Uniform Wiener-Wintner theorems for Lamperti representations of amenable groups

Analysis Seminar at
University of Jena

Based on joint work with Micky Barthmann
https://arxiv.org/abs/2404.05877

Sohail Farhangi
Slides available on sohailfarhangi.com

March 27, 2025

Overview

Birkhoff's pointwise ergodic theorem

Theorem ([1])

Let $(X, \mathscr{B}, \mu, \varphi)$ be a probability measure preserving system, and let $f \in L^1(X, \mu)$. For a.e. $x \in X$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\varphi^n x) = f^*(x), \tag{1}$$

where $f^*(x) \in L^1(X, \mu)$ is such that $f^*(\varphi x) = f^*(x)$ for a.e. $x \in X$ and $\int_A f^* d\mu = \int_A f d\mu$ for every $A \in \mathscr{B}$ satisfying $A = \varphi^{-1}A$. In particular, if φ is ergodic, then for a.e. $x \in X$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\varphi^n x) = \int_X f d\mu. \tag{2}$$

Operatorial generalizations of Birkhoff

Doob [5] and Kakutani [8] produced a pointwise ergodic theorem for Markoff processes. Later Hopf [7] proved a general operator theoretic pointwise ergodic theorem, which was further refined by Dunford and Schwartz [6], and then extended to operators on Bochner spaces by Chacon. Yoshimoto [14] extended Chacon's result to more general operators and to a larger class of functions. Similar results were also found independently by Chilin and Litvinov [4].

Dunford-Schwartz+Chacon ergodic theorem

Theorem (Chacon, [3, Theorem 1])

Let E be a reflexive Banach space, let $1 \leq p < +\infty$, let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space, and let $T : L^1(X, \mu; E) \to L^1(X, \mu; E)$ be a linear contraction for which we also have $\|Tg\|_\infty \leq \|g\|_\infty$ whenever $g \in L^1(X, \mu; E) \cap L^\infty(X, \mu; E)$. For any $f \in L^p(X, \mu; E)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f(x) \tag{3}$$

converges in the norm topology of E for a.e. $x \in X$. Furthermore, if $1 < p < +\infty$, then there exists a $f^* \in L^p(X, \mu; E)$ such that for a.e. $x \in X$ we have

$$\sup_{N\in\mathbb{N}} \left\| \frac{1}{N} \sum_{n=1}^{N} T^n f(x) \right\| \leq \|f^*(x)\|. \tag{4}$$

The uniform Wiener-Wintner theorem of Bourgain

Theorem ([13, 2])

Let $\mathcal{X} := (X, \mathscr{B}, \mu, \varphi)$ be a probability measure preserving system and let $f \in L^1(X, \mu)$. There exists $X_f \in \mathscr{B}$ with $\mu(X_f) = 1$, such that for $x \in X_f$ and $\lambda \in \mathbb{S}^1$ we have existence of the following limit:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\varphi^n x)\lambda^n. \tag{5}$$

If $\mathcal{X}$ is ergodic and $f \in L^1(X, \mu)$ is weakly mixing, i.e.,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int_X T_\varphi^n f g d\mu \right| = 0, \tag{6}$$

for all $g \in L^\infty(X, \mu)$, then for $x \in X_f$ we have

$$\lim_{N\to\infty} \sup_{\lambda\in\mathbb{S}^1} \frac{1}{N} \left| \sum_{n=1}^{N} f(\varphi^n x)\lambda^n \right| = 0. \tag{7}$$

Lamperti operators

For $0 < p < \infty$ and a σ-finite measure space $(X, \mathscr{B}, \mu)$, a bounded linear operator $T : L^p(X, \mu) \to L^p(X, \mu)$ is a **Lamperti operator** if for any $f, g \in L^p(X, \mu)$ with $fg = 0$, we have $(Tf)(Tg) = 0$. Lamperti [10] showed that if T is an isometry of $L^p(X, \mu)$ for $p \neq 2$, then T is a Lamperti operator. Kan [9] observed that every Lamperti operator is of the form $(Tf)(x) = h(x)f(\varphi x)$ for some $h : X \to \mathbb{C}$ and some nonsingular $\varphi : X \to X$. If E is a Banach space then an operator $T : L^p(X, \mu; E) \to L^p(X, \mu; E)$ is a **Lamperti operator** if for any $f, g \in L^p(X, \mu; E)$ with $||f(x)|| \cdot ||g(x)|| = 0$ μ-a.e., we have $||Tf(x)|| \cdot ||Tg(x)|| = 0$ μ-a.e. A typical example of such an operator is $(Tf)(x) = H(x)(f(\varphi x))$, where $\mathcal{L}_1(E)$ is the unit ball of the bounded linear operators on E, and $H : X \to \mathcal{L}_1(E)$ is sufficiently measurable. A **Lamperti representation** T of a group G on $L^p(X, \mu; E)$ is representation of G in which each T_g is a Lamperti operator.

