Undecidability in the Ramsey theory of polynomial equations

Warwick summer seminar
University of Warwick

Based on joint work with Steve Jackson and William Mance

https://arxiv.org/abs/2412.14917

Sohail Farhangi
Slides available on sohailfarhangi.com

July 25, 2025

Overview

Partition regularity

Definition

Let R be an integral domain, let $S \subseteq R$, let $n, m \in \mathbb{N}$ and $p_1, \cdots, p_m \in R[x_1, \cdots, x_n]$ be arbitrary. The system of equations

$$\begin{aligned} p_1(x_1, \cdots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \cdots, x_n) &= 0 \end{aligned} \tag{1}$$

is ℓ-**partition regular (p.r.) over** S if for any partition $S = \bigcup_{i=1}^{\ell} C_i$, there is some $1 \leq i_0 \leq \ell$ for which C_{i_0} contais a solution to the system of equations in (1). The system of equations is **partition regular** if it is ℓ-partition regular for all $\ell \in \mathbb{N}$.

Positive results 1/2

The following systems of equations are partition regular over $\mathbb{N}$.

1) $x + y = z$, Schur 1916 [25]

2) van der Waerden 1927 [26] (arithmetic progressions or A.P.s)

$$\begin{aligned} x_1 - x_2 &= x_2 - x_3 \\ &\vdots \\ x_{n-2} - x_{n-1} &= x_{n-1} - x_n, \text{ or equivalently,} \end{aligned}$$

$$\sum_{i=1}^{n-2} (x_{i+2} - 2x_{i+1} + x_i)^2 = 0.$$

3) Brauer 1928 [7] (A.P.s and their common difference)

$$\begin{aligned} x_1 - x_2 &= x_0 \\ &\vdots \\ x_{n-1} - x_n &= x_0 \end{aligned}$$

Positive results 2/2

4) Rado 1933 [23] classified which finite systems of linear equations are p.r.

5) $x - y = p(z)$ with $p(z) \in z\mathbb{Z}[z]$, Bergelson 1996 [4, page 53]

6) Bergelson, Moreira, and Johnson 2017 [5], for $p_i(x) \in x\mathbb{Z}[x]$

$$\begin{aligned} x_1 - x_2 &= p_1(x_0) \\ &\vdots \\ x_{n-1} - x_n &= p_{n-1}(x_0) \end{aligned}$$

7) $x^2 - y^2 = z$, Moreira 2017 [20]

8) $z = x^y$, Sahasrabudhe 2018 [24]

Negative results

The following systems of equations are not partition regular over $\mathbb{N}$.

1) $2x + 3y = z$, Rado 1933 [23]

2) Rado 1933 [23]

$$\begin{aligned} x + 3y &= z_1 \\ x + 2y &= 2z_2 \end{aligned}$$

3) $x + y = z^2$ (ignoring $2 + 2 = 2^2$), Csikvári, Gyarmati, and Sárközy 2012 [10] (see also [18, 3])

4) $x - 2y = z^2$, Di Nasso and Luperi Baglini 2018 [13]

5) $x^2 - 2y^2 = z$, Di Nasso and Luperi Baglini 2018 [13]

6) $x + y = w^3 z^2$, F. and Magner 2022 [15]

7) $2x + 3y = wz^2$, F. and Magner 2022 [15]

8) F. and Magner 2022 [15]

$$\begin{aligned} x_1 + 17y_1 &= w_1 z_1^{100} \\ 9x_2 + 18y_2 &= w_2 z_2^2 \end{aligned}$$

Open problems

The partition regularity of the following systems of equations over $\mathbb{N}$ is not known.

