> **This is a recreation of the original chat by Codex using the data export supplied by the user.**

# Correctness Reassessment

Conversation ID: 6adca739-a260-83eb-95b4-85bae563a4d1

6 recorded user and assistant messages • Dates shown in Asia/Shanghai (UTC+08:00), with source timestamps.

Source: ChatGPT data export

Source: supplied ChatGPT data export dated 2026-10-25. Conversation URL: https://chatgpt.com/c/6adca739-a260-83eb-95b4-85bae563a4d1
This PDF reproduces all 6 saved user questions and assistant text responses: 6 on the saved active conversation path.
Messages retain their original wording. Mathematics and Markdown are typeset; citation chips are reconstructed from exported reference metadata.
Message dates use the original creation timestamps, displayed in Asia/Shanghai (UTC+08:00); UTC source timestamps are also shown.
The exact source message text is embedded in this PDF as a UTF-8 transcript and JSON record. Original uploaded files and available cited file assets are also embedded; use a PDF reader with an attachments panel.
This is a record of the conversation, including any errors and later corrections in the original answers.
Message date range: 2026-10-24 20:40:29 (Asia/Shanghai, UTC+08:00) to 2026-10-24 20:48:08 (Asia/Shanghai, UTC+08:00)
Video tab title: Verify skew product answer. Exported title: Correctness Reassessment.

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# Main conversation

### 1. User

2026-10-24 20:40:29 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-10-24T12:40:29.430000Z

Below is one of the answers you gave me from an older chat. Find the chat to refresh yourself on context, and then tell me whether or not you still think that this answer is correct.

Perfect — that restriction makes sense. We need a **distal but non-nilpotent system**, e.g., a **strictly ergodic distal skew product over a rotation with nontrivial cocycle**. I'll give a step-by-step explicit construction, with **quantitative estimates** for the super-polynomial growth of $P(N,\epsilon)$.

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## Step 1: Base dynamics

We want a system more complicated than a nilrotation, so let's start with a **Liouville rotation on the circle**:

$$ R_\alpha: \mathbb{T}^1 \to \mathbb{T}^1, \quad R_\alpha(x) = x + \alpha \mod 1, $$

where $\alpha$ is Liouville: there exists a sequence of integers $q_n$ such that

$$ 0 < \Big|\alpha - \frac{p_n}{q_n}\Big| < \frac{1}{q_n^{n}}, \quad \text{for some integers } p_n. $$

This ensures that the orbit of multiples of $\alpha$ stays extremely close to rational points for super-polynomially long intervals.

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## Step 2: Non-nilpotent distal skew product

Define a skew product over this rotation:

$$ T: \mathbb{T}^1 \times \mathbb{T}^1 \to \mathbb{T}^1 \times \mathbb{T}^1, \quad T(x,y) = (x + \alpha, y + f(x)) \mod 1, $$

where:

- $f \in C^\infty(\mathbb{T}^1, \mathbb{T}^1)$, non-constant, and **not cohomologous to a constant**.
- Magnitude of $f$ is small: $\|f\|_\infty < \epsilon_0$.

Properties:

1. **Distal**: obvious, since it's a compact group extension of a rotation.
2. **Minimality**: by Gottschalk–Hedlund theorem, if $f$ is not a coboundary, the skew product is minimal.
3. **Ergodicity** w.r.t. Lebesgue measure: generic choice of $f$ ensures no invariant measurable set under $T$.

This is no longer a nilrotation — it is a **nontrivial distal system with a cocycle**.

## Step 3: Orbit along multiples $kn$

Fix a point $(x_0,y_0)$ and consider

$$\mathcal{O}_{N,k} := \{T^{kn}(x_0,y_0)\}_{n=1}^N = \Big\{ \Big(x_0 + kn\alpha, y_0 + \sum_{j=0}^{kn-1} f(x_0 + j\alpha) \Big) \Big\}_{n=1}^N.$$

Focus on the **first coordinate** $x_0 + kn\alpha$.

- Because $\alpha$ is Liouville, the orbit along multiples $kn$ remains very close to $x_0 + kp_n/q_n$ for $n \approx q_n^{n/2}$ (super-polynomial in $q_n$).
- Therefore, for these $N$, all points are confined to an interval of length $< \epsilon$.

Hence, the **number of $\epsilon$-tubes along the $y$-coordinate** must be roughly the number of distinct $n$ in this interval — which grows **super-polynomially** in $N$, because $q_n$ grows faster than any polynomial in $n$.

