UNDECIDABILITY IN THE RAMSEY THEORY OF POLYNOMIAL EQUATIONS

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Our goal is to classify the computability of the sets of polynomials with certain Ramsey theoretic properties. For that, we will first introduce the notion of density regularity, the Lightface Hierarchy and Hilbert's 10th problem. After the statement and proof of the main result, we will also give an application of this classification, which is a bridge between sets of polynomials of Ramsey theoretic interest and sets of polynomials with roots in some fixed set.

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Contents

1. Basic notions 2
2. Density Regularity 3
3. Hilbert's 10th Problem 4
4. The Lightface Hierarchy 5
5. Main Result 7
6. Partition Regularity 8
7. Proof of the Main Result 9
8. Bridge between $\mathrm{IADR}_R(\delta) \cap \mathrm{T}_R$ and roots in K 10

References 11

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[1] `https://sites.google.com/view/isem28/isem-format`

1. Basic notions

For $A \subseteq \mathbb{N}$, the upper density is given by:

$$\bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap \{1, 2, \ldots, N\}|}{N}.$$

Roughly speaking, Density Ramsey Theory is the study of what structures can be found in sets $A \subseteq \mathbb{N}$ satisfying $\overline{d}(A) > 0$.

Definition. Let $(S, +)$ be a commutative semigroup.

- S is called *cancellative* if for all $a, b, c \in S$, we have
$$a + b = a + c \Rightarrow b = c.$$
- The set of finite subsets is denoted as
$$\mathcal{P}_f(S) := \{A \subseteq S \mid |A| < \infty\}.$$
- A *Følner sequence* $\mathcal{F} = (F_n)_{n\geq 1} \subseteq \mathcal{P}_f(S)$ satisfies:
$$\lim_{n\to\infty} \frac{|(s + F_n)\Delta F_n|}{|F_n|} = 0 \quad \forall s \in S,$$
where Δ denotes the symmetric difference.

Now, let $\mathcal{F}$ be a Følner sequence and $A \subseteq S$.

- The $\mathcal{F}$-*upper density* of A is defined as
$$\overline{d}_{\mathcal{F}}(A) := \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$
- If the limit exists, we define the $\mathcal{F}$-*density* of A as
$$d_{\mathcal{F}}(A) := \lim_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$
We also define the *upper Banach density* of A as
$$d^*(A) := \sup\{\overline{d}_{\mathcal{F}}(A) \mid \mathcal{F} \text{ is a Følner sequence}\}.$$

Remark. In commutative semigroups, there exists a Følner sequence, so the upper Banach density is well-defined.

When we work with an integral domain $(R, +, \cdot)$, we define

$$d^* := \text{upper Banach density for } (R, +),$$

and

$$d^*_\times := \text{the upper Banach density in the semigroup } (R \setminus \{0\}, \cdot).$$

Definition (Measure-preserving S-systems)**.** Let $(S, +)$ be a commutative, cancellative and countable semigroup. An S-*system* is a tuple $(X, \mathcal{B}, \mu, (T_s)_{s\in S})$, where:

- $(X, \mathcal{B}, \mu)$ is a probability space.
- For all $s \in S$ the map T_s is measurable.

- For all $A \in \mathcal{B}$ one has and $\mu(A) = \mu(T_s^{-1}(A))$.
- For all $s, t \in S$, we have $T_s T_t = T_{s+t}$.
- If S has an identity element e, then $T_e = \mathrm{id}$.

