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Nilsequence for Amenable Groups

Conversation ID: 69b76ea8-9574-8325-8c6a-0fbce8df38a3

8 recorded user and assistant messages • Dates shown in Asia/Shanghai (UTC+08:00), with source timestamps.

Source: ChatGPT data export

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This PDF reproduces all 8 saved user questions and assistant text responses: 8 on the saved active conversation path. Messages retain their original wording. Mathematics and Markdown are typeset; citation chips are reconstructed from exported reference metadata.
Message dates use the original creation timestamps, displayed in Asia/Shanghai (UTC+08:00); UTC source timestamps are also shown.
The exact source message text is embedded in this PDF as a UTF-8 transcript and JSON record. Original uploaded files and available cited file assets are also embedded; use a PDF reader with an attachments panel.
This is a record of the conversation, including any errors and later corrections in the original answers.
Message date range: 2026-03-16 10:45:00 (Asia/Shanghai, UTC+08:00) to 2026-03-16 12:16:28 (Asia/Shanghai, UTC+08:00)

Main conversation

1. User

2026-03-16 10:45:00 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T02:45:00.940000Z

is there a natural notion of nilsequence for a countable amenable group?

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2. Assistant

2026-03-16 10:45:02 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T02:45:02.058691Z

Yes, but with an important caveat:

for an arbitrary countable amenable group, there is a natural notion of "nilsequence," but it is not as canonical or as rich as in the $\mathbb{Z}$-case unless you put extra structure on the group.

Here is the clean way to think about it.

1. The most natural bare definition

Let G be a countable amenable group.

A very natural definition of a **basic G-nilsequence** is a function

$$a : G \to \mathbb{C}$$

of the form

$$a(g) = F(T_g x),$$

where:

- $X = N/\Gamma$ is a compact nilmanifold,
- $T : G \to \mathrm{Homeo}(X)$ is an action by niltranslations, typically coming from a homomorphism

$$\rho : G \to N$$

via

$$T_g(y\Gamma) = \rho(g)y\Gamma,$$

- $x \in X$,
- $F \in C(X)$.

Then a ** G-nilsequence** is a uniform limit of such basic ones.

This is the direct analogue of the classical definition for $G = \mathbb{Z}$, where

$$a(n) = F(g^n x).$$

For $G = \mathbb{Z}^d$, this gives the standard multivariable nilsequences

$$a(\mathbf{n}) = F(g_1^{n_1} \cdots g_d^{n_d} x).$$

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2. Why this is natural

Because it says exactly:

> a nilsequence is a continuous observable along an orbit in a nilsystem.

That philosophy does not depend on $G = \mathbb{Z}$. If G acts on a nilmanifold, then evaluating a continuous function along a G-orbit is the obvious generalization.

So in that sense, **yes**, there is a natural notion.

---

3. But why this is not the whole story

The subtle point is that in the classical theory, nilsequences are often built from **polynomial orbits**

$$a(n) = F(p(n)\Gamma),$$

where $p : \mathbb{Z} \to N$ is a polynomial sequence in the sense of Leibman.

That extra richness comes from the fact that $\mathbb{Z}$ has a very natural additive structure, so "discrete polynomial in n" makes sense.

For a general countable amenable group G, **amenability alone does not give you a canonical notion of polynomial map** $G \to N$. So if you want a true analogue of polynomial nilsequences, you usually need to equip G with extra structure, such as:

- a filtration,
- a cubespace structure,
- or some other notion of higher-order differentiation.

Without that extra structure, the definition above only captures the "linear orbit" version.

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4. A good slogan

A good slogan is:

- **For arbitrary G:** nilsequence $=$ continuous function along an orbit in a G-nilsystem.
- **For G with extra algebraic structure:** one can also talk about polynomial nilsequences.

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5. A more flexible formulation

Sometimes it is better to define a G-nilsequence using a **pronilsystem** rather than a single nilmanifold:

$$a(g) = F(g \cdot x),$$

where (X, G) is an inverse limit of s-step nilsystems.

This is often the right topological-dynamical formulation if one wants a class closed under natural operations.

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6. The main limitation

If G is a very general amenable group, this notion may be **too small** to play all the roles that nilsequences play for $\mathbb{Z}$ or $\mathbb{Z}^d$. For example, for general amenable groups, the characteristic factors for multiple ergodic averages are not known in general to be controlled by ordinary nilsystems in the same way as in the $\mathbb{Z}$-theory.

So:

- as a **definition**, it is natural;
- as a **fully satisfactory structural theory**, it is much less settled.

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7. Bottom line

My answer would be:

> **Yes.** A natural notion is to call $a : G \to \mathbb{C}$ a basic G-nilsequence if
>
> $$a(g) = F(T_g x)$$
>
> for a continuous G-action by niltranslations on a compact nilmanifold (or pronilmanifold), and a G-nilsequence is a uniform limit of these.
>
> **However**, if you want the analogue of classical polynomial nilsequences, then for a general countable amenable group you need more than amenability; you need some extra algebraic/cubic structure on G.

