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Failure of Green-Tao for distals

Conversation ID: 69d62c4d-5af8-8329-93a2-cd617d8323e3

15 recorded user and assistant messages • Dates shown in Asia/Shanghai (UTC+08:00), with source timestamps.

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Messages retain their original wording. Mathematics and Markdown are typeset; citation chips are reconstructed from exported reference metadata.
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This is a record of the conversation, including any errors and later corrections in the original answers.
Message date range: 2026-04-08 18:23:17 (Asia/Shanghai, UTC+08:00) to 2026-04-12 11:00:30 (Asia/Shanghai, UTC+08:00)

Main conversation

1. User

2026-04-08 18:23:17 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-08T10:23:17.582000Z

Can you give me an example of a distal system, perhaps a 2-step distal system, in which every point equidistributes slowly? I want to somehow find a distal system that has opposite equidistribution behavior to that of nilrotation on a nilmanifold, although I don't know of a precise way to phrase the question.

2. Assistant

2026-04-08 18:25:26 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-08T10:25:26.972525Z

Yes. In fact, the simplest example is already a **1-step distal** system:

$$T_\alpha : x \mapsto x + \alpha \pmod 1$$

with α chosen **Liouville**.

Why this works: irrational circle rotations are minimal, uniquely ergodic, and distal, so every point equidistributes to Lebesgue measure. For Diophantine α, Hölder observables have polynomial-type quantitative bounds, but for suitably Liouville α the convergence can be made **arbitrarily slow**. Klein–Liu–Melo explicitly note that for any prescribed rate $\rho_N \to 0$, one can choose a Liouville frequency ω and even an analytic observable (\phi

3. User

2026-04-08 18:26:46 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-08T10:26:46.365000Z

this is a special case of nilrotation, so this answer is awful. I am willing to settle for almost every point equidistributing slowly instead of every point. please try again.

4. Assistant

2026-04-08 18:28:34 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-08T10:28:34.045046Z

You are right — my previous answer was too close to the nilrotation world you were trying to get away from.

A much better example is:

$$X = \mathbb{T} \times \Delta, \qquad T(x, \delta) = (x + \alpha,\ \delta + 1),$$

where α is a **Liouville** irrational and Δ is an **odometer** (for instance the 2-adic adding machine).

This has the features you seem to want:

1. **It is distal.**
   In fact it is equicontinuous, since it is just a translation on the compact abelian group $\mathbb{T} \times \Delta$.
2. **It is not a nilmanifold rotation.**
   The phase space is not even a manifold; Δ is a Cantor-type compact abelian group, so this lies outside the usual nilmanifold/nilrotation class.
3. **Every point equidistributes** (so certainly "almost every point" does).
   Since T is a translation by an element whose projections are minimal on both factors, it is uniquely ergodic, with invariant measure $m_{\mathbb{T}} \otimes m_{\Delta}$.
4. **The equidistribution can be as slow as you want.**
   For Liouville rotations, one can choose a continuous, even analytic, zero-mean observable g on $\mathbb{T}$ whose Birkhoff averages converge arbitrarily slowly. Then for

$$F(x, \delta) := g(x),$$

the averages on X are exactly the same:

$$\frac{1}{N} \sum_{n=0}^{N-1} F(T^n(x, \delta)) = \frac{1}{N} \sum_{n=0}^{N-1} g(x + n\alpha).$$

So whatever slow rate you arrange on the circle factor survives unchanged on the distal system X. Arbitrarily slow convergence for Liouville rotations is explicitly discussed in recent literature. arXiv

So this gives a **distal, non-nilmanifold** system in which **every point** equidistributes, but the convergence can be made **arbitrarily slow**.

The reason I think this is the right "opposite behavior" to keep in mind is: nilsystems often come with strong quantitative equidistribution mechanisms coming from their algebraic structure, whereas here you have a very tame distal system whose topological structure is still extremely regular, but whose quantitative equidistribution is as bad as the Liouville base allows.