An S-system $(X, \mathcal{B}, \mu, (T_s)_{s \in S})$ is *ergodic* if for $A \in \mathcal{B}$, we have:

$$\mu(T_s^{-1} A \Delta A) = 0 \quad \forall s \in S \;\Rightarrow\; \mu(A) \in \{0, 1\}\,.$$

2. Density Regularity

Definition. Let $(S, +)$ be a commutative, cancellative and countable semigroup. A collection $\mathcal{A} \subseteq \mathcal{P}_f(S)$ is *translation invariant* if for all $A \in \mathcal{A}$ and $s \in S$, we have

$$s + A \in \mathcal{A}.$$

Definition ((weakly) δ-density regular)**.** Let S be a commutative, cancellative and countable semigroup. Given $\delta \in [0, 1)$, the collection $\mathcal{A}$ is *weakly δ-density regular* if:

$$\forall B \subseteq S \text{ with } d^*(B) > \delta,\ \exists A \in \mathcal{A} \text{ such that } A \subseteq B.$$

The collection $\mathcal{A}$ is *δ-density regular* if:

$$\forall B \subseteq S \text{ with } d^*(B) \geq \delta, \exists A \in \mathcal{A} \text{ such that } A \subseteq B.$$

If $\mathcal{A}$ is weakly 0-density regular, then we say $\mathcal{A}$ is *density regular*.

Example. The collection $\mathcal{A} = \{\{x, y, z\} \mid x + y = z\} \subseteq \mathcal{P}(\mathbb{Z})$ is weakly $\frac{1}{2}$-density regular but not $\frac{1}{2}$-density regular.

Proof. Let $B \subseteq \mathbb{Z}$ satisfy

$$d^*(B) > \frac{1}{2}\,,$$

then there is a Følner sequence $\mathcal{F}$ such that we can take an arbitrary $x \in B$ and will have

$$d_{\mathcal{F}}(B - x) = d_{\mathcal{F}}(B) > \frac{1}{2}\,.$$

This implies $d_{\mathcal{F}}(B \cap (B - x)) > 0$, because otherwise $d_{\mathcal{F}}(B \cup (B - x))$ would be greater than 1. Let $y \in B \cap (B - x)$ be arbitrary and observe that $z := x + y \in B$. So we have $x, y, z \in B$ and thus $\{x, y, z\} \in \mathcal{A}$. Hence, B is weakly $\frac{1}{2}$-density regular.

However, we see that $d^*(2\,\mathbb{Z} + 1) = \frac{1}{2}$ and $2\,\mathbb{Z} + 1$ does not contain any member of A, so B is not $\frac{1}{2}$-density regular. □

We give another example with translation invariance.

Example. Let $\mathcal{A} = \{\{x, y\} \mid x - y \equiv 1 \pmod 2\}$. We see that $\mathcal{A}$ is translation invariant in $(\mathbb{Z}, +)$. Similar to the first example, let $B \subseteq \mathbb{Z}$ such that

$$d^*(B) > \frac{1}{2}$$

holds, then there is a Følner sequence $\mathcal{F}$ for which we have $d_{\mathcal{F}}(B + 1) = d_{\mathcal{F}}(B) > \frac{1}{2}$, which implies $d_{\mathcal{F}}(B \cap (B + 1)) > 0$. Let $x \in B \cap (B + 1)$

be arbitrary, then $\{x, x-1\} \in \mathcal{A}$. However, $2\mathbb{Z}+1$ does not contain any member of $\mathcal{A}$.

2.1. **The Conversion Lemma.**

Later in the talk, we will discuss the complexity of sets of polynomials with a root in every set with positive density. The following lemma will allow us to establish a correspondence between the polynomials over R that have a root in the field of fractions K with the polynomials that are density regular over R.

Lemma 1. *Let R be a countably infinite integral domain with the field of fractions K. For any m and any $k_1, \dots, k_m \in K$, we have the following:*

(i) If $A \subseteq R$ is such that $d^*(A) > 0$, then A contains a solution to the system of equations:

$$\frac{z_{4i-3} - z_{4i-2}}{z_{4i-1} - z_{4i}} = k_i, \quad \forall i \in \{1, \dots, m\}.$$

Furthermore, the solution can be taken such that $z_i \neq z_j$ for $i \neq j$.

(ii) If $A \subseteq R \setminus \{0\}$ is such that $d^*_\times(A) > 0$, then A contains a solution to system in (i) such that $z_i \neq z_j$ for $i \neq j$.

