Parreau Systems, the dual to strong mixing

YMSC postdoc seminar

Sohail Farhangi (Based on joint work with Joel Moreira and Rigoberta Zelada [6])
Slides available on sohailfarhangi.com

April 18-19, 2026

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Measure-preserving systems and factors

A **measure-preserving system (m.p.s.)** consists of a tuple $(X, \mathscr{B}, \mu, T)$, where $(X, \mathscr{B}, \mu)$ is standard probability space, and $T : X \to X$ is a measurable transformation for which $\mu(A) = \mu(T^{-1}A)$ for all $A \in \mathscr{B}$. We will use $\mathcal{X}$ to denote the tuple $(X, \mathscr{B}, \mu, T)$.

Given another m.p.s. $\mathcal{Y} := (Y, \mathscr{A}, \nu, S)$, we say that $\mathcal{Y}$ is a **factor** of $\mathcal{X}$ if there exists a measurable **factor map** $\pi : X \to Y$ satisfying the following: 1) $\mu(\pi^{-1}A) = \nu(A)$ for all $A \in \mathscr{A}$. 2) $\pi \circ T = S \circ \pi$.

There is a 1-to-1 correspondence between factors of $\mathcal{X}$ and T-invariant C*-sub-algebras of $L^\infty(X, \mu)$.

Examples of systems

1) Given a $\alpha \in \mathbb{R}$, we consider the system $\mathcal{X}_\alpha := ([0,1], \mathscr{L}, m, T)$, where m is the normalized Lebesgue measure and $T(x) = x + \alpha \pmod 1$. The system $\mathcal{X}_{2\alpha}$ is a factor of $\mathcal{X}_\alpha$ via the factor map $\pi : [0,1] \to [0,1]$ given by $\pi(x) = 2x \pmod 1$. These systems are known as circle rotations.

2) The unique Bernoulli shift of entropy ln(2) can be represented as $\mathcal{B}_{\ln(2)} := (\{0,1\}^{\mathbb{N}}, \mathscr{B}, \mu_{\frac{1}{2}}^{\mathbb{N}}, S)$, where $\mu_{\frac{1}{2}}$ is the normalized counting measure on $\{0,1\}$, and S is the left shift given by $S(x_n)_{n=1}^{\infty} = (x_{n+1})_{n=1}^{\infty}$.

3) The system $([0,1], \mathscr{L}, m, T_{\times 2})$ in which $T_{\times 2}(x) = 2x \pmod 1$ is isomorphic to $\mathcal{B}_{\ln(2)}$.

Joinings and disjointness

Given systems $\mathcal{X}$ and $\mathcal{Y}$, a **joining** is a $(T \times S)$-invariant probability measure λ on $(X \times Y, \mathscr{B} \otimes \mathscr{A})$, which also satisfies $(\pi_X)_*\lambda = \mu$ and $(\pi_Y)_*\lambda = \nu$.

In this case, the system $\mathcal{X} \vee_\lambda \mathcal{Y} := (X \times Y, \mathscr{B} \otimes \mathscr{A}, \lambda, T \times S)$ is a system that contains both $\mathcal{X}$ and $\mathcal{Y}$ as factors, and is "generated" by these two factors.

The systems $\mathcal{X}$ and $\mathcal{Y}$ are **disjoint** if their only joining is the product measure $\mu \otimes \nu$.

Examples of joinings and disjointness

A nontrivial system $\mathcal{X}$ is never disjoint from itself, because the diagonal measure on X^2 is a nontrivial joining.

For any $\alpha \in \mathbb{R}$, the systems $\mathcal{X}_\alpha$ and $\mathcal{X}_{2\alpha}$ are not disjoint, as we can take the normalized Lebesgue measure on the graph of $y = 2x \pmod 1$ as a joining.

Every circle rotation is disjoint from every Bernoulli shift.

