Undecidability in the Ramsey theory of polynomial equations

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Overview

Partition regularity

Definition

Let R be an integral domain, let $S \subseteq R$, let $n, m \in \mathbb{N}$ and $p_1, \cdots, p_m \in R[x_1, \cdots, x_n]$ be arbitrary. The system of equations

$$\begin{aligned} p_1(x_1, \cdots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \cdots, x_n) &= 0 \end{aligned} \tag{1}$$

is ℓ-**partition regular (p.r.) over** S if for any partition $S = \bigcup_{i=1}^{\ell} C_i$, there is some $1 \leq i_0 \leq \ell$ for which C_{i_0} contais a solution to the system of equations in (1). The system of equations is **partition regular** if it is ℓ-partition regular for all $\ell \in \mathbb{N}$.

Positive results 1/2

The following systems of equations are partition regular over $\mathbb{N}$.

1) $x + y = z$, Schur 1916 [25]

2) van der Waerden 1927 [26] (arithmetic progressions or A.P.s)

$$\begin{aligned} x_1 - x_2 &= x_2 - x_3 \\ &\vdots \\ x_{n-2} - x_{n-1} &= x_{n-1} - x_n, \text{ or equivalently,} \end{aligned}$$

$$\sum_{i=1}^{n-2} (x_{i+2} - 2x_{i+1} + x_i)^2 = 0.$$

3) Brauer 1928 [7] (A.P.s and their common difference)

$$\begin{aligned} x_1 - x_2 &= x_0 \\ &\vdots \\ x_{n-1} - x_n &= x_0 \end{aligned}$$

Positive results 2/2

4) Rado 1933 [24] classified which finite systems of linear equations are p.r.

5) $x - y = p(z)$ with $p(z) \in z\mathbb{Z}[z]$, Bergelson 1996 [4, page 53]

6) Bergelson, Moreira, and Johnson 2017 [5], for $p_i(x) \in x\mathbb{Z}[x]$

$$\begin{aligned} x_1 - x_2 &= p_1(x_0) \\ &\vdots \\ x_{n-1} - x_n &= p_{n-1}(x_0) \end{aligned}$$

7) $x^2 - y^2 = z$, Moreira 2017 [21]

Negative results

The following systems of equations are not partition regular over $\mathbb{N}$.

1) $2x + 3y = z$, Rado 1933 [24]

2) $x + y = z^2$ (ignoring $2 + 2 = 2^2$), Csikvári, Gyarmati, and Sárközy 2012 [10] (see also [18, 3])

3) $x - 2y = z^2$, Di Nasso and Luperi Baglini 2018 [13]

4) $x^2 - 2y^2 = z$, Di Nasso and Luperi Baglini 2018 [13]

Open problems

The partition regularity of the following systems of equations over $\mathbb{N}$ is not known.

1) $x^2 + y^2 = z^2$ (**VERY** popular, [14, 19, 16, 17])
2) $a(x^2 - y^2) = bz^2 + dw$ (important, cf. [23])
3) $x^3 + y^3 + z^3 = w^3$ (cf. [9])
4) $x^3 + y^3 + z^3 - 3xyz = w^3$
5) $x^4 + y^4 + z^4 = w^4$ (cf. [9])
6) (**VERY** popular, cf. [21, 1, 2, 6])

$$\begin{aligned} w &= xy \\ z &= x + y \end{aligned}$$

7) $2x - 8y = wz^3$ (cf. [15])
8) $16x + 17y = wz^8$ (cf. [15])

Computable and computably enumerable sets

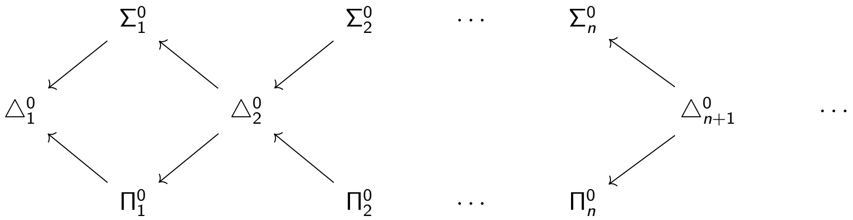
Definition

A set $A \subseteq \mathbb{N}$ is **computable** if there exists an algorithm (Turing machine) that halts on every input, and outputs 1 if and only if the input is an element of A. A set $A \subseteq \mathbb{N}$ is **computably enumerable** if there exists an algorithm (Turing machine) such that the set of inputs for which the algorithm halts is exactly A.

If $S = \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$, and $A \subseteq S$ is those polynomials that possess an integer root, then A is seen to be computably enumerable from the first of the given definitions.

The lightface hierarchy

We denote the collection of computable subsets of $\mathbb{N}$ by $\triangle_1^0$, the collection of computably enumerable subsets of $\mathbb{N}$ by Σ_1^0, and we let Π_1^0 denote the collection of sets whose complement is in Σ_1^0. We inductively define Σ_{n+1}^0 to be the collection of sets that are reducible to a set in Π_n^0 in a "computably enumerable fashion", Π_{n+1}^0 is the complement of Σ_{n+1}^0, and $\triangle_{n+1}^0 = \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$.