[1, Lemma 2.15]

2.2. **Characterization of density regularity.**

Theorem (Characterization of density regularity)**.** *Let $(S,+)$ be a countably infinite cancellative commutative semigroup, G be the group of differences of S, $\mathcal{F} = (F_n)_{n\geq 1}$ be a Følner sequence in S, $\delta \in (0,1)$ and $\mathcal{A} \subseteq \mathcal{P}_f(S)$ be translation invariant. Then the following statements are equivalent:*

(i) $\mathcal{A}$ is δ-density regular.

(ii) For all $B \subseteq S$ with $\overline{d}_\mathcal{F}(B) \geq \delta$, there exists $A \in \mathcal{A}$ such that $A \subseteq B$.

(iii) For any ergodic G-system $(X, \mathcal{B}, \mu, (T_s)_{s\in G})$ and any $B \in \mathcal{B}$ with $\mu(B) \geq \delta$, there exists an $A \in \mathcal{A}$ with

$$\mu\left(\bigcap_{a\in A} T_a^{-1} B\right) > 0.$$

[1, Theorem 3.5]

3. Hilbert's 10th Problem

In 1902, David Hilbert stated the following problem.

Question (Hilbert's 10th Problem)**.** Is the set of polynomials $p \in \mathbb{Z}[x_1, \dots, x_n]$ with an integer root computable?

Here, *computable* means that there is an algorithm that can determine in finite time whether an object is an element of the set or not.

Answer. No: In 1971 Matiyasevich, drawing on earlier work of Davis, Putnam and Robinson showed that such an algorithm does not exist by relating this problem to the halting problem.

However, it's easy to find an algorithm which stops after finite time *if* p has an integer root: We just have to check for each $z \in \mathbb{Z}$ whether $p(z) = 0$. However, if p does not have an integer root, then this algorithm will not stop.

We will now introduce the so-called Lightface Hierarchy which shall put the ending-after-finite-time-ness in precise terms.

4. The Lightface Hierarchy

4.1. First Layer.

Definition ($\Delta_1^0(\mathbb{Z})$)**.** The lowest position in the Lightface Hierarchy is denoted by $\Delta_1^0(\mathbb{Z})$ and consists of computable sets.

There are a few examples.

- $\emptyset$ and $\mathbb{Z}$ are trivially in $\Delta_1^0(\mathbb{Z})$.
- All finite sets are in $\Delta_1^0(\mathbb{Z})$.
- The set of square numbers is in $\Delta_1^0(\mathbb{Z})$ because for a given $n \in \mathbb{Z}$, one has only have to check for $m \in \{0, \dots, n\}$ whether $m^2 = n$.
- The set of prime numbers is in $\Delta_1^0(\mathbb{Z})$.

4.2. Generalization of computable sets.

We now gerneralize this definition for other domains than $\mathbb{Z}$: Given a set S and a computable bijection $\phi\colon \mathbb{Z} \to S$, one can define $\Delta_1^0(S)$ as those $A \subseteq S$ satisfying $\phi^{-1}(A) \in \Delta_1^0(\mathbb{Z})$. Because of this identification, we will denote $\Delta_1^0(S)$ by Δ_1^0 from now on.

Remark. Usually, the Lightface Hierarchy is first defined on $\mathbb{N}$ and then generalized to $\mathbb{Z}$ in this way.

4.3. Second Layer.

As discussed earlier, one can say that an algorithm is Δ_1^0 if it stops after finite time. Now, given a set, what is the next lowest level of complexity after computability that it can have?

Answer. Given an element, it could at least stop in finite time *if* the object is in the set. Conversely, it could stop in finite time if the object is *not* in the set.

We will call the first set Σ_1^0 and the second set Π_1^0. Using the same procedure as for Δ_1^0, one can similary generalize those sets to arbitrary (computable) domains.

Remark. The three sets are closely connected: One has $\Delta_1^0 = \Sigma_1^0 \cap \Pi_1^0$, and if A is in Σ_1^0, if and only if A^c is in Π_1^0.