If you want, I can next try to formulate a version that is best suited for **ergodic averages over amenable groups**, or a version specialized to **abelian amenable groups**.

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3. User

2026-03-16 10:47:26 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T02:47:26.288000Z

is the theory more fully developed if I assume that G is abelian?

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4. Assistant

2026-03-16 10:47:26 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T02:47:26.075245Z

Yes — **very much so.** If G is **abelian**, the theory becomes substantially more developed and behaves much closer to the classical $\mathbb{Z}$ theory.

The reason is that several key ingredients of the nilsequence/nilspace theory rely on having a **good notion of discrete derivatives**, which works naturally for abelian groups.

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1. Polynomial sequences make sense for abelian groups

If G is abelian, one can define **polynomial maps**

$$p : G \to N$$

into a filtered nilpotent Lie group N using discrete derivatives.

For $h \in G$, define the difference operator

$$\Delta_h p(g) := p(g+h)p(g)^{-1}.$$

Then p is polynomial of degree $\leq s$ if all $(s+1)$-fold iterated differences vanish in the appropriate level of the filtration.

This is exactly the framework used by **Leibman** for polynomial sequences.

Thus for abelian G one obtains **polynomial nilsequences**

$$a(g) = F(p(g)\Gamma).$$

This includes:

- $G = \mathbb{Z}$
- $G = \mathbb{Z}^d$
- any countable abelian group.

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2. Host–Kra / Gowers theory extends naturally to abelian groups

The higher-order uniformity norms $U^k(G)$ and the associated structure theory are well developed for **abelian groups**.

Important results include:

- inverse theorems for U^k,
- characteristic factors for multiple ergodic averages,
- description using nilsystems.

Key contributors include:

- **Host–Kra**
- **Ziegler**
- **Green–Tao**
- **Candela–Szegedy**
- **Gutman–Manners–Varjú**

Many results are formulated for **locally compact abelian groups** or **finite abelian groups**, and extend to countable abelian groups.

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3. Nilspace theory works particularly well for abelian actions

Modern formulations of the theory (nilspaces and cubes) treat **abelian group actions** very cleanly.

For example:

- polynomial maps $G \to X$ into nilspaces are defined via cubes,
- the theory of **Host–Kra factors** and **characteristic factors** extends naturally.

This viewpoint is developed in work of:

- **Camarena–Szegedy**
- **Candela–Szegedy**
- **Gutman–Manners–Varjú**

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4. Example: $G = \mathbb{Z}^d$

The theory is essentially as complete as the $\mathbb{Z}$ case.

Polynomial sequences look like

$$p(n_1, \dots, n_d) = g_1^{n_1} \cdots g_d^{n_d} g_{12}^{n_1 n_2} g_{111}^{n_1^3} \cdots$$

Nilsequences are

$$a(\mathbf{n}) = F(p(\mathbf{n})\Gamma).$$

These play the same role in:

- multidimensional Szemerédi theory
- polynomial ergodic averages
- Gowers norms.

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5. What breaks for non-abelian groups

For a general amenable group G:

- the difference operator Δ_h does not behave well,
- polynomial maps become much harder to define,
- the Host–Kra cube machinery becomes complicated.

This is why the theory for **general amenable groups is much less developed**.

6. References where the abelian theory appears

Polynomial sequences

Leibman
Polynomial sequences in groups
Journal of Algebra 2012

(no corresponding link found)

Earlier version: https://arxiv.org/abs/math/0210048

Nilsequences and ergodic theory

Host–Kra
Nilsequences and multiple ergodic averages
Advances in Mathematics (2005)

https://doi.org/10.1016/j.aim.2005.01.003

Nilspaces

Camarena–Szegedy
Nilspaces, nilmanifolds and their morphisms

https://arxiv.org/abs/1009.3825

Structure theory

Gutman–Manners–Varjú
The structure theory of nilspaces

https://arxiv.org/abs/1604.00664

7. Summary

Group G	**Nilsequence theory**
$\mathbb{Z}$	fully developed
$\mathbb{Z}^d$	essentially complete
countable abelian	mostly works
general amenable	largely open

So your intuition is correct:

> **If G is abelian, the nilsequence theory is much more complete and behaves similarly to the classical $\mathbb{Z}$ theory.**

If you're interested, there is also a **very elegant nilspace-based definition of nilsequences that works for *all abelian groups***, which avoids explicit polynomial formulas. I can explain that version if you'd like.

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5. User

2026-03-16 10:48:31 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T02:48:31.951000Z

why does the difference operator not behave well when you lose commutativity?

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6. Assistant

2026-03-16 10:48:31 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T02:48:31.692976Z

The short answer is that **in non-abelian groups the difference operators stop commuting**, and this breaks the whole discrete calculus that underlies the polynomial/nilsequence theory.

Let me explain precisely where the problem appears.