1) $x^2 + y^2 = z^2$ (**VERY** popular, [14, 19, 16, 17])
2) $a(x^2 - y^2) = bz^2 + dw$ (important, cf. [22])
3) $x^3 + y^3 + z^3 = w^3$ (cf. [9])
4) $x^3 + y^3 + z^3 - 3xyz = w^3$
5) $x^4 + y^4 + z^4 = w^4$ (cf. [9])
6) (**VERY** popular, cf. [20, 1, 2, 6])

$$\begin{aligned} w &= xy \\ z &= x + y \end{aligned}$$

7) $2x - 8y = wz^3$ (cf. [15])
8) (cf. [15])

$$\begin{aligned} 16x_1 + 17y_1 &= w_1 z_1^8 \\ 33x_2 - 17y_2 &= w_2 z_2^8 \end{aligned}$$

Computable sets

Definition

A set $A \subseteq \mathbb{N}$ is **computable** if there exists an algorithm (Turing machine) that halts on every input, and outputs 1 if and only if the input is an element of A. If S is a countably infinite set, then $A \subseteq S$ is **computable** if there exists a a computable bijection $\phi : S \to \mathbb{N}$ for which the set $\phi(A)$ is computable.

Examples of computable subsets of $\mathbb{N}$ include the set of squares, the set of primes, the set of powers of 2, and the set of square free numbers. If $S \subseteq \mathbb{Z}[x]$ denotes the collection of polynomials of degree at most 2, and $A \subseteq S$ is those polynomials that have an integer root, then A is a computable set. Rado's Theorem [23] shows us that if S is the set of finite systems of linear equations with coefficients in $\mathbb{Z}$, and $A \subseteq S$ consists of those systems that are partition regular over $\mathbb{N}$, then A is computable.

Computably enumerable sets

Definition

A set $A \subseteq \mathbb{N}$ is **computably enumerable** if there exists an algorithm (Turing machine) such that the set of inputs for which the algorithm halts is exactly A. Equivalently, A is computably enumerable if there exists an algorithm that enumerates the members of A.

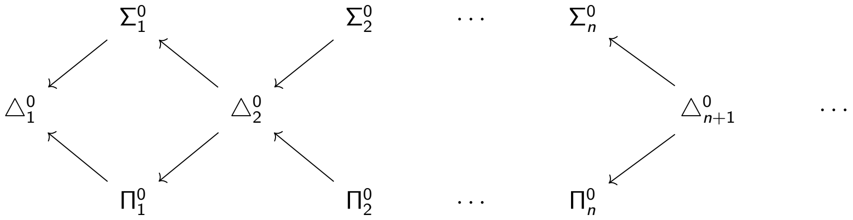
Given a polynomial $p \in \mathbb{Z}[x_1, \cdots, x_n]$, the set Z_p of integer roots of p is seen to be computably enumerable from the second of the given definitions. If $S = \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$, and $A \subseteq S$ is those polynomials that possess an integer root, then A is seen to be computably enumerable from the first of the given definitions.

Lemma

A set $A \subseteq \mathbb{N}$ is computable if and only if A and A^c are both computably enumerable.

The lightface hierarchy

We denote the collection of computable subsets of $\mathbb{N}$ by Δ^0_1, the collection of computably enumerable subsets of $\mathbb{N}$ by Σ^0_1, and we let Π^0_1 denote the collection of sets whose complement is in Σ^0_1. We inductively define Σ^0_{n+1} to be the collection of sets that are reducible to a set in Π^0_n in a "computably enumerable fashion", Π^0_{n+1} is the complement of Σ^0_{n+1}, and $\Delta^0_{n+1} = \Sigma^0_{n+1} \cap \Pi^0_{n+1}$.



Hilbert's 10th problem (HTP)

At the International Congress of Math in 1900, David Hilbert presented 10 important problems in mathematics, and 2 years later published a completed list of 23 problems now known as Hilbert's problems. The 10^{th} of the 23 problems (published but not presented) asked if the set $A \subseteq S := \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ of polynomials that have an integer root is computable.

Theorem (Matiyasevič, 1971)

The set A is not computable.

See the survey of Davis [11] for an exposition of the proof of this result and the history.

Open Problem: Is the set $A_{\mathbb{Q}} \subseteq S$ of polynomials that have a root in $\mathbb{Q}$ computable?

