact box, so there is $C_0 > 0$ with

$$d_Y\big(Q(t)\Gamma, Q(0)\Gamma\big) \le C_0 \frac{t}{J} \qquad (0 \le t \le J). \tag{5.12}$$

One may obtain the Lipschitz estimate by integrating the derivative along the straight segment between two coordinate vectors inside the box.

Step 5: return to the original sampled curve. From (5.8), (5.5), and $k \ge 4W + 2$, we have

$$qJ \ge k - 1 - b - q \ge k - 1 - 2W \ge k/2. \tag{5.13}$$

If $L \in L_0\mathbb{N}$, then $q \mid L$. For $jL \le k/2$, the number $t = jL/q$ is therefore an integer in $[0, J]$. Using (5.6) at this integer and at 0 yields

$$\begin{aligned} d_Y\big(p(b + jL)\Gamma, p(b)\Gamma\big) &= d_Y\big(Q(jL/q)\Gamma, Q(0)\Gamma\big) \\ &\le C_0 \frac{jL}{qJ} \le 2C_0 \frac{jL}{k}. \end{aligned}$$

Taking $C = 2C_0$ proves the lemma. □

6 The two-syndetic encoding and the proof of the theorem

We first record the purely combinatorial part of the construction.

Lemma 6.1 (Binary encoding)**.** *For any coloring $c : \mathbb{N} \to \{0, 1\}$, define*

$$A_c = \{2n - 1 : c(n) = 0\} \ \cup \ \{2n : c(n) = 1 \text{ or } c(n + 1) = 1\}. \tag{6.1}$$

Then A_c is two-syndetic. If A_c contains a progression $\{a, a + d, \dots, a + (2k - 2)d\}$, there are $u \in \mathbb{N}$, $e \in \{0, 1\}$, and $\epsilon_0, \dots, \epsilon_{k-1} \in \{0, 1\}$ such that

$$c(u + nd + \epsilon_n) = e \qquad (0 \le n < k). \tag{6.2}$$

If a is odd, one may take $e = 0$ and every $\epsilon_n = 0$. If a is even, one may take $e = 1$.

Proof. For the pair $\{2n-1, 2n\}$, the odd member belongs to A_c if $c(n) = 0$, and the even member belongs if $c(n) = 1$. For the pair $\{2n, 2n+1\}$, the odd member belongs if $c(n+1) = 0$, and the even member belongs if $c(n+1) = 1$. Thus every pair of consecutive positive integers meets A_c.

From the given progression retain the terms $a, a+2d, \ldots, a+2(k-1)d$. They all have the parity of a. If $a = 2u-1$, these terms equal $2(u+nd)-1$, so their membership in A_c gives $c(u+nd) = 0$. If $a = 2u$, they equal $2(u+nd)$, so membership gives either $c(u+nd) = 1$ or $c(u+nd+1) = 1$ at each index. Choose ϵ_n accordingly. In both cases the common difference in the color indices is exactly d. □

Proof of Theorem 1.3. Fix a nil-Bohr neighborhood B. If $B = \mathbb{N}$, take $A = \mathbb{N}$ and $K = 2$. Otherwise choose the model Y, the element τ, the invariant set Z, the integer ℓ, and the number $\nu > 0$ from Section 2. Fix any $x \in Z$ and choose a representative $\widetilde{x} \in G$ of x.

Apply the shell construction and Lemma 5.1 to this fixed ambient nilmanifold. This fixes F, h, β, W, C, and L_0. By continuity on compact Y,

$$\sup_{y\in Y} |F(\tau^t y) - F(y)| \longrightarrow 0 \qquad (t \longrightarrow 0).$$

Choose a positive multiple L of L_0 so large that

$$\sup_{y\in Y} |F(\tau^{1/L} y) - F(y)| < \beta. \tag{6.3}$$

Define the coloring

$$c(n) = \kappa(\tau^{n/L}x) = \left\lfloor \frac{F(\tau^{n/L}x)}{h} \right\rfloor \pmod 2, \qquad n \in \mathbb{N}, \tag{6.4}$$

and form $A = A_c$ as in (6.1). Lemma 6.1 makes A two-syndetic.

Choose an integer k such that

$$k \geq 4W + 2, \qquad k \geq 2\ell L, \qquad \frac{C\ell L}{k} < \nu, \tag{6.5}$$

and set $K = 2k-1$. All these objects have now been fixed using only B and the auxiliary choices made from it.