Ergodic properties of Lamperti operators

Kan [9] proved dominated ergodic estimates and pointwise ergodic theorems for Lamperti operators on $L^p(X, \mu)$ with $1 < p < \infty$. Tempelman [11] as well as Tempelman and Shulman [12] extended these results to Lamperti representations of an amenable group.

Theorem (Tempelman [11])

Let G be a locally compact second countable (l.c.s.c.) amenable group with left Haar measure λ, and let $(F_n)_{n=1}^{\infty}$ be a tempered left Følner sequence in G. If T is a bounded Lamperti representation of G on $L^p(X, \mu)$ with $(X, \mathscr{B}, \mu)$ a σ-finite measure space and $1 < p < \infty$, then for a.e. $x \in X$ we have

$$\lim_{N\to\infty} \frac{1}{\lambda(F_N)} \int_{F_N} T_g f(x) d\lambda(g) = Pf(x), \tag{8}$$

where P is the projection onto the T-invariant subspace.

The spaCb property

Definition

Let G be a l.c.s.c. amenable group, and let $\mathcal{F} = (F_n)_{n=1}^{\infty}$ be a left-Følner sequence. A weakly relatively compact representation T of G on $L^p(X, \mu; E)$ is **$\mathcal{F}$-pointwise absolutely Cesàro bounded ($\mathcal{F}$-paCb)** if there exists a $C > 0$ such that for every $f \in L^1(X, \mu; E)$ we have

$$\limsup_{n\to\infty} \frac{1}{\lambda(F_n)} \int_{F_n} \|T_g f(x)\|_E d\lambda(g) \leq C||f||_p, \tag{9}$$

for μ-a.e. $x \in X$. The representation T is **$\mathcal{F}$-strongly pointwise absolutely Cesàro bounded ($\mathcal{F}$-spaCb)** if it is $\mathcal{F}$-paCb and for every $f \in L^p(X, \mu; E) \cap L^\infty(X, \mu; E)$ we have $\sup_{g\in G} \|T_g f\|_\infty \leq C\|f\|_\infty$.

Examples

Let $(X, \mathscr{B}, \mu)$ be a standard probability space. Let E be an arbitrary Banach space, and let $1 \leq p < \infty$ be arbitrary.

1. If $\varphi : X \to X$ is measure preserving and ergodic, and $H : X \to \mathcal{L}_1(E)$ is measurable, then the Lamperti operator T given by $(Tf)(x) = H(x)(f(\varphi x))$ is a $([1, N])_{N=1}^{\infty}$-spaCb operator on $L^p(X, \mu; E)$.
2. If φ is an ergodic measure preserving action of G on $(X, \mathscr{B}, \mu)$, and $h : G \times X \to \mathcal{L}_1(E)$ is a bounded cocycle, i.e., $H(g_1 g_2, x) = H(g_2, x)H(g_1, \varphi_{g_2} x)$, then the Lamperti representation T of G given by $(T_g f)(x) = H(g, x)f(\varphi_g x)$ is $\mathcal{F}$-spaCb for any tempered Følner sequence $\mathcal{F}$.
3. If $p > 1$, then there are situations in which we can allow the cocycle $h : X \to \mathbb{C}_{\leq 1}$ to be an unbounded Radon-Nikodym derivative of pushforwards of μ with respect to a nonsingular action.