The latter problem is referred to as Hilbert's 10th problem over $\mathbb{Q}$. It is generally believed that $A_{\mathbb{Q}}$ is not computable.

Variations of Hilbert's 10th problem

Given a **computable** integral domain R, we let $HTP(R)$ refer to the following statement:

HTP(R): There does not exist a computable procedure to determine if a given $p \in R[x_1, \cdots, x_n]$ has a root in R.

The statement HTP(R) can be true, or false depending on the integral domain R.

Theorem ([27, 21, 12])

Suppose that R is a finite degree algebraic extension of $\mathbb{F}_{p^k}(t_1, \cdots, t_n)$ for some prime $p > 2$ and some $n, k \in \mathbb{N}$.

1. *$HTP(R)$ is true.*
2. *There does not exist a computable procedure for determining whether or not a given polynomial $p \in R[x_1, \cdots, x_n]$ has an integer root $(z_1, \cdots, z_n) \in (R \setminus \{0\})^n$.*

First main result

Theorem (F., Jackson, Mance, 2025+)

1. *Let us assume that $HTP(\mathbb{Q})$ is true. For $\ell \in \mathbb{N}$, the set $A_\ell \subseteq \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ is ℓ-partition regular over $\mathbb{Z} \setminus \{0\}$ is computably enumerable but not computable, so it is Σ_1^0-complete. The set $A := \bigcap_{\ell=1}^{\infty} A_\ell$ is Π_2^0-complete.*
2. *Suppose that R is as in the Theorem on the last slide (or just $R = \mathbb{F}_p(t)$). For $\ell \in \mathbb{N}$, the set $A_\ell \subseteq \bigcup_{n=1}^{\infty} R[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ is ℓ-partition regular over $R \setminus \{0\}$ is Σ_1^0-complete. The set $A := \bigcap_{\ell=1}^{\infty} A_\ell$ is Π_2^0-complete.*

Reducing partition regularity to HTP

Lemma (cf. Krawczyk,Byszewski, 2021 [8])

Let R be an integral domain with field of fractions K. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K$, the system of equations

$$\frac{z_{3i-2} - z_{3i-1}}{z_{3i}} = k_i \text{ for all } 1 \leq i \leq m, \tag{2}$$

is partition regular over $R \setminus \{0\}$.

Corollary

Given an integral domain R, and a polynomial $p \in R[x_1, \cdots, x_n]$, p has a root in K if and only if the equation $p'(x_1, \cdots, x_{3n}) = 0$ with

$$p'(x_1, \cdots, x_{3n}) := p\left(\frac{x_1 - x_2}{x_3}, \cdots, \frac{x_{3n-2} - x_{3n-1}}{x_{3n}}\right)\left(\prod_{i=1}^{n} x_{3i}\right)^{deg(p)}$$

is partition regular over $R \setminus \{0\}$.

Implications 1/3

Observe that if R is a countably infinite integral domain, then there are only countably many polynomials p with coefficients in R. Consequently, the set B of such polynomials that are not partition regular over $R \setminus \{0\}$ is countable, so for each $p \in B$ there exists a partition P_p of $R \setminus \{0\}$ that does not contain a root of p. Consequently, to determine whether or not a polynomial p is partition regular over $R \setminus \{0\}$, it suffices check whether or not p has a root in some cell of each partition in the family $\{P_b\}_{b \in B}$. The fact that the set of "partition regular polynomials" is Π_2^0-complete, means that there does not exist a simpler method of determining whether or not the equation $p(x_1, \cdots, x_n) = 0$ is partition regular over $R \setminus \{0\}$.

Implications 2/3

Conjecture

Let R be a computable integral domain. There exists a computable collection of finite partitions $\{\mathcal{C}_n\}_{n=1}^{\infty}$ of $R \setminus \{0\}$ such that p is partition regular over $R \setminus \{0\}$ if and only if for every $n \in \mathbb{N}$, p has a root in some cell of $\mathcal{C}_n$.