If you want something **genuinely 2-step distal** rather than equicontinuous, the same trick still works: start with the base $(\mathbb{T} \times \Delta, T)$ above and take a compact-group skew extension

$$\widetilde{T}(x, \delta, u) = (x + \alpha,\ \delta + 1,\ u + \Phi(x, \delta))$$

for a continuous cocycle Φ. Such compact extensions are distal, and Furstenberg-type skew-product constructions provide many strictly ergodic distal examples of this sort. Any observable depending only on the x-coordinate still has the same slow averages as above, so the slow equidistribution persists in the 2-step distal extension as well. Semantic Scholar Cambridge University Press & Assessment

So the clean takeaway is:

$$\boxed{\text{Liouville rotation} \times \text{odometer}}$$

is already a good example, and it is genuinely outside the nilmanifold world.

If you want, I can next try to formulate a more precise notion of "opposite to nilrotation" and turn this into a theorem-style statement.

5. User

2026-04-12 10:43:28 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:43:28.152000Z

is there an example of a uniquely ergodic distal system on some n-dimensional torus for which you cannot bound the rate of equidistribution of a given point as a function of how close it is to rational points and/or rational submanifolds?

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6. Assistant

2026-04-12 10:43:28 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:43:28.943873Z

I'm checking whether there is a torus example that is genuinely distal and uniquely ergodic, but where rational-approximation data of the starting point is too crude to control the discrepancy. The likely candidates are Furstenberg-type skew products on $\mathbb{T}^2$, and I'm verifying which properties are actually true.

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7. Assistant

2026-04-12 10:44:26 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:44:26.193291Z

Yes — if you allow the obstruction to be stated for **almost every** starting point, then a torus translation already gives a strong example.

Take a translation on $\mathbb{T}^2$,

$$T_\omega(x) = x + \omega, \qquad \omega = (\omega_1, \omega_2) \in \mathbb{T}^2$$

with $1, \omega_1, \omega_2$ linearly independent over $\mathbb{Q}$. Then T_ω is uniquely ergodic and distal. Klein–Liu–Melo recall a theorem of Yoccoz showing that for **any prescribed rate** $\rho_N \downarrow 0$, one can choose such a rationally independent ω and an **analytic** observable φ so that for **almost every** $x \in \mathbb{T}^2$,

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} \varphi(x + n\omega) - \int_{\mathbb{T}^2} \varphi \right| \geq \rho_N$$

for all sufficiently large N. arXiv

So in that sense there is **no upper bound on the rate of equidistribution that depends only on the starting point's "arithmetic position"** relative to rational points or rational subtori. The reason is that the lower bound above holds for almost every starting point, while those points have wildly varying approximation behavior with respect to rational points/subtori. The real obstruction is the arithmetic of the **frequency vector** ω together with the observable, not the "rational closeness" of the initial point. arXiv

There is one subtlety, though. If you literally mean "distance to rational points/submanifolds," that quantity is not the right invariant on the torus, because rational points and rational subtori are dense, so naive distance data is too coarse. A more meaningful version would use finite-scale Diophantine complexity, for example bounds on

$$\min_{q \leq Q,\, p \in \mathbb{Z}^n} \|x - p/q\|$$

or analogous approximation data to rational affine subtori. Yoccoz's example still strongly suggests that no bound depending only on such pointwise data can control discrepancy in general, because the slow behavior can occur for almost every starting point. arXiv

If you want the system to be **non-equicontinuous** but still distal and uniquely ergodic on a torus, then the natural next place to look is a **Furstenberg skew product** on $\mathbb{T}^2$. Furstenberg's 1961 paper gives strictly ergodic torus skew products, and these are the standard nontrivial distal examples on tori. JSTOR
What I do **not** know offhand is a clean published statement, in that non-equicontinuous Furstenberg setting, that says exactly the same "no pointwise rational-closeness bound" phenomenon. For the translation example above, though, the answer is clearly yes. arXiv JSTOR

So the clean answer is:

- **Yes**, already on $\mathbb{T}^2$, via a rationally independent translation.
- It is uniquely ergodic and distal.
- For suitable ω and analytic φ, equidistribution can be made **arbitrarily slow for almost every point**.
- Hence no quantitative bound can depend only on how the initial point sits relative to rational points or rational subtori. arXiv

If you want, I can next try to formulate a precise "negative theorem" in terms of Diophantine approximation of the initial point, rather than the informal phrase "close to rational submanifolds."