We already mentioned some examples for Σ_1^0:

- The set of polynomials with an integer root is Σ_1^0.
- The set of algorithm-input-pairs whose algorithm halts on its input is $\Sigma_1^0 \setminus \Delta_1^0$: Just plugging the input into the algorithm indeed ends in finite time if the algorithm halts on the input. This is called the *Halting Problem.*

4.4. **Further Layers.**

For an intuition of the definition of further layers of the Lightface Hierarchy, we first might ask what kinds of sets are Σ_1^0 or Π_1^0: If A is Σ_1^0, then it has the form of

$$A = \{a \in * \mid \exists x \in * \text{ such that a computable condition holds for } x \text{ and } a\},$$

where $*$ is a placeholder for arbitrary computable sets, which do not have to be the same. Here, a computable condition is a formula whose output can be calculated in finite time. Conversely, a set B of Π_1^0 has the form of

$$\begin{aligned} B &= \{b \in * \mid \neg\exists x \in * \text{ such that a computable condition holds for } x \text{ and } b\} \\ &= \{b \in * \mid \forall x \in * \text{ such that a computable condition does } \textit{not} \text{ hold for } x \text{ and } b\} \\ &= \{b \in * \mid \forall x \in * \text{ such that } \textit{another} \text{ computable condition holds for } x \text{ and } b\}. \end{aligned}$$

The intuition for Σ_2^0 is now that a set C is Σ_2^0 if

$$C = \{c \in * \mid \exists x \in * \forall y \in * \text{ such that a computable condition holds for } x, y \text{ and } c\}$$

and a set D is Σ_3^0 if

$$D = \{d \in * \mid \exists x \in * \forall y \in * \exists z \in * \text{ such that a computable condition holds for } x, y, z \text{ and } d\}.$$

Conversely, a set E is Π_2^0 if

$$E = \{e \in * \mid \forall x \in * \exists y \in * \text{ such that a computable condition holds for } x, y \text{ and } e\}.$$

This extends by induction.

Definition. We set $\Delta_n^0 \coloneqq \Sigma_n^0 \cap \Pi_n^0$.

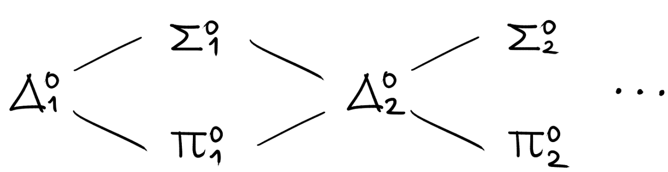

FIGURE 1. Structure of the Lightface Hierarchy.

4.5. Completeness and Universality.

$*$-Completeness means that $*$ is the narrowest category it fits in:

Definition ($*$-complete)**.** A set A is called Σ^0_n-*complete* if it is an element of $\Sigma^0_n \setminus \Delta^0_n$. Similary, A is called Π^0_n-*complete* if it is an element of $\Pi^0_n \setminus \Delta^0_n$.

$*$-Universality means that the set is as complex as $*$-sets can be.

Definition ($*$-universal)**.** A set A is $*$-*universal* if all $B \in *$ are *computably reducible* to A, i. e. there is a computable $\phi\colon \mathbb{Z} \to \mathbb{Z}$ with $n \in B \Leftrightarrow \phi(n) \in A$.

5. Main Result

The main result is about the hierarchy position of the following sets.

Definition (IADR_R and IMDR_R)**.** Let $p = (p_1, \dots, p_n) \in R[x_1, \dots, x_n]$ be a polynomial. It is in $\mathrm{IADR}_R(\delta)$ if every $B \subseteq R$ with $d^*(B) > \delta$ contains an *injective root* of p, i. e. there are $x_1, \dots, x_n \in B$ such that all x_i are different from one another, with $p(x_1, \dots, x_n) = 0$. We set

$$\mathrm{IADR}_R \coloneqq \bigcap_{\delta>0} \mathrm{IADR}_R(\delta)\,.$$

Similary, p is $\mathrm{IMDR}_R(\delta)$ if every $B \subseteq R$ with $d^*_\times(B) > \delta$ contains an injective root of p, and IMDR_R is defined as

$$\mathrm{IMDR}_R \coloneqq \bigcap_{\delta>0} \mathrm{IMDR}_R(\delta)\,.$$

Before we can state the main result, we shortly need to generalize Hilbert's 10th Problem on integral domains.