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## Step 4: Dense $G_\delta$ set

Define

$$X_{\text{slow}} := \Big\{ x_0 \in \mathbb{T}^1 \mid \text{orbit of } kn\alpha \text{ has super-polynomial return times to intervals of length } \epsilon \Big\}.$$

- Standard Baire category arguments show that $X_{\text{slow}}$ is **dense $G_\delta$**.
- Every $(x_0,y_0) \in X_{\text{slow}} \times \mathbb{T}^1$ inherits the same slow orbit property.

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## Step 5: Quantitative estimate of $P(N,\epsilon)$

Let $q_n$ be the denominator sequence from $\alpha$'s Liouville property, growing faster than any polynomial in $n$. For $N \approx q_n$, the first coordinates of $\{T^{kn}(x_0,y_0)\}$ lie in an interval of length $< 1/q_n^n$.

- Fix $\epsilon>0$. Choose $n$ large so that $1/q_n^n < \epsilon$.
- Then the orbit segment of length $N \approx q_n$ lies in a single $\epsilon$-interval.
- To cover it **fully along all $k\in \mathbb{Z}$**, one needs $P(N,\epsilon) \gtrsim n^{c}$ for **any fixed $c>0$**, since $q_n$ grows faster than any polynomial in $n$.

Hence $P(N,\epsilon)$ grows **super-polynomially in $N$**.

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## ✅ Step 6: Summary of the construction

- **System**: $T(x,y) = (x+\alpha, y+f(x)) \mod 1$ on $\mathbb{T}^2$.
- **Properties**: distal, minimal, ergodic w.r.t. Lebesgue.
- **Set of points**: $G_\delta$ dense in $X$.

- **Super-polynomial orbit complexity**: along multiples \(kn\), the orbit requires **super-polynomially many \(\epsilon\)-tubes** to cover.

This is a **fully explicit counterexample** to polynomial orbit covering in distal systems, beyond nilrotations.

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If you want, I can **draw a picture of this orbit behavior** and **write a fully formal lemma + theorem with all constants, formulas, and the precise growth bound** for \(P(N,\epsilon)\), like a small rigorous paper section.

Do you want me to do that next?

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## 2. Assistant

2026-10-24 20:41:07 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-10-24T12:41:07.777013Z

I'll use the personal-context skill to find the earlier chat and recover the definition of $P(N, \epsilon)$. Then I'll check the construction and growth argument against that definition.

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## 3. Assistant

2026-10-24 20:42:25 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-10-24T12:42:25.776211Z

I found the matching exchange from May 6, 2026, including your definition: $P(N, \epsilon)$ covers the family of tuples $(T^k x, T^{2k} x, \ldots, T^{Nk} x)$ as $k$ varies. The earlier answer is not valid: it confuses slow returns with a lower bound on this covering number, and its final estimate does not imply super-polynomial growth. I'm checking the skew-product claims as well so I can explain precisely what must be withdrawn.

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## 4. Assistant

2026-10-24 20:44:24 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-10-24T12:44:24.403880Z

The checks confirm that the old proof must be withdrawn. I also found a promising correction: a published theorem about these exact arithmetic-progression tuples gives exponential covering growth under a suitable condition on a minimal distal system. I'm verifying a known smooth skew-product example against that condition, so I can tell you whether the original goal survives despite the faulty argument.

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## 5. Assistant

2026-10-24 20:47:23 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-10-24T12:47:23.846427Z

The corrected argument checks out. A known smooth, strictly ergodic torus skew product satisfies the needed condition, giving $P_x(N, \epsilon) \geq 2^N$ on a dense $G_\delta$ set for every sufficiently small fixed $\epsilon$. One distinction matters: this example is measurably isomorphic to a rotation, but is topologically neither a rotation nor an inverse limit of nilsystems. Your covering number detects the topological structure.

### 6. Assistant

2026-10-24 20:48:08 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-10-24T12:48:08.153613Z

**No. The answer you quoted is mathematically incorrect, and I withdraw its claim to have supplied a "fully explicit counterexample."** I recovered the matching exchange from **May 6, 2026**, including your original definition of $P(N,\epsilon)$. The search did not return the chat's title or a direct link.

There is, however, a substantial correction: **the phenomenon you were asking for does exist in the topological setting. A different, rigorous argument gives exponential growth on a dense $G_\delta$ set for suitable smooth, strictly ergodic distal skew products.** I will first explain why the old answer fails, then give the valid replacement.

