# On the 2-large is large conjecture

Yau postdoc seminar

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# Van der Waerden's Theorem and $r$-large sets

### Theorem (van der Waerden [15], 1927)

*In any finite partition* $\mathbb{N} = \bigsqcup_{i=1}^{r} C_i$, *there exists a* $1 \leq i_0 \leq r$ *for which* $C_{i_0}$ *contains arbitrarily long arithmetic progressions.*

### Definition

Given $r \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the set $D$ is $r$-**large** if for any finite partition $\mathbb{N} = \sqcup_{i=1}^{r} C_i$, there exists $1 \leq i_0 \leq r$ for which $C_{i_0}$ contains arbitrarily long arithmetic progressions with common difference in $D$. More concretely, for any $\ell \in \mathbb{N}$ there exists $a \in \mathbb{N}$ and $d \in D$ for which $\{a + id\}_{i=0}^{\ell} \subseteq C_{i_0}$. If $D$ is $r$-large for all $r \in \mathbb{N}$, then $D$ is **large**.

### Conjecture (Brown, Graham, Landman [4], 1999)

*If* $D \subseteq \mathbb{N}$ *is 2-large, then it is large.*

# Examples

The following examples of sets are known to be large.

1. $n\mathbb{N}$ for some $n \in \mathbb{N}$ (Exercise).
2. $p(\mathbb{N})$ where $p \in x\mathbb{N}[x]$.
3. $D$ is an IP set, i.e., there exists $(x_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ such that for any finite set $F \subseteq \mathbb{N}$ we have $\sum_{n\in F} x_n \in D$.
4. $p(D)$ where $D$ is an IP set and $p \in x\mathbb{N}[x]$.

The following examples of sets are known to not be 2-large.

1. $D_{\alpha,\epsilon} := \{n \mid n\alpha \pmod 1 \in (\epsilon, 1-\epsilon)\}$, with $\alpha \in \mathbb{R}$ and $\epsilon > 0$ (see [13]).
2. Lacunary sequences, i.e., $D = \{d_n\}_{n=1}^{\infty}$ with $\inf_{n\geq 1} \frac{d_{n+1}}{d_n} > 1$ (see [13, 5]).
3. $D$ is a finite union of lacunary sequences (see [6]).

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# Dynamical van der Waerden

## Definition

A **topological dynamical system** is a pair $(X, T)$ with $X$ a compact metric space and $T : X \to X$ a continuous transformation. The system is **minimal** if for every $x \in X$ the orbit $\{T^n x\}_{n=1}^{\infty}$ is dense. Equivalently, the system is minimal if the only $T$-invariant closed sets are $\emptyset$ and $X$.

The following topological version of van der Waerden's Theorem follows from the Furstenberg-Weiss correspondence principle [8, 9].

## Theorem (Topological van der Waerden)

*Let $(X, T)$ be a minimal topological dynamical system. For any open set $\emptyset \neq U \subseteq X$ and any $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which*

$$U \cap T^{-n}U \cap \cdots \cap T^{-n\ell}U \neq \emptyset. \tag{1}$$

# Dynamical formulation of $r$-largeness

**Lemma**

*The set $D \subseteq \mathbb{N}$ is $r$-large if and only if for any topological dynamical system $(X, T)$, any open cover $\{U_i\}_{i=1}^r$ of $X$, and any $\ell \in \mathbb{N}$ there exists $d \in D$ and $1 \leq i_0 \leq r$ for which*

$$U_{i_0} \cap T^{-d} U_{i_0} \cap \cdots \cap T^{-\ell d} U_{i_0} \neq \emptyset. \tag{2}$$

Yet another characterization of minimal topological dynamical systems is that for any $\neq U \subseteq X$, there exists $n$ for which $X \subseteq \bigcup_{i=1}^n T^{-i} U$. Let us (informally) call an open set $U \subseteq X$ $r$-syndetic if there exists $\{n_i\}_{i=1}^r$ for which $X \subseteq \bigcup_{i=1}^r T^{-n_i} U$. We now obtain another characterization of $r$-large sets as those sets for which Equation (2) holds for any $r$-syndetic open set of any minimal system, for some $d \in D$.

# 2-large sets are Bohr recurrent 1/2

In 2016 I conjectured [5] that 2-large sets are Bohr recurrent, and recently Ryan Alweiss proved my conjecture.

**Theorem (Alweiss [1], 2025+)**

*For any* $\vec{\alpha} = (\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k$ *and* $\epsilon > 0$*, the set* $D_{\vec{\alpha},\epsilon} := \{n \mid n\alpha_i \pmod 1 \in (\epsilon, 1-\epsilon) \text{ for all } i\}$ *is not 2-large.*

**Proof sketch:** Given $m \in \mathbb{N}$, there exists $N = N(m) \in \mathbb{N}$ such that for any $\vec{\beta} \in \mathbb{R}^k$, the orbit $\{n\vec{\beta} \pmod 1\}_{n=1}^N$ in $\mathbb{T}^k$ is within $m^{-2}$ of a sequence that is $m^{-1}$-equidistributed on a subtorus of $\mathbb{T}^k$ and some of its rational translates. The set of possible translates of the trivial subtorus is bounded in size by $M = n^{O(1)}$, so we consider rotation by $\vec{\beta} = \frac{1}{M!}\vec{\alpha}$. We take $m > M!\epsilon^{-1}$, and it follows that the orbit $\{n\vec{\alpha}\}_{n=1}^N$ must be close to being equidistributed on a finite union of nontrivial subtori.