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1. The abelian case: discrete derivatives commute

Suppose G is abelian and $p : G \to N$ (where N is a filtered nilpotent group).
Define

$$\Delta_h p(g) = p(g+h)p(g)^{-1}.$$

Then we can take higher differences:

$$\Delta_{h_1}\Delta_{h_2}p(g)$$

and an important identity holds:

$$\Delta_{h_1}\Delta_{h_2}p = \Delta_{h_2}\Delta_{h_1}p.$$

So the difference operators **commute**.

Because of this:

- iterated derivatives are independent of order,
- the notion of "degree $\leq s$" is well defined,
- the derivative calculus behaves like ordinary polynomial calculus.

For example one can define

$$\Delta_{h_1} \cdots \Delta_{h_{s+1}} p \equiv e$$

as the definition of a polynomial of degree s.

This is exactly how Leibman defines polynomial maps on abelian groups.

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2. What goes wrong in non-abelian groups

If G is not abelian, the natural derivative is

$$\Delta_h p(g) = p(hg)p(g)^{-1}.$$

Now compute two derivatives.

First:

$$\Delta_k(\Delta_h p)(g) = p(khg)p(hg)^{-1}p(kg)p(g)^{-1}.$$

Second:

$$\Delta_h(\Delta_k p)(g) = p(hkg)p(kg)^{-1}p(hg)p(g)^{-1}.$$

These are **not equal unless** $hk = kh$.

So

$$\Delta_k \Delta_h p \neq \Delta_h \Delta_k p.$$

Thus the difference operators **do not commute**.

---

3. Why this destroys the polynomial definition

Because of the non-commutation:

- the value of an iterated difference depends on the **order of the derivatives**,
- the usual notion

$$\Delta_{h_1} \cdots \Delta_{h_{s+1}} p \equiv e$$

is ambiguous.

In fact there are many possible “degrees” depending on which order you use.

So the elegant definition of polynomial maps via discrete derivatives **breaks down**.

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4. Another way to see the problem

In the abelian case, differences behave like additive derivatives:

$$\Delta_h(f + g) = \Delta_h f + \Delta_h g.$$

In the non-abelian case the group operation introduces **commutator errors**.

Higher differences produce expressions involving nested commutators such as

$$[p(hg), p(kg)].$$

This rapidly becomes complicated.

---

5. Why the theory still works for $\mathbb{Z}$

For $G = \mathbb{Z}$, all derivatives are taken along the **same direction**, so no commutation issues arise.

We only ever use

$$\Delta_1^k p.$$

Thus the non-abelian problem does not appear.

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6. How Leibman solves this for general groups

Leibman introduces a more complicated notion:

A map $p : G \to N$ is polynomial if

$$\Delta_{g_1} \cdots \Delta_{g_k} p$$

lands in progressively deeper levels of a **filtration** of N.

This definition:

- keeps track of the **order of derivatives**,
- uses the nilpotent filtration to absorb commutator errors.

But the theory becomes much more technical.

Reference:

Leibman
Polynomial sequences in groups
J. Algebra 2012
https://arxiv.org/abs/math/0210048

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7. Intuition

The key intuition is:

Setting	Derivatives
abelian G	commute
non-abelian G	produce commutators

Polynomial calculus requires commuting derivatives, so losing commutativity forces a much more complicated structure.

✅ **Summary**

The difference operator behaves poorly in non-abelian groups because

$$\Delta_k \Delta_h \neq \Delta_h \Delta_k,$$

so higher discrete derivatives depend on their order.
This destroys the simple derivative-based definition of polynomial maps that works for abelian groups.

If you'd like, I can also show you a **very concrete example where the second difference of a map depends on derivative order**, which makes the failure extremely transparent.

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7. User

2026-03-16 12:16:24 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T04:16:24.552000Z

can you find me the title of the paper in which furstenberg proves his structure theorem for distal systems?

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8. Assistant

2026-03-16 12:16:28 (Asia/Shanghai, UTC+08:00)
Source timestamp: 2026-03-16T04:16:28.447107Z

Yes. The paper where Furstenberg proved the structure theorem for distal systems is:

H. Furstenberg — "The structure of distal flows."
American Journal of Mathematics, **85** (1963), 477–515. The Hebrew University of Jerusalem

You can access it here:
https://doi.org/10.2307/2373137

What the paper proves

In that paper Furstenberg proves the **structure theorem for minimal distal flows**, showing that:

- Every **minimal distal flow** can be obtained by a **transfinite tower of isometric (equicontinuous) extensions** starting from a trivial one-point system. Department of Mathematics
- In modern language, minimal distal systems are exactly **PI-systems** (inverse limits of iterated equicontinuous extensions).

This theorem became one of the foundational results of topological dynamics and inspired later structure theories such as:

- the **Furstenberg–Zimmer structure theorem** in ergodic theory,

- the **Host–Kra–Ziegler theory** for characteristic factors,
- the structural theory of minimal flows.

If you want, I can also tell you the **best modern references where the theorem is explained in detail** (for example in Auslander–Glasner or Glasner's book), which are often easier to read than the original 1963 paper.