# 1. Your original covering number

The earlier question concerned the family

$$A_N(x) = \left\{ \left(T^k x, T^{2k} x, \dots, T^{Nk} x\right) : k \in \mathbb{Z} \right\}.$$

Interpreting an $\epsilon$-tube in the usual coordinatewise sense, the relevant metric on $X^N$ is

$$d_N\big((x_1,\dots,x_N),(y_1,\dots,y_N)\big) = \max_{1\le j\le N} d(x_j,y_j),$$

and $P_x(N,\epsilon)$ is the number of radius- $\epsilon$ balls needed to cover $A_N(x)$.

The roles of the two indices are essential:

- $N$ is the length of each progression tuple.
- $k$ varies over all integers and indexes the different tuples being covered.

Consequently, to prove a lower bound $P_x(N,\epsilon) \ge M$, it suffices to find $M$ integers $k_1,\dots,k_M$ such that

$$\max_{1\le j\le N} d\big(T^{jk_a}x, T^{jk_b}x\big) > 2\epsilon \qquad (a \ne b).$$

**The old answer never produced such a separated family.** Its discussion of how many points lie inside one progression does not establish how many different progression tuples are needed.

# 2. The Liouville argument is wrong

## The rational approximation does not put the progression in one small interval

Use $r$ for the approximation index, so that

$$\left|\alpha - \frac{p_r}{q_r}\right| < q_r^{-r}.$$

What actually follows is

$$\left\| x + jk\alpha - \left(x + \frac{jkp_r}{q_r}\right)\right\|_{\mathbb{T}} \le |k| N q_r^{-r}, \qquad 1 \le j \le N.$$

This compares the orbit with a **rational progression**. The rational comparison point depends on $j$.

For example, when $k = 1$ and $N = q_r$, the rational progression generally visits all $q_r$ points of a rational grid around the circle. Those points are not confined to a tiny interval.

For the special choice $k = q_r$, we do obtain clustering:

$$\|x + jq_r\alpha - x\|_{\mathbb{T}} \le Nq_r^{1-r}.$$

But this gives one progression whose base coordinates are nearly constant. It does not give many mutually separated progression tuples.

### Clustering in the base does not force separation in the fiber

For the skew product,

$$T^{jk}(x, y) = \left(x + jk\alpha,\, y + \sum_{\ell=0}^{jk-1} f(x + \ell\alpha)\right)$$

when $jk > 0$.

To obtain the proposed covering lower bound, we would need estimates showing that the fiber coordinates for many different values of $k$ produce distinguishable patterns as $j$ runs from $1$ to $N$.

The earlier answer gave no such estimates. Its assertion that the number of tubes "must be roughly the number of distinct $n$" is an unjustified change from counting coordinates inside a tuple to counting separated tuples.

### The final growth comparison uses the wrong variable

The answer also confused growth in the approximation index $r$ with growth in the progression length $N \asymp q_r$.

Even growth faster than every polynomial in $r$ can be slower than every positive power of $N$. For example, let

$$N_r = 2^{r^2}, \qquad M_r = 2^r.$$

Then

$$\frac{M_r}{r^c} \longrightarrow \infty \quad \text{for every } c > 0,$$

but

$$\frac{M_r}{N_r^A} = 2^{r - Ar^2} \longrightarrow 0 \quad \text{for every } A > 0.$$

Thus, even if the old answer had proved its asserted lower bounds in terms of $r$, those bounds would not establish super-polynomial growth in $N$.

## 3. A direct calculation shows why slow rotation returns cannot be the explanation

For the circle rotation $R_\alpha(u) = u + \alpha$, fix $x$ and define

$$F_N(t) = (x + t, x + 2t, \dots, x + Nt).$$

Because $\{k\alpha : k \in \mathbb{Z}\}$ is dense in $\mathbb{T}$,

$$\overline{A_N(x)} = F_N(\mathbb{T}).$$

Moreover,

$$\begin{aligned} d_N(F_N(t), F_N(s)) &= \max_{1 \le j \le N} \|j(t-s)\|_{\mathbb{T}} \\ &\le N\|t-s\|_{\mathbb{T}}. \end{aligned}$$

An $\epsilon/N$-mesh of the parameter circle therefore gives

$$\boxed{P_x(N, \epsilon) = O_\epsilon(N).}$$

This upper bound is independent of the Diophantine properties of $\alpha$. It holds for Liouville rotations as well.

This calculation does **not** give a polynomial upper bound for the full skew product. It does show that the old answer's proposed mechanism—exceptionally slow returns in the rotation factor—cannot by itself produce the claimed covering complexity.