Implications 3/3

The following statement is false since it is describing a Σ_1^0 set, but the set of "partition regular polynomials" is (conditionally) a Π_2^0-complete set.

False Statement: Let R be a countably infinite integral domain. For each $p \in R[x_1, \cdots, x_n]$, there exists $q \in R[x_1, \cdots, x_m]$ that is a computable function of p such that $p(x_1, \cdots, x_n) = 0$ is partition regular over $R \setminus \{0\}$ if and only if q has a root in K.

However, the following result is true.

Theorem

For each $p \in R[x_1, \cdots, x_n]$ and each $r \in \mathbb{N}$, there exists $q_r \in R[x_1, \cdots, x_m]$ that is a computable function of p and r such that $p(x_1, \cdots, x_n) = 0$ is r-partition regular over $R \setminus \{0\}$ if and only if q_r has a root in K.

Second main result

Theorem (F., Jackson, Mance, 2024+)

1. *Let us assume that HTP($\mathbb{Q}$) is true. For $\delta \in (0,1)$, the set $A_\delta \subseteq \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ has an injective solution in any set $B \subseteq \mathbb{Z}$ with $\overline{d}(B) \geq \delta$ is Σ_1^0-complete. The set $A := \bigcap_{\delta > 0} A_\delta$ is Π_2^0-complete. Analogous results hold when $\overline{d}$ is replaced with d^* or $d_\times^*$.*
2. *Suppose that R is the ring of integers of an algebraic function field over a finite field of constants. For $\delta > 0$, the set $A_\delta \subseteq \bigcup_{n=1}^{\infty} R[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ has an injective solution in any set $B \subseteq R$ with $d^*(B) \geq \delta$ is Σ_1^0-complete. The set $A := \bigcap_{\delta > 0} A_\delta$ is Π_2^0-complete. Analogous results hold when d^* is replaced with $\overline{d}_{\mathcal{F}}$ or $d_\times^*$.*

Reduction to HTP for density Ramsey theory 1/2

Lemma

Let R be a countably infinite integral domain with field of fractions K. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K^\times$ we have the following:

1. *If $A \subseteq R$ is such that $d^*(A) > 0$, then A contains a solution to the system of equations*

$$\frac{z_{4i-3} - z_{4i-2}}{z_{4i-1} - z_{4i}} = k_i \text{ for all } 1 \leq i \leq m. \quad (3)$$

   Furthermore, the solution can be taken such that $z_i \neq z_j$ when $i \neq j$.

2. *If $A \subseteq R \setminus \{0\}$ is such that $d^*_\times(A) > 0$, then A contains a solution $(z_1, \cdots, z_{4m})$ to the system (3), such that $z_i \neq z_j$ for $i \neq j$.*

Reduction to HTP for density Ramsey theory 2/2

Corollary

Let R be a countably infinite integral domain with field of fractions K, and let $p \in R[x_1, \cdots, x_n]$.

1. *p has a root in K if and only if for any $A \subseteq R$ with $d^*(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$, where*

$$p'(z_1, \cdots, z_{4n}) = p\left(\frac{z_1 - z_2}{z_3 - z_4}, \cdots, \frac{z_{4n-3} - z_{4n-2}}{z_{4n-1} - z_{4n}}\right)\left(\prod_{i=1}^{n}(z_{4n-1} - z_{4n})\right)^{deg(p)}.$$

2. *p has a root in K if and only if for any $A \subseteq R \setminus \{0\}$ with $d^*_{\times}(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$.*

Implications

As in the case of partition Ramsey Theory, the following conjecture would yield the simplest possible characterization of the collection of density regular polynomial equations.

Conjecture

Let R be a computable integral domain. There exists a computable collection of subsets $\{B_n\}_{n=1}^{\infty}$ of $R \setminus \{0\}$ with $d^*(B_n) > 0(d^*_\times(B_n) > 0)$ such that p is additively (multiplicatively) density regular over $R \setminus \{0\}$ if and only if for every $n \in \mathbb{N}$, p has a root in B_n.