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8. User

2026-04-12 10:46:00 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:46:00.751000Z

I am still looking for a way in which the Green-Tao equidistribution theorem for nilmanifolds fails for a general distal system, so any example you give that is a nilsystem (such as the example you just gave), cannot possibly work. Please try again

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9. Assistant

2026-04-12 10:46:01 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:46:01.549009Z

I'm looking specifically for a **non-nilsystem distal torus example** now, not just a slow torus translation. The right family is probably Furstenberg-type skew products on $\mathbb{T}^2$, and I'm checking which ones are uniquely ergodic and where the nilsystem-style quantitative control actually breaks.

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10. Assistant

2026-04-12 10:47:40 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:47:40.601003Z

Yes — the right family to look at is **Furstenberg skew products on** $\mathbb{T}^2$, not translations and not affine skew-shifts.

A model example is

$$T_{\alpha,g}(x,y) = (x+\alpha,\ y+g(x)) \pmod 1,$$

with α irrational and $g : \mathbb{T} \to \mathbb{T}$ real-analytic (or smooth). Furstenberg's 1961 paper constructed strictly ergodic torus skew products of this kind, and modern work explicitly treats uniquely ergodic analytic skew products on $\mathbb{T}^2$ as **non-algebraic** smooth systems, i.e. outside the nilsystem/algebraic world. Annals of Mathematics Semantic Scholar

This is why I think this is the correct answer to the spirit of your question:

- it lives on a finite-dimensional torus;
- it is distal, because it is a circle-isometric extension of the irrational rotation $x \mapsto x+\alpha$;
- for suitable choices it is uniquely ergodic;
- and, unlike nilsystems, its orbit structure is controlled by the cocycle sums

$$T^n(x,y) = \big(x+n\alpha,\ y+S_n g(x)\big), \qquad S_n g(x) = \sum_{j=0}^{n-1} g(x+j\alpha),$$

so the quantitative problem is no longer governed by finitely many horizontal characters or rational subtori, but by the full small-divisor behavior of α together with the Fourier structure of g. Annals of Mathematics

So the **Green–Tao paradigm fails in form** here: on nilmanifolds, quantitative equidistribution is detected by horizontal characters and rational/algebraic obstructions; for these Furstenberg skew products, the obstruction is instead hidden in the cocycle sums $S_n g(x)$. Two points that are equally "close to rational subtori" can still have very different finite-time behavior, because the relevant object is not geometric proximity to rational sets, but the resonance of the base orbit $x+n\alpha$ with the Fourier modes of g.

The one thing I do **not** want to overclaim is this:

I do **not** currently know a clean published theorem saying, in exactly your language, that for some uniquely ergodic distal non-nilsystem on $\mathbb{T}^2$, **no bound on discrepancy can be expressed purely in terms of the initial point's closeness to rational points or rational subtori**. I think Furstenberg skew products are the right place to prove such a statement, but I cannot honestly tell you I have a citation that already proves that exact formulation.

What I *can* verify is an even more dramatic nearby failure if you drop unique ergodicity: Liu–Sarnak study analytic skew products on $\mathbb{T}^2$ that are distal and non-algebraic, and they emphasize that these flows can be **irregular**, meaning Birkhoff averages need not even exist for all points. That is already a very strong way in which nilsystem-style equidistribution theory breaks for general distal systems. arXiv Project Euclid

So my honest bottom line is:

$$\boxed{\text{Furstenberg skew products } T_{\alpha,g}(x,y) = (x+\alpha,\ y+g(x))}$$

are the correct non-nilsystem distal torus examples to investigate. They are the natural place where a Green–Tao-type quantitative theory should fail. But I cannot presently give you a theorem, with citation, that exactly matches your requested "no rational-submanifold bound for a given point" formulation in the uniquely ergodic case.

A plausible next step would be to try to prove it for observables like $e^{2\pi i y}$ by choosing α Liouville and g with a lacunary Fourier series, so that $S_n g(x)$ has prescribed large deviations on sparse scales.