# 2-large sets are Bohr recurrent 2/2

**Proof sketch continued:** Since the subtori are nontrivial, each of them intersects at least a linear number of squares in the grid on $\mathbb{T}^k = [0, 1)^k$ consisting of squares of sidelength $m^{-1}$. It follows that a randomly coloring almost surely yields the desired result. An equidistributed orbit on a subtori will visit linearly many squares, so there is an exponentially small probability that all of the squares are the same color. There are only polynomially many in $m$ possible subtori for a given $N = N(m)$.

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# Nilmanifolds and nilsystems

A **nilmanifold** is a homogeneous space $G/\Gamma$ where $G$ is a finite dimensional real nilpotent Lie group and $\Gamma \leq G$ is a discrete cocompact subgroup (lattice). For convenience, we assume that $G$ is connected and simply connected. $G$ acts on $G/\Gamma$ on the left by translation, and for any fixed $g \in G$ the system $(G/\Gamma, T_g)$ given by $T_g h\Gamma = gh\Gamma$ is a **nilsystem** and $T_g$ is a **nilrotation**. Nilrotations can be seen as generalizations of toral rotations, as they are also dynamical systems strongly controlled by algebra. Recently, nilsystems have played a crucial role in the understanding of multiple recurrence in ergodic Ramsey theory [12, 7, 3, 10, 11].

# The Heisenberg Nilmanifold 1/2

The simplest nontrivial example of a nilmanifold is the Heisenberg nilmanifold. In this case, $G$ is the space of real upper triangular 3 by 3 matrices with 1s on the diagonal, and $\Gamma$ is the subgroup of such matrices with integer entries above the diagonal. We see that every element of $G/\Gamma$ has a representative in which all entries above the diagonal are in $[0, 1)$, and this gives us a fundamental domain for $G/\Gamma$ in $G$.

# The Heisenberg Nilmanifold 2/2

Using this fundamental domain (i.e., $x, y, z \in [0, 1)$), we obtain the following formula for niltranslations:

$$\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & y\\ 0 & 1 & z\\ 0 & 0 & 1\end{pmatrix}\Gamma = \begin{pmatrix}1 & a+x & y+az+b\\ 0 & 1 & c+z\\ 0 & 0 & 1\end{pmatrix}\Gamma$$

$$= \begin{pmatrix}1 & a+x \pmod 1 & y+az+b-(a+x)\lfloor c+z\rfloor \pmod 1\\ 0 & 1 & c+z \pmod 1\\ 0 & 0 & 1\end{pmatrix}\Gamma$$

# Properties of nilrotations 1/2

**Theorem**

*For a nilrotation $(X, T)$, the following are equivalent:*

1. *There exists $x \in X$ for which $\{T^n x\}_{n=1}^{\infty}$ is dense.*
2. *$(X, T)$ is minimal.*
3. *$(X, T)$ is uniquely ergodic, i.e., there exists a unique $T$-invariant Borel probability measure supported on $X$.*

# Properties of nilrotations 2/2

**Theorem ( [13, 14])**

*If $(X, T)$ is a s-step minimal nilsystem and $R \subseteq \mathbb{N}$, then the following are equivalent:*

1. *For any $x \in X$ and any $\epsilon > 0$, there exists $r \in R$ with $d(T^r x, x) < \epsilon$.*
   - *$R$ is a set of pointwise recurrence*
2. *For any open $\emptyset \neq U \subseteq X$ there exists $r \in R$ with $U \cap T^{-r}U \cap \cdots \cap T^{-sr}U \neq \emptyset$.*
   - *$R$ is a set of s-recurrence*
3. *For any open $\emptyset \neq U \subseteq X$ and any $\ell \in \mathbb{N}$ there exists $r \in R$ with $U \cap T^{-r}U \cap \cdots \cap T^{-\ell r}U \neq \emptyset$.*
   - *$R$ is a set of multiple recurrence*

# A Hierarchy of minimal systems

In 2016 [14] it was asked whether or not sets of multiple recurrence for nilrotations are sets of multiple recurrence for all minimal systems. It was only recently in 2025+ that Ryan Alweiss [2] constructed a set of multiple recurrence for nilrotations that is not a set of 2-recurrence for distal systems. The following is a hierarchy of classes of minimal topological dynamical systems of interest. It starts from the most structured systems and becomes progressively more complex.

1. The one point system
2. Toral rotations
3. nilrotations
4. distal systems
5. point distal systems
6. zero entropy systems
7. all systems

# 2-large sets are (probably) NilBohr recurrent

**Theorem (Farhangi, Alweiss, 2026+)**

*If $D \subseteq \mathbb{N}$ is 2-large, then it is a set of multiple recurrence for nilrotations.*

Difficulties that we need to overcome that are not present when dealing with the torus:

1. When $\mathbb{R}$ acts on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, it is the same as $\mathbb{T}$ acting on $\mathbb{T}$, so it is a compact action. When $G$ acts on $G/\Gamma$, it is not a compact action.
2. $\mathbb{T}$ acts on $\mathbb{T}$ by isometries, while the action of $G$ on $G/\Gamma$ is highly non-isometric.

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