Compactness in Density Ramsey Theory

Theorem (F., Jackson, Mance, 2025+)

Let S be a countably infinite cancellative left-amenable semigroup, let $\delta \in (0,1)$, and let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be right-translation invariant. The following are equivalent:

1. *If $B \subseteq S$ satisfies $d^*(B) \geq \delta$, then $\exists\ A \in \mathcal{A}$ with $A \subseteq B$.*
2. *There exists $K \in \mathscr{P}_f(S)$ and $\epsilon > 0$ such that for any (K,ϵ)-invariant set $F \in \mathscr{P}_f(S)$ and for all $B \subseteq F$ with $|B| \geq \delta|F|$, there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
3. *There exists a $H \in \mathscr{P}_f(S)$ such that for all $B \subseteq H$ with $|B| \geq \delta|H|$, there exists $A \in \mathcal{A}$ with $A \subseteq B$.*

A fun corollary

Corollary (F., Jackson, Mance, 2024)

Let S be a countably infinite cancellative left-amenable semigroup and let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be right-translation invariant. There exists $\delta \in [0,1)$ for which the following hold:

1. *If $B \subseteq S$ satisfies $d^*(B) > \delta$, then there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
2. *There exists a $B \subseteq S$ with $d^*(B) = \delta$, such that B does not contain any member of $\mathcal{A}$.*

Uniformity of Density regularity 1/2

Theorem (F., Jackson, Mance, 2024+)

*Let S be a countably infinite cancellative left-amenable semigroup. Let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be δ-**density regular**, i.e., for any measure preserving system $(X, \mathscr{B}, \mu, (T_s)_{s\in S})$ and any $B \in \mathscr{B}$ with $\mu(B) \geq \delta$, there exists $A \in \mathcal{A}$ for which*

$$\mu\left(\bigcap_{a\in A} T_a^{-1}B\right) > 0.$$

Then there exists a finite collection $\mathcal{A}_\delta \subseteq \mathcal{A}$ that is also δ-density regular. In fact, we can say even more.

Uniformity of Density regularity 2/2

Theorem (Uniformity of Density Regularity)

Let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be right-translation invariant and δ-density regular. Then there exists a finite subcollection $\mathcal{A}' \subseteq \mathcal{A}$ and a $\gamma > 0$ such that the following hold:

1. *There exists $K \in \mathscr{P}_f(S)$ and $\epsilon > 0$ such that for any (K, ϵ)-invariant set $F \in \mathscr{P}_f(S)$, and any $B \subseteq F$ with $|B| \geq (\delta - \epsilon)|F|$, there exists $A \in \mathcal{A}'$ for which $|F \cap \bigcap_{a \in A} a^{-1}B| > \gamma|F|$.*
2. *If $\mathcal{F}$ is a left-Følner sequence in S, and $B \subseteq S$ satisfies $\overline{d}_{\mathcal{F}}(B) \geq \delta$, then there exists $A \in \mathcal{A}'$ for which $\overline{d}_{\mathcal{F}}(\bigcap_{a \in A} a^{-1}B) \geq \gamma$.*

References I

[1] R. Alweiss.
Monochromatic sums and products over *mathbbQ*.
arXiv preprint arXiv:2307.08901, 2023.

[2] R. Alweiss.
Monochromatic sums and products of polynomials.
Discrete Anal., pages Paper No. 5, 7, 2024.

[3] Z. Baja, D. Dobák, B. Kovács, P. P. Pach, and D. Pigler.
Towards characterizing the 2-ramsey equations of the form ax+ by= p (z).
Discrete Mathematics, 346(5):113324, 2023.

References II

[4] V. Bergelson.
Ergodic Ramsey theory—an update.
In *Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 1–61. Cambridge Univ. Press, Cambridge, 1996.