Operatorial uniform Wiener-Wintner for $\mathbb{N}$

Theorem

Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space, let E be a Banach space, and let $T : L^1(X, \mu; E) \to L^1(X, \mu; E)$ be a bounded linear spaCb operator. Then for any weakly mixing $f \in L^1(X, \mu; E)$, i.e., any f satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle T^n f, g' \rangle| = 0, \tag{10}$$

for all $g' \in L^1(X, \mu; E)'$, we have for a.e. $x \in X$ that

$$\lim_{N\to\infty} \sup_{\lambda\in\mathbb{S}^1} \left\| \frac{1}{N} \sum_{n=1}^{N} T^n f(x)\lambda^n \right\| = 0. \tag{11}$$

Operatorial amenable uniform Wiener-Wintner

Theorem

Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space, let E be a Banach space, and let $T : L^1(X, \mu; E) \to L^1(X, \mu; E)$ be a bounded linear $\mathcal{F}$-spaCb representation of G. Then for any weakly mixing $f \in L^1(X, \mu; E)$, i.e., any f satisfying

$$\lim_{N\to\infty} \frac{1}{\lambda(F_N)} \int_{F_N} |\langle T_g f, \eta' \rangle| = 0, \tag{12}$$

for all $\eta' \in L^1(X, \mu; E)'$, we have for a.e. $x \in X$ that

$$\lim_{N\to\infty} \sup_{\phi\in\Phi_d} \left|\left| \frac{1}{\lambda(F_N)} \int_{F_N} T_g f(x)\phi(g) d\lambda(g) \right|\right| = 0. \tag{13}$$

A counterexample

Let $e(x) = e^{2\pi ix}$ and consider the multiplication operator $M_e : L^1([0,1],m) \to L^1([0,1],m)$ given by $(M_e f)(x) = e(x)f(x)$. The operator is weakly mixing (in fact, strongly mixing) since for any $g \in L^\infty([0,1],m) = (L^1([0,1],m))'$ we have

$$\lim_{N\to\infty} \langle M_e^n f, g\rangle = \lim_{N\to\infty} \int_0^1 e(nx)f(x)g(x)dx = \lim_{N\to\infty} \widehat{fg}(-n) = 0,$$

where the final equality follows from the Riemman-Lebesgue Lemma. However, we see that for $\lambda_x := e(-x)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (M_e^n f)(x)\lambda_x^n = 1. \tag{14}$$

We also see that M_e is not paCb, as $C - \lim_{N\to\infty} |(M_e \mathbb{1}_A)(x)| = 1$ for $x \in A$, regardless of the measure of A.

References I

[1] G. D. Birkhoff.
Proof of the ergodic theorem.
Proc. Natl. Acad. Sci. USA, 17:656–660, 1931.

[2] J. Bourgain.
Double recurrence and almost sure convergence.
J. Reine Angew. Math., 404:140–161, 1990.

[3] R. V. Chacon.
An ergodic theorem for operators satisfying norm conditions.
J. Math. Mech., 11:165–172, 1962.

[4] V. Chilin and S. Litvinov.
The validity space of Dunford-Schwartz pointwise ergodic theorem.
J. Math. Anal. Appl., 461(1):234–247, 2018.

References II

[5] J. L. Doob.
Asymptotic properties of Markoff transition prababilities.
Trans. Amer. Math. Soc., 63:393–421, 1948.

[6] N. Dunford and J. T. Schwartz.
Convergence almost everywhere of operator averages.
J. Rational Mech. Anal., 5:129–178, 1956.

[7] E. Hopf.
The general temporally discrete Markoff process.
J. Rational Mech. Anal., 3:13–45, 1954.

[8] S. Kakutani.
Ergodic theorems and the Markoff process with a stable distribution.
Proc. Imp. Acad. Tokyo, 16:49–54, 1940.

[9] C. H. Kan.
Ergodic properties of Lamperti operators.
Canadian J. Math., 30(6):1206–1214, 1978.

[10] J. Lamperti.
On the isometries of certain function-spaces.
Pacific J. Math., 8:459–466, 1958.

[11] A. Tempelman.
Pointwise ergodic theorems for bounded lamperti representations of amenable groups.
Proceedings of the American Mathematical Society, 143(11):4989–5004, 2015.

[12] A. Tempelman and A. Shulman.
Dominated and pointwise ergodic theorems with "weighted" averages for bounded lamperti representations of amenable groups.
Journal of Mathematical Analysis and Applications, 474(1):23–58, 2019.

[13] N. Wiener and A. Wintner.
Harmonic analysis and ergodic theory.
Amer. J. Math., 63:415–426, 1941.

[14] T. Yoshimoto.
Vector-valued ergodic theorems for operators satisfying norm conditions.
Pacific J. Math., 85(2):485–499, 1979.