Definition (Hilbert's 10th Problem – on R)**.** Let R be a computable integral domain. Given a polynomial $p \in R[x_1, \dots, x_n]$, is there an algorithm that can determine in finite time whether p has a root in R? If so, Hilbert's 10th Problem is called *decidable* for R, if not, is called *undecidable* for R.

One can generalize this definition to any subset $A \subseteq R$ by requiring the root to be in A instead of R.

We are now able to state one of the main results: To precisely position the density sets introduced in the beginning of this section into the Lightface Hierarchy.

Theorem 2 (Density Results)**.** *Let R be a computable integral domain, K its field of fractions and let Hilbert's 10th Problem for K be undecidable. Then we have:*

- *(i) For each $\delta > 0$, $\mathrm{IMDR}_R(\delta) \cap \mathrm{H}_R$ and $\mathrm{IADR}_R(\delta) \cap \mathrm{T}_R$ are Σ^0_1-complete and even Σ^0_1-universal.*
- *(ii) The set $\mathrm{IMDR}_R \cap \mathrm{H}_R$ is Π^0_2-complete and even Π^0_2-universal.*

If Hilbert's 10th Problem on $\mathbb{Q}$ *is undecidable, then* $\mathrm{IADR}_{\mathbb{Z}} \cap \mathrm{T}_{\mathbb{Z}}$ *is* Π^0_2-*complete.* [1, Theorems 4.1, 4.2 and 4.3]

The last two points should be clear intuitively, as

$$\mathrm{IMDR}_R \cap \mathrm{H}_R = \{p \mid \forall \delta > 0 : p \in \mathrm{IMDR}_R(\delta) \cap \mathrm{H}_R\}$$

and instead of counting for all $\delta > 0$, one can switch to $\{\frac{1}{n} \mid n \in \mathbb{N}\}$, which is countable, so it fits the formula we presented in section 4.4 because $\mathrm{IMDR}_R(\delta) \cap \mathrm{H}_R$ is Σ^0_1.

6. Partition Regularity

We will now turn to another field of Ramsey Theory: Partition Ramsey Theory. This field is concerned with colorings. The following definition is central.

Definition ((r-)Partition Regular)**.** Let $r \in \mathbb{N}$. A finite collection of polynomials $p_1, \dots, p_n \in R[x_1, \dots, x_d]$ is called r-*partition regular* on the domain R if for all partitions $C_1, \dots, C_r$ of $R \setminus \{0\}$ there is a *monochrome solution* of $p_1, \dots, p_n$, i.e. there is an i and $x_1, \dots, x_d$ such that

$$p_1(x_1, \dots, x_d) = \cdots = p_n(x_1, \dots, x_d) = 0 \text{ and } x_1, \dots, x_d \in C_i\,.$$

In this case, we write $(p_1, \dots, p_n) \in \mathrm{PR_R}(r)$.

A finite collection of polynomials $p_1, \dots, p_n \in R[x_1, \dots, x_d]$ *partition regular on the domain* R, if it is r-partition regular for all $r \in \mathbb{N}$. In this case, we write $(p_1, \dots, p_n) \in \mathrm{PR_R}$.

We have $\mathrm{PR_R} = \bigcap_{r \in \mathbb{N}} \mathrm{PR_R}(r)$.

A common question in partition Ramsey theory is whether a given finite collection of polynomials is partition regular. Here are a few examples.

- If $d = 1$, i.e. if the polynomials only take one argument, then partition regularity is equivalent to the question whether the polynomials have a solution other than 0.
- The equation $x + y = z$ is partition regular, which corresponds to the polynomial $p(x, y, z) = x + y - z$. However:
- The equation $x + 2y = z$ is *not* partition regular.
- A characterization of partition regularity of a system of linear equations is given by *Rado's Theorem*, which we will not cover in our presentation.