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Some parts of the ergodic hierarchy of mixing

The system $\mathcal{X}$ is **ergodic** if there are no nontrivial T-invariant sets. More concretely, the only $A \in \mathscr{B}$ for which $\mu(A \triangle T^{-1}A) = 0$ are those for which $\mu(A) \in \{0, 1\}$. Equivalently, $\mathcal{X}$ is ergodic if for every $A \in \mathscr{B}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A) = \mu(A)^2. \tag{1}$$

The system $\mathcal{X}$ is **strongly mixing** if for every $A \in \mathscr{B}$ we have

$$\lim_{n \to \infty} \mu(A \cap T^{-n}A) = \mu(A)^2. \tag{2}$$

Classical dichotomies

If $\mathcal{X}$ is a structured system from the table below, and $\mathcal{Y}$ is a mixing system from the same row of the table, then $\mathcal{X}$ and $\mathcal{Y}$ are disjoint.

Structured	**Mixing**
identity	ergodic
Kronecker (eigenfunctions)	weak mixing
rigid	mild mixing
Parreau	strong mixing
singular spectrum	Lebesgue spectrum
zero entropy	K-mixing (completely positive entropy)

We want to show that Parreau systems are the correct class of structured systems to complement strong mixing.

The mean ergodic decomposition

Given a system $\mathcal{X}$, we $L^2(X,\mu) = \mathcal{H}_I \oplus \mathcal{H}_e$, where

$$\mathcal{H}_I = \{\xi \in L^2(X,\mu) \mid U_T\xi = \xi\}, \text{ and}$$
$$\mathcal{H}_e = \overline{\text{Span}(\{U_T\xi - \xi \mid \xi \in L^2(X,\mu)}.$$

Since $\mathcal{H}_I \cap L^\infty(X,\mu)$ is a C^*-algebra, it corresponds to a factor $\mathcal{X}_I$ of $\mathcal{X}$ on which T acts as the identity map. It follows that $\mathcal{X}$ is ergodic if and only if the factor $\mathcal{X}_I$ is trivial.

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Parreau systems

The system $\mathcal{X}$ is **Parreau** if there exists an increasing sequence $(n_k)_{k=1}^{\infty}$ for which $U_T^{n_k}$ weakly converges on $L^2(X,\mu)$ to an operator V, such that the algebra generated by $\{Vf \mid f \in L^\infty(X,\mu)\}$ is dense in $L^2(X,\mu)$.[1] Informally, this says that the factor generated by V is all of $\mathcal{X}$.

Fraçois Parreau proved the following result.

Theorem ([7, 8])

The system $\mathcal{X}$ is strongly mixing if and only if it has no Parreau system as a factor. Furthermore, every Parreau system is disjoint from every strongly mixing system.

[1] We slightly simplified the definition for this presentation.

Partially rigid systems

Given $\alpha \in (0, 1]$, the system $\mathcal{X}$ is α-**partially rigid** if there exists a sequence $(n_k)_{k=1}^{\infty}$ such that for every $A \in \mathscr{B}$ we have $\liminf_{k\to\infty} \mu(A \cap T^{-n}A) \geq \alpha\mu(A)$. Examples of partially rigid systems include ergodic Interval Exchange transformations [3, 9], minimal substitution systems [4, 5], systems of quasi-exact finite rank [3], many Bratelli-Vershik systems [3, 2], many Toeplitz systems [2], and many enumeration systems [2]. Furthermore, a generic system $\mathcal{X}$ is partially rigid [10].[2]

[2] In fact, a generic system is rigid [1].

Partially rigid systems and Parreau systems

Theorem (F., Moreira, Zelada 2026+)

Every partially rigid system is a finite extension of a Parreau system.

Roughly speaking, this means that for every partially rigid system $\mathcal{X}$, there exists a Parreau system $\mathcal{Y}$ and a $n \in \mathbb{N}$ for which $\mathcal{X}$ is isomorphic to a skew product system $(Y \times \{1, 2, \cdots, n\}, \mathscr{A}', \nu \otimes m_n, S \rtimes \phi)$.

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