11. User

2026-04-12 10:52:46 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:52:46.484000Z

I like the direction you are going in now. Please continue in this direction following your own proposal.

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12. Assistant

2026-04-12 10:52:47 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:52:47.429043Z

I've narrowed it to a concrete construction strategy on $\mathbb{T}^2$: a Furstenberg skew product with a Liouville base rotation and a very sparse Fourier cocycle. The key is that the Birkhoff sums for $e^{2\pi i y}$ are then controlled by small divisors coming from α, not by how x sits near rational points or rational subtori.

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13. Assistant

2026-04-12 10:53:38 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T02:53:38.830180Z

Yes. There is a clean way to push this through.

The right move is to separate two goals:

1. build a **strictly ergodic distal torus map that is not a torus nilrotation**, and
2. make the quantitative failure come from a point that is itself as Diophantine as you like, so the obstruction clearly does **not** come from proximity to rational points or rational subtori.

Here is a concrete statement.

Claim. Let $\rho_N \downarrow 0$ be any prescribed rate. Then there exists a homeomorphism

$$T : \mathbb{T}^2 \to \mathbb{T}^2, \qquad T(x,y) = (x+\alpha,\ y + x + h(x)) \pmod 1,$$

with the following properties:

- T is distal and uniquely ergodic.
- T is not a torus nilrotation, because it is not equicontinuous when h is nonconstant.
- There is a continuous observable $F \in C(\mathbb{T}^2)$ and a point $z = (x_0, y_0)$ that may be chosen Diophantine, such that

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} F(T^n z) - \int F \, d\mu \right|$$

is at least ρ_N along infinitely many N up to an absolute constant.

Furstenberg proved strict ergodicity for a large class of torus skew products of exactly this type, namely when the fiber map has nonzero degree and a mild regularity condition. ResearchGate

The key point is that the slow convergence can be forced while z itself remains very far, in the Diophantine sense, from rational affine subtori.

Here is the construction.

Take α Liouville and choose a rapidly growing sequence $q_k \to \infty$ such that

$$\|q_k\alpha\| \le q_k^{-3}$$

and also so large that

$$\rho_{q_k} \le 2^{-k}.$$

Now fix a point $x_0 \in \mathbb{T}$. Later we will choose x_0 Diophantine. Define

$$u(x) := \sum_{k=1}^{\infty} a_k \cos(2\pi q_k (x - x_0)), \qquad a_k := \rho_{q_k}.$$

Since $a_k \le 2^{-k}$, this defines a continuous zero-mean function on $\mathbb{T}$.

Now look at the Birkhoff sums for the irrational rotation $R_\alpha x = x + \alpha$:

$$S_N u(x) := \sum_{n=0}^{N-1} u(x + n\alpha).$$

At the special times $N = q_k$, the k-th Fourier mode is almost resonant because $q_k\alpha$ is extremely close to an integer. At $x = x_0$,

$$\frac{1}{q_k} S_{q_k}\big(\cos(2\pi q_k(\cdot - x_0))\big)(x_0) = \Re\!\left(\frac{1 - e(q_k^2\alpha)}{q_k(1 - e(q_k\alpha))}\right),$$

where $e(t) := e^{2\pi i t}$.

Because $\|q_k\alpha\| \le q_k^{-3}$, the last quantity tends to 1. So the k-th mode contributes essentially $a_k = \rho_{q_k}$ to the average at time q_k.

The other modes are errors:

- for $j > k$, the trivial bound gives contribution at most a_j, so the tail contributes at most $\sum_{j>k} a_j$;
- for $j < k$, the geometric-series bound gives

$$\left|\frac{1}{q_k} S_{q_k}\big(e(q_j\cdot)\big)(x_0)\right| \le \frac{2}{q_k\,\|q_j\alpha\|},$$

and since we made q_k grow very fast, the total contribution of all $j < k$ can be forced to be much smaller than a_k.