The covering number concerns the entire family indexed by unrestricted $k$. How slowly the parameter $k\alpha$ visits parts of the circle does not change the closure of that parameter family.

# 4. What survives from the qualitative claims?

Some parts were correct, and the correction needs to preserve them.

## Distality is correct

For

$$T(u, v) = (u + \alpha, v + f(u)),$$

two points with distinct base coordinates retain their base-coordinate distance under every iterate. If they have the same base coordinate, their fiber-coordinate difference remains constant.

Thus this skew product is distal. It also preserves Lebesgue measure.

## Minimality and ergodicity need different cohomological conditions

The appropriate criteria involve the equation

$$g(u + \alpha) = e^{2\pi i m f(u)} g(u), \qquad m \in \mathbb{Z} \setminus \{0\}.$$

Minimality is governed by the absence of **continuous** circle-valued solutions; Lebesgue ergodicity is governed by the absence of **measurable** solutions. These criteria are described in the appendix of Kułaga-Przymus–Lemańczyk, *The Möbius function and continuous extensions of rotations*. Springer Nature Link

The statement I wrote, "by Gottschalk–Hedlund, not a coboundary implies minimal," was inaccurate. However, the stronger assumption appearing elsewhere in that answer—**not continuously cohomologous to any constant**—can imply minimality for this circle-over-circle setting. Therefore, I should not replace the old error with the claim that its minimality conclusion is necessarily false under that stronger hypothesis.

The real unresolved assertion there was Lebesgue ergodicity. Ruling out continuous transfer functions does not rule out measurable ones, and saying “a generic choice” supplies neither a specific cocycle nor a proof.

There is also a useful qualification about strict ergodicity: in this compact group extension of a uniquely ergodic base, Lebesgue ergodicity would imply unique ergodicity. This is Proposition 2.5.16 in the same reference. The missing step was establishing Lebesgue ergodicity for an actual choice of $f$. Springer Nature Link

### The claimed dense $G_\delta$ set was not established

The old definition of $X_{\text{slow}}$ did not specify the relevant return-time quantifiers.

For a fixed nonzero $k$ and a fixed nonempty open interval $I$, an irrational rotation by $k\alpha$ has uniformly bounded gaps between visits to $I$. Indeed, the sets

$$R_{k\alpha}^{-j} I, \qquad j \geq 0,$$

cover the circle, and compactness gives a finite subcover.

Slow recurrence can concern shrinking intervals or a varying step $k$, but those are different statements and need precise formulations. No Baire-category argument establishing your desired covering property appeared in the old answer.

Finally, no particular $f$ was actually defined. Calling that a fully explicit construction was unjustified.

## 5. A rigorous replacement gives exponential growth

Here is the stronger result that can actually be justified.

> **Proposition.** Let $(X, T)$ be a compact metric minimal distal system that is **not an inverse limit of nilsystems**. Then there exist $\epsilon_0 > 0$ and a dense $G_\delta$ set $\Omega \subset X$ such that
>
> $$\boxed{P_x(N, \epsilon) \geq 2^N}$$
>
> for every $x \in \Omega$, every $N \geq 2$, and every $0 < \epsilon < \epsilon_0$.

The assumption “not an inverse limit of nilsystems” is stronger than merely “not a nilrotation.” It is the assumption used in the proof below.

The input is a theorem of Glasner–Huang–Shao–Weiss–Ye, *Topological characteristic factors and nilsystems*. In the distal case, their Theorem 4.3 says that the maximal $(N-1)$-step pro-nilfactor is a topological characteristic factor for the action

$$T \times T^2 \times \cdots \times T^N.$$

Their definition gives a dense $G_\delta$ set of fixed starting points whose progression orbit closures are saturated over that factor. arxiv.org

Here is the covering-number consequence, written out.

### Step A: Obtain uniformly separated points in the same pro-nilfactor fiber

Let

$$\pi_\infty : X \longrightarrow X_\infty$$

be the maximal infinite-step pro-nilfactor map.

Since $X$ is not an inverse limit of nilsystems, this map is not injective. Choose $a \neq b$ with

$$\pi_\infty(a) = \pi_\infty(b).$$

Distality gives

$$\delta := \inf_{n \in \mathbb{Z}} d(T^n a, T^n b) > 0.$$

Now fix any $x \in X$. By minimality, choose integers $n_j$ such that

$$T^{n_j} a \longrightarrow x.$$

After passing to a subsequence, compactness gives

$$T^{n_j} b \longrightarrow y$$

for some $y \in X$.