Suppose $\{a, a+d, \ldots, a+(K-1)d\} \subseteq A$. Apply Lemma 6.1 to obtain u, e, and ϵ_n as in (6.2). Consider the polynomial-coordinate curve

$$p(t) = \tau^{(u+dt)/L}\widetilde{x}. \tag{6.6}$$

It belongs to $\mathcal{P}$: it is of the form $g^t v$ with $g = \tau^{d/L}$ and $v = \tau^{u/L}\widetilde{x}$. Put $y_n = p(n)\Gamma = \tau^{(u+nd)/L}x$. Since $c(u+nd+\epsilon_n) = e$, we have

$$F(\tau^{\epsilon_n/L} y_n) \in \mathcal{I}_e.$$

If $\epsilon_n = 0$, this already gives $F(y_n) \in \mathcal{I}_e$. If $\epsilon_n = 1$, estimate (6.3) gives $\operatorname{dist}(F(y_n), \mathcal{I}_e) < \beta$. Thus, in both cases,

$$\operatorname{dist}(F(p(n)\Gamma), \mathcal{I}_e) < \beta \qquad (0 \leq n < k).$$

Lemma 5.1 supplies $0 \leq b < W$ such that

$$d_Y\big(p(b+jL)\Gamma, p(b)\Gamma\big) \leq \frac{CjL}{k} < \nu \qquad (1 \leq j \leq \ell), \tag{6.7}$$

where (6.5) ensures $jL \le k/2$. Set $y = p(b)\Gamma$. The crucial identity is

$$p(b + jL)\Gamma = \tau^{(u+d(b+jL))/L}x = \tau^{jd}(\tau^{(u+db)/L}x) = \tau^{jd}y. \tag{6.8}$$

Also $y = \tau^{(u+db)/L}x \in Z_*$ by (2.4). Equations (6.7) and (6.8) satisfy Lemma 2.2, so $d \in B$. This proves the finite witness assertion (1.4).

Finally, let S be two-syndetic-large. Apply its defining property to the constructed A and the length K. It produces a progression in A with common difference $d \in S$. The finite witness assertion gives $d \in B$, hence $S \cap B \neq \varnothing$. As B was arbitrary, S is a set of nil-Bohr recurrence. □

7 Parameter order and exact difference recovery

For clarity, the choices in the proof have the following order. This also records why the slow root introduces no circular dependence.

Objects chosen	Dependence and purpose
Y, τ, Z, ℓ, ν	Chosen from the target neighborhood B; detect membership in B from a short cluster of iterates.
ψ, D_i, D	Fixed coordinates and polynomial degree bounds for the ambient nilmanifold.
$F, \rho, \eta, h, \beta, T_0$	Fixed continuous size function and the uniform oscillation and sampling parameters.
ξ, M, W, L_0, C	Fixed interpolation, lattice-correction, and slow-motion parameters.
L	A sufficiently large multiple of L_0 satisfying the uniform one-step estimate (6.3).
c, A, k, K	The integer coloring, its two-syndetic encoding, and a length satisfying (6.5), with $K = 2k - 1$.
a, d	An arbitrary K-term progression in A; the preceding choices force its difference d into B.

There are two separate changes of spacing. First, keeping every second term of the progression in A doubles its physical spacing to $2d$, while undoing the binary encoding divides this spacing by 2. The resulting color indices have spacing d. Second, evaluating the slow orbit at index increments L cancels the factor $1/L$ in the root flow, as shown in (6.8). The final nilrotation increment is exactly τ^d.

References

[1] B. Green and T. Tao, *The quantitative behaviour of polynomial orbits on nilmanifolds*, Annals of Mathematics (2) **175** (2012), no. 2, 465–540. doi:10.4007/annals.2012.175.2.2. The coordinate facts used here are in Definition 2.1 and Appendix A, especially Lemmas A.2, A.3, and A.14.

[2] W. Huang, S. Shao, and X. Ye, *Nil Bohr-sets and almost automorphy of higher order*, Memoirs of the American Mathematical Society **241** (2016), no. 1143. doi:10.1090/memo/1143. Preprint: arXiv:1407.1179. The inputs used here are Theorem A(1) and the connected, simply connected ambient realization in Section 3.1.2.