Theorem (Partition Compactness Principle)**.** *Let R be an integral domain, let $S \subseteq R$, and let $p \in R[x_1, \dots, x_n]$.*

(i) Given $r \in \mathbb{N}$, the equation $p(x_1, \dots, x_n) = 0$ is r-partition regular over S if and only if it is also r-partition regular over some finite set $F_r \subseteq S$.

(ii) *The equation* $p(x_1, \ldots, x_n) = 0$ *is partition regular over* S *if and only if for every* $r \in \mathbb{N}$ *it is* r*-partition regular over some finite set* $F_r \subseteq S$*.*

[1, Theorem 2.24]

Recall that, for a finite $K \subseteq S$ and $\epsilon > 0$, a finite set $F \subseteq S$ is called (K, ϵ)*-invariant* if for any $k \in K$ we have $|kF\Delta F| < \epsilon|F|$.

Theorem 3 (Density Compactness Principle)**.** *Let* S *be a commutative, cancellative and countable semigroup, let* $\delta \in (0, 1]$*, and let* $\mathcal{A} \subseteq \mathcal{P}_f(S)$ *be translation invariant. Then, the following statements are equivalent:*

(i) *If* $B \subseteq S$ *satisfies* $d^*(B) \geq \delta$*, then there exists some* $A \in \mathcal{A}$ *with* $A \subseteq B$*.*
(ii) *There exists a finite set* $K \subseteq S$ *and* $\varepsilon > 0$ *such that for every* (K, ε)*-invariant finite subset* $F \subseteq S$*, and every subset* $B \subseteq F$ *with*
$$|B| \geq \delta|F|\,,$$
there exists $A \in \mathcal{A}$ *with* $A \subseteq B$*.*
(iii) *There exists a finite set* $H \subseteq S$ *such that for every* $B \subseteq H$ *with*
$$|B| \geq \delta|H|\,,$$
there exists $A \in \mathcal{A}$ *with* $A \subseteq B$*.*

[1, Theorem 3.1]

Remark. The first statement of Theorem 3 pertains to the structures studied in Ramsey theory. Unfortunately, it is too complex to be used in classifying into the Lightface Hierarchy levels introduced in these notes. However, the condition in (iii) is Σ^0_1 and thus can be used to reduce the complexity of H. The condition in (ii) is more complex, thus the second statement may not be of set theoretic use, but is still of independent interest.

7. Proof of the Main Result

Here, we will only prove that $\mathrm{IADR}_R \cap \mathrm{T}_R$ is Σ^1_0-complete. For that, we will give a computable reduction to $\mathrm{HTP}(K^\times)$. Let $P \in R[x_1, \ldots, x_k]$ be arbitrary.

We construct a new polynomial $P_1 \in R[z_1, \ldots, z_{4k}]$ as follows:
$$P_1(z_1, \ldots, z_{4k}) := P\left(\frac{z_1 - z_2}{z_3 - z_4}, \ldots, \frac{z_{4k-3} - z_{4k-2}}{z_{4k-1} - z_{4k}}\right) \cdot \prod_{i=1}^{k}(z_{4i-1} - z_{4i})^{\deg(P)}.$$

Step 1: If $P \in \mathrm{HTP}(K^\times)$**, then** $P_1 \in \mathrm{IADR}_R(\delta) \cap \mathrm{T}_R$**.**

Suppose there exists a solution $s_1, \ldots, s_k \in K^\times$ such that $P(s_1, \ldots, s_k) = 0$. By Lemma 1 (i), for any set $B \subset R\backslash\{0\}$ with additive upper Banach density $d^*(B) > 0$, there exist *distinct* elements $z_1, \ldots, z_{4k} \in B$ such that
$$\frac{z_{4i-3} - z_{4i-2}}{z_{4i-1} - z_{4i}} = s_i \quad \text{for } i = 1, \ldots, k\,.$$

Then,

$$P_1(z_1,\ldots,z_{4k}) = P(s_1,\ldots,s_k)\cdot\prod_{i=1}^{k}(z_{4i-1}-z_{4i})^{\deg(P)} = 0\,.$$

Thus, P_1 has a root on a set B of positive additive density, and $z_1,\ldots,z_{4k}$ are injective. This implies $P_1 \in \mathrm{IADR}_R(\delta)$. Furthermore, P_1 is translation invariant. Hence, $P_1 \in \mathrm{IADR}_R(\delta)\cap \mathrm{T}_R$.