Hence, after choosing the sequence q_k inductively fast enough, we get

$$\left|\frac{1}{q_k} S_{q_k} u(x_0)\right| \ge \frac{1}{2}\, a_k = \frac{1}{2}\, \rho_{q_k}$$

for all k large enough.

So for the rotation R_α, the point x_0 has arbitrarily slow equidistribution for the observable u, with the slow times prescribed by the near-resonances of α.

Now embed this into a genuinely non-nilsystem torus map by setting

$$T(x, y) = (x + \alpha,\ y + x + h(x)) \pmod 1,$$

where h is any nonconstant Lipschitz circle function. Because the fiber action is by circle rotations depending continuously on the base, this is distal. Because the fiber cocycle $x + h(x)$ has degree 1, Furstenberg's theorem gives minimality and unique ergodicity under mild regularity assumptions. ResearchGate

Take the observable

$$F(x, y) := u(x).$$

Then F ignores the fiber, so for every (x, y),

$$\frac{1}{N}\sum_{n=0}^{N-1} F(T^n(x, y)) = \frac{1}{N}\sum_{n=0}^{N-1} u(x + n\alpha).$$

Therefore, at $z = (x_0, y_0)$,

$$\left|\frac{1}{q_k}\sum_{n=0}^{q_k-1} F(T^n z) - \int F\, d\mu\right| \geq \frac{1}{2}\,\rho_{q_k}$$

for all large k.

This already gives the failure you want.

Why does it answer your rational-submanifold question? Because we may choose x_0 and y_0 to be Diophantine, even so that the vector (x_0, y_0) is Diophantine in $\mathbb{T}^2$. Then z is quantitatively far from rational points and rational affine subtori in the usual sense, yet its equidistribution can still be made as slow as we like. So the rate is not controlled by the arithmetic position of the starting point. The real obstruction is the small-divisor structure of the **base frequency** α and the Fourier support of the observable.

This is exactly the way the nilmanifold picture breaks.

For nilrotations, Green–Tao-type quantitative equidistribution is driven by algebraic obstructions, ultimately detected by horizontal characters and rational factor data. For this Furstenberg skew product, even after choosing a very non-rational starting point, one can force slow convergence through a sparse resonance mechanism in the base rotation. The starting point's Diophantine behavior does not see that mechanism.

There is one caveat. This construction is still "factor-driven": the bad observable F comes from the base coordinate. So it proves that no Green–Tao-style bound depending only on the point's proximity to rational subtori can hold for general distal systems, but it does not yet show a purely fiber-sensitive failure.

The stronger version would use the fiber observable

$$G(x, y) = e^{2\pi i y}.$$

Then

$$G(T^n(x, y)) = e^{2\pi i y}\, e^{2\pi i\left(nx + \frac{n(n-1)}{2}\alpha + S_n h(x)\right)}.$$

If one chooses h with a very sparse Fourier series tuned to the same Liouville denominators, one should be able to make the phases

$$nx + \frac{n(n-1)}{2}\alpha + S_n h(x)$$

nearly stationary on long sparse scales, producing slow equidistribution for a genuinely fiber-sensitive observable. That would be a more intrinsic non-nilsystem failure, and it is very close to the mechanism behind Furstenberg's irregular examples and later analytic skew-product work. Liu–Sarnak emphasize that analytic skew products on $\mathbb{T}^2$ can be distal and even irregular, showing how far the general distal world is from the homogeneous nilsetting. arXiv arXiv

So my recommendation is:

- if you want a clean theorem now, use the $F(x,y)=u(x)$ construction above;
- if you want the sharpest philosophical contrast with Green–Tao, the next target is to carry out the same sparse-resonance construction for $G(x,y)=e^{2\pi iy}$.

I can write out the sparse-resonance proof in a theorem-proof format next, or try to push the genuinely fiber-sensitive version.

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14. User

2026-04-12 11:00:28 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T03:00:28.731000Z

I just want to let you know that this is exactly the direction that I wanted to take this in. Give yourself a pat on the back and then wait for my next question.

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15. Assistant

2026-04-12 11:00:30 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-04-12T03:00:30.188253Z

Glad we landed in the right spot — that's a really nice direction to explore ☺

I'll hold here and wait for your next question.