[5] V. Bergelson, J. H. Johnson, Jr., and J. Moreira.
New polynomial and multidimensional extensions of classical partition results.
J. Combin. Theory Ser. A, 147:119–154, 2017.

[6] M. Bowen and M. Sabok.
Monochromatic products and sums in the rationals.
Forum Math. Pi, 12:Paper No. e17, 12, 2024.

References III

[7] R. Brauer.
Untersuchungen über die arithmetischen Eigenschaften von Gruppen linearer Substitutionen.
Math. Z., 28(1):677–696, 1928.

[8] J. Byszewski and E. Krawczyk.
Rado's theorem for rings and modules.
J. Combin. Theory Ser. A, 180:105402, 28, 2021.

[9] S. Chow, S. Lindqvist, and S. Prendiville.
Rado's criterion over squares and higher powers.
J. Eur. Math. Soc. (JEMS), 23(6):1925–1997, 2021.

References IV

[10] P. Csikvári, K. Gyarmati, and A. Sárközy.
Density and Ramsey type results on algebraic equations with restricted solution sets.
Combinatorica, 32(4):425–449, 2012.

[11] M. Davis.
Hilbert's tenth problem is unsolvable.
Amer. Math. Monthly, 80:233–269, 1973.

[12] J. Denef.
The Diophantine problem for polynomial rings and fields of rational functions.
Trans. Amer. Math. Soc., 242:391–399, 1978.

References V

[13] M. Di Nasso and L. Luperi Baglini.
Ramsey properties of nonlinear Diophantine equations.
Adv. Math., 324:84–117, 2018.

[14] P. Erdős and R. L. Graham.
Old and new problems and results in combinatorial number theory, volume 28 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*.
Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[15] S. Farhangi and R. Magner.
On the partition regularity of $ax + by = cw^m z^n$.
Integers, 23:Paper No. A18, 52, 2023.

[16] N. Frantzikinakis, O. Klurman, and J. Moreira.
Partition regularity of pythagorean pairs.
arXiv preprint arXiv:2309.10636, 2023.

[17] N. Frantzikinakis, O. Klurman, and J. Moreira.
Partition regularity of generalized pythagorean pairs.
arXiv preprint arXiv:2407.08360, 2024.

[18] B. J. Green and S. Lindqvist.
Monochromatic solutions to $x + y = z^2$.
Canad. J. Math., 71(3):579–605, 2019.

[19] M. J. H. Heule, O. Kullmann, and V. W. Marek.
Solving and verifying the Boolean Pythagorean triples problem via cube-and-conquer.
In *Theory and applications of satisfiability testing—SAT 2016*, volume 9710 of *Lecture Notes in Comput. Sci.*, pages 228–245. Springer, [Cham], 2016.

[20] J. Moreira.
Monochromatic sums and products in $\mathbb{N}$.
Ann. of Math. (2), 185(3):1069–1090, 2017.

[21] T. Pheidas.
Hilbert's tenth problem for fields of rational functions over finite fields.
Invent. Math., 103(1):1–8, 1991.

[22] S. Prendiville.
Counting monochromatic solutions to diagonal diophantine equations.
Discrete Anal., pages Paper No. 14, 47, 2021.

[23] R. Rado.
Studien zur Kombinatorik.
Math. Z., 36(1):424–470, 1933.

[24] J. Sahasrabudhe.
Exponential patterns in arithmetic Ramsey theory.
Acta Arith., 182(1):13–42, 2018.

[25] I. Schur.
Uber die kongruenz $x^m + y^m = z^m \pmod{p}$.
Jahresber. Dtsch. Math, 25:114–117, 1916.

References IX

[26] B. van der Waerden.
Beweis einer baudetschen vermutung.
Nieuw Arch. Wiskd, 15:212–216, 1927.

[27] C. R. Videla.
Hilbert's tenth problem for rational function fields in characteristic 2.
Proc. Amer. Math. Soc., 120(1):249–253, 1994.