The factor relation is closed and invariant, so

$$\pi_\infty(x) = \pi_\infty(y).$$

Also,

$$d(x, y) \geq \delta.$$

Thus, for every $x$, there is a point $y$ in the same $\pi_\infty$-fiber separated from $x$ by at least the same fixed positive constant $\delta$.

## Step B: Use saturation of the progression orbit closure

Write

$$L_N(x) = \overline{A_N(x)}$$

and let

$$\pi_{N-1} : X \longrightarrow X_{N-1}$$

be the maximal $(N-1)$-step pro-nilfactor map.

For each $N \geq 2$, the characteristic-factor theorem gives a dense $G_\delta$ set $\Omega_N$ such that $L_N(x)$ is saturated under $\pi_{N-1}^{\times N}$ whenever $x \in \Omega_N$.

Concretely, saturation means that if a tuple belongs to $L_N(x)$, then replacing any of its coordinates by another point with the same $\pi_{N-1}$-image produces another tuple in $L_N(x)$.

Set

$$\Omega = \bigcap_{N \geq 2} \Omega_N.$$

This is still a dense $G_\delta$ set.

Fix $x \in \Omega$, and choose $y$ as in Step A. Since $\pi_{N-1}$ factors through $\pi_\infty$,

$$\pi_{N-1}(x) = \pi_{N-1}(y).$$

The diagonal tuple $(x, \dots, x)$ belongs to $A_N(x)$ by taking $k = 0$. Therefore saturation gives

$$\boxed{\{x, y\}^N \subseteq L_N(x).}$$

This is the missing source of complexity: **every one of the $2^N$ coordinatewise choices between $x$ and $y$ occurs in the progression orbit closure.**

### Step C: Turn these choices into a covering lower bound

Two distinct tuples in $\{x, y\}^N$ differ in at least one coordinate. Hence their maximum-metric distance is at least

$$d(x, y) \geq \delta.$$

For each of these $2^N$ tuples, choose an actual tuple from $A_N(x)$ within $\delta/8$. The chosen actual tuples are pairwise separated by at least

$$\delta - \frac{\delta}{8} - \frac{\delta}{8} = \frac{3\delta}{4}.$$

Consequently, a radius-$\epsilon$ ball cannot contain two of them when $\epsilon < \delta/4$. Taking

$$\epsilon_0 = \frac{\delta}{4}$$

proves

$$P_x(N, \epsilon) \geq 2^N.$$

This holds for every $N \geq 2$, not just along a sparse subsequence. In particular,

$$\frac{P_x(N, \epsilon)}{N^A} \longrightarrow \infty \qquad \text{for every } A > 0$$

at each fixed $0 < \epsilon < \epsilon_0$.

The restriction to sufficiently small $\epsilon$ is necessary: for $\epsilon$ larger than the diameter of the space, one tube covers everything.

## 6. There is a smooth, strictly ergodic torus example to which this applies

Example 6.1 of Glasner–Huang–Shao–Ye, *Regionally proximal relation of order $d$ along arithmetic progressions and nilsystems*, constructs a $C^\infty$ skew product

$$T(u, v) = (u + \alpha, v + f(u))$$

that is distal, minimal, and uniquely ergodic for Lebesgue measure. Its measurable system is isomorphic to

$$(u, v) \longmapsto (u + \alpha, v + \beta),$$

where $1, \alpha, \beta$ are rationally independent, but its maximal **topological** equicontinuous factor is only the base rotation $R_\alpha$. arxiv.org

This example is not an infinite-step pro-nilsystem. Indeed, it has the measurable eigenvalue $e^{2\pi i\beta}$, while its continuous eigenvalues are only $e^{2\pi im\alpha}$, $m \in \mathbb{Z}$. In a minimal inverse limit of nilsystems, measurable eigenfunctions have continuous representatives; the finite-order statement and inverse-limit representation are given in Lemmas 5.6–5.7 of that paper. Thus the preceding proposition applies. arxiv.org

For completeness, the passage to an inverse limit is straightforward: the conditional expectation of a nonzero eigenfunction onto some finite-stage factor is nonzero and remains an eigenfunction with the same eigenvalue. That factor supplies a continuous representative, and the eigenspace in an ergodic system is one-dimensional.

So the answer to the original **topological** existence question is affirmative, with an exponential lower bound. The example is measurably isomorphic to a rotation, which is compatible with its much richer topological progression orbit closures. Its covering complexity is established by the separated tuples supplied by the saturation theorem; the Liouville clustering argument in my old answer does not establish it.