Hilbert's 10th problem (HTP)

At the International Congress of Math in 1900, David Hilbert presented 10 important problems in mathematics, and 2 years later published a completed list of 23 problems now known as Hilbert's problems. The 10^{th} of the 23 problems (published but not presented) asked if the set $A \subseteq S := \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ of polynomials that have an integer root is computable.

Theorem (Matiyasevič [20], 1971)

The set A is not computable.

See the survey of Davis [11] for an exposition of the proof of this result and the history.

Open Problem: Is the set $A_{\mathbb{Q}} \subseteq S$ of polynomials that have a root in $\mathbb{Q}$ computable? This problem is referred to as Hilbert's 10th problem over $\mathbb{Q}$. It is generally believed that $A_{\mathbb{Q}}$ is not computable.

Variations of Hilbert's 10th problem

Given a **computable** integral domain R, we let $HTP(R)$ refer to the following statement:

HTP(R): There does not exist a computable procedure to determine if a given $p \in R[x_1, \cdots, x_n]$ has a root in R.

The statement HTP(R) can be true, or false depending on the integral domain R.

Theorem ([27, 22, 12])

Suppose that R is a finite degree algebraic extension of $\mathbb{F}_{p^k}(t_1, \cdots, t_n)$ for some prime $p > 2$ and some $n, k \in \mathbb{N}$.

1. *$HTP(R)$ is true.*
2. *There does not exist a computable procedure for determining whether or not a given polynomial $p \in R[x_1, \cdots, x_n]$ has an integer root $(z_1, \cdots, z_n) \in (R \setminus \{0\})^n$.*

First main result

Theorem (F., Jackson, Mance, 2024+)

1. *Let us assume that HTP($\mathbb{Q}$) is true. For $\ell \in \mathbb{N}$, the set $A_\ell \subseteq \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ is ℓ-partition regular over $\mathbb{Z} \setminus \{0\}$ is computably enumerable but not computable, so it is Σ_1^0-complete. The set $A := \bigcap_{\ell=1}^{\infty} A_\ell$ is Π_2^0-complete.*
2. *Suppose that R is as in the Theorem on the last slide (or just $R = \mathbb{F}_p(t)$). For $\ell \in \mathbb{N}$, the set $A_\ell \subseteq \bigcup_{n=1}^{\infty} R[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ is ℓ-partition regular over $R \setminus \{0\}$ is Σ_1^0-complete. The set $A := \bigcap_{\ell=1}^{\infty} A_\ell$ is Π_2^0-complete.*

Reducing partition regularity to HTP

Lemma (cf. Krawczyk,Byszewski, 2021 [8])

Let R be an integral domain with field of fractions K. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K$, the system of equations

$$\frac{z_{3i-2} - z_{3i-1}}{z_{3i}} = k_i \text{ for all } 1 \leq i \leq m, \tag{2}$$

is partition regular over $R \setminus \{0\}$.

Corollary

Given an integral domain R, and a polynomial $p \in R[x_1, \cdots, x_n]$, p has a root in K if and only if the equation $p'(x_1, \cdots, x_{3n}) = 0$ with

$$p'(x_1, \cdots, x_{3n}) := p\left(\frac{x_1 - x_2}{x_3}, \cdots, \frac{x_{3n-2} - x_{3n-1}}{x_{3n}}\right)\left(\prod_{i=1}^{n} x_{3i}\right)^{deg(p)}$$

is partition regular over $R \setminus \{0\}$.

Implications 1/2

Conjecture

Let R be a computable integral domain. There exists a computable collection of finite partitions $\{\mathcal{C}_n\}_{n=1}^{\infty}$ of $R \setminus \{0\}$ such that p is partition regular over $R \setminus \{0\}$ if and only if for every $n \in \mathbb{N}$, p has a root in some cell of $\mathcal{C}_n$.

This conjecture describes a Π_2^0 set, so it is the simplest possible description of the set of partition regular polynomials for the cases from 2 slides ago.

Implications 2/2

The following statement is false since it is describing a Σ_1^0 set, but the set of "partition regular polynomials" is (conditionally) a Π_2^0-complete set.

False Statement: Let R be a countably infinite integral domain. For each $p \in R[x_1, \cdots, x_n]$, there exists $q \in R[x_1, \cdots, x_m]$ that is a computable function of p such that $p(x_1, \cdots, x_n) = 0$ is partition regular over $R \setminus \{0\}$ if and only if q has a root in K.

However, the following result is true.

Theorem

For each $p \in R[x_1, \cdots, x_n]$ and each $r \in \mathbb{N}$, there exists $q_r \in R[x_1, \cdots, x_m]$ that is a computable function of p and r such that $p(x_1, \cdots, x_n) = 0$ is r-partition regular over $R \setminus \{0\}$ if and only if q_r has a root in K.

Everything that we have considered so far for partition regularity, we have also considered for density Ramsey theory. See the paper for more details.

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