Step 2: If $P_1 \in \mathrm{IADR}_R(\delta)\cap \mathrm{T}_R$, then $P \in \mathrm{HTP}(K^\times)$. Suppose $P_1 \in \mathrm{IADR}_R(\delta)\cap \mathrm{T}_R$. Because we have $d^*(R) = 1 \geq \delta$, there exist distinct elements $z_1,\ldots,z_{4k} \in R$ such that

$$P_1(z_1,\ldots,z_{4k}) = 0\,.$$

But then the rational numbers

$$s_i := \frac{z_{4i-3}-z_{4i-2}}{z_{4i-1}-z_{4i}} \in K^\times$$

satisfy $P(s_1,\ldots,s_k) = 0$. Therefore, $P \in \mathrm{HTP}(K^\times)$.

Having shown that the map $P \mapsto P_1$ is computable, we have reduced the problem of deciding membership in $\mathrm{HTP}(K^\times)$ to deciding membership in $\mathrm{IADR}_R(\delta)\cap \mathrm{T}_R$. Moreover, since the set $\mathrm{IADR}_R(\delta)\cap \mathrm{T}_R$ is recursively enumerable, it is Σ^0_1-complete. □

The proof for $\mathrm{IMDR}_R(\delta)\cap \mathrm{H}_R$ is similar. One just has to use Lemma 1 (ii) instead and see that P_1 is also homogeneous.

8. Bridge between $\mathrm{IADR}_R(\delta)\cap \mathrm{T}_R$ and roots in K

We will now look at some applications of Theorem 2, namely a reduction of the polynomials discussed in the previous sections.

False Statement. *Suppose that R is a computable integral domain such that the set $\mathrm{IADR}_R \cap T_R$ (or $\mathrm{IMDR}_R \cap H_R$) is Π^0_2-complete. Then for each polynomial $p \in R[x_1,\ldots,x_n]$, there exists a polynomial $q \in R[x_1,\ldots,x_m]$, computable as a function of p, such that:*

$$p \in \mathrm{IADR}_R \cap T_R \qquad \Longleftrightarrow \qquad q \text{ has a root in } K$$

(respectively, $p \in \mathrm{IMDR}_R \cap H_R$) for some fixed computable set K.

The statement is false because it attempts to give a computable reduction of a Π^0_2-set (namely, $\mathrm{IADR}_R \cap T_R$) to a Σ^0_1-set (the set of polynomials with root in K), which is impossible.

However, $\mathrm{IADR}_R(\delta)\cap T_R$ and the set of polynomials with root in K are both Σ^0_1, and we are able to prove the following positive result.

Theorem 4. *For every polynomial $p \in R[x_1, \ldots, x_n]$ and every $\delta \in (0, 1]$, there exists a polynomial $q \in R[x_1, \ldots, x_m]$, computable from p and δ, such that:*

$$p \in \mathrm{IADR}_R(\delta) \cap T_R \iff q \text{ has a root in } K$$

(respectively, $p \in \mathrm{IMDR}_R(\delta) \cap H_R$). [1, Theorem 5.5]

Remark. In all known cases of interest in which $\mathrm{HTP}(R)$ is undecidable, it is actually Σ^0_1-universal, so there exists a computable reduction from the Σ^0_1-set $\mathrm{IADR}_R(\delta) \cap T_R$ to $\mathrm{HTP}(R)$.

References

[1] Sohail Farhangi, Steve Jackson, and Bill Mance. Undecidability in the ramsey theory of polynomial equations and hilbert's tenth problem, 2025.