When normality and distribution normality coincide for nice classes of Cantor series

University of Texas at Austin Dynamical systems seminar

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Base-b normality 1/2

Definition

For $b \in \mathbb{N}_{\geq 2}$, a number $x \in [0,1]$ is **normal base-b** if for $\ell \in \mathbb{N}$ and any word $w \in \{0,1,\cdots,b-1\}^{\ell}$, the word w appears in the base-b expansion of x with the correct frequency, and we denote the set of such numbers by $\mathcal{N}_b$. More explicitly, if $x = 0.x_1x_2\cdots x_n\cdots_b$, then

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N \mid w = x_nx_{n+1}\cdots x_{n+\ell-1}\} = b^{-\ell}. \quad (1)$$

Equivalently, x is **normal base-b** if the sequence $(b^n x)_{n=1}^{\infty}$ is uniformly distributed in $[0,1]$. More explicitly, if for any $0 < a < c < 1$ we have

$$\lim_{N\to\infty} \frac{1}{N}\#\{1 \leq n \leq N \mid b^n x \pmod 1 \in (a,c)\} = c - a. \quad (2)$$

Base-b normality 2/2

We observe that for $x \in [0,1]$ with a base-2 expansion of $x = 0.x_1x_2\cdots x_n\cdots$, we have

$$2^n x \pmod 1 \in \begin{cases} [0, \frac{1}{2}) & \text{iff } x_{n+1} = 0 \\ [\frac{1}{2}, 1) & \text{iff } x_{n+1} = 1 \\ [0, \frac{1}{4}) & \text{iff } (x_{n+1}, x_{n+2}) = (0,0) \\ [\frac{1}{4}, \frac{2}{4}) & \text{iff } (x_{n+1}, x_{n+2}) = (0,1) \\ [\frac{2}{4}, \frac{3}{4}) & \text{iff } (x_{n+1}, x_{n+2}) = (1,0) \\ [\frac{3}{4}, \frac{4}{4}) & \text{iff } (x_{n+1}, x_{n+2}) = (1,1). \end{cases}$$

More generally, if $x = 0.x_1x_2\cdots x_n\cdots_b$ and $w = (w_1, \cdots, w_\ell) \in \{0,1,\cdots,b-1\}^\ell$, then

$$b^n x \pmod 1 \in [\sum_{j=1}^{\ell} \frac{w_j}{b^j}, \sum_{j=1}^{\ell} \frac{w_j}{b^j} + \frac{1}{b^\ell}) \text{ iff } (x_{n+1}, \cdots, x_{n+\ell}) = w.$$

Cantor series

Given a basic sequence (sequence of bases) $Q = (q_n)_{n=1}^{\infty} \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$ and some $x \in [0,1]$, the base Q expansion $x = 0.x_1 x_2 \cdots x_n \cdots_Q$ with $0 \leq x_i < q_i$ is defined by

$$x = \sum_{n=1}^{\infty} x_i \left(\prod_{j=1}^{n} q_j \right)^{-1} = \frac{x_1}{q_1} + \frac{x_2}{q_1 q_2} + \frac{x_3}{q_1 q_2 q_3} + \cdots \qquad (3)$$

It is unique for all but countably many points in $[0,1]$.

Normality for a Cantor series

Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$ and an $x = 0.x_1x_2\cdots x_n\cdots_Q \in [0,1]$, we say that x is **Q-normal** if for any block of digits $D = (d_1, \cdots, d_\ell) \in \mathbb{Z}_{\geq 0}^{\ell}$ appears in the base Q expansion of x with the correct frequency, and we denote the set of such x by $\mathcal{N}(Q)$. More precisely, if D satisfies

$$M_D(N) := \sum_{n=1}^{N} \left(\prod_{j=1}^{\ell} \frac{1}{q_{n+j}} \mathbb{1}_{[0,q_{n+j})}(d_j) \right) \underset{N\to\infty}{\longrightarrow} \infty, \text{ then}$$

$$\lim_{N\to\infty} \#\{1 \leq n \leq N \mid x_n x_{n+1} \cdots x_{n+\ell-1} = D\}/M_D(N) = 1.$$

Distribution normality for a Cantor series

Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$ and an $x \in [0,1]$, we say that x is **Q-distribution normal** if the sequence $(x, q_1x, q_2q_1x, \cdots, q_nq_{n-1}\cdots q_2q_1x, \cdots)$ is uniformly distributed, and the set of such x is denoted by $\mathcal{DN}(Q)$. For a general basic sequence Q, the notions of Q-normality and Q-distribution normality don't need to be the same.

Theorem (Airey, Jackson, and Mance [1])

If $Q = (q_n)_{n=1}^{\infty}$ is such that $\lim_{n\to\infty} q_n = \infty$ and $\sum_{n=1}^{\infty} q_n^{-1} = \infty$, then $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ and $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ are $D_2(\Pi_3^0)$-complete.

See also [3] for results regarding the Hausdorff dimension of the difference sets, which are usually 1.

Remark: For any Q, the set $\mathcal{DN}(Q)$ has Lebesgue measure 1, and if Q is such that $\sum_{n=1}^{\infty} q_n^{-1} < \infty$, then $\mathcal{N}(Q) = [0,1]$.

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Dynamically generated basic sequences

Definition

A basic sequence $Q = (q_n)_{n=1}^{\infty}$ is **dynamically generated** if there exists a (continuous) measure preserving system $(X, \mathscr{B}, \mu, T)$ on a (not necessarily compact) polish space X, a continuous function $f : X \to \mathbb{N}_{\geq 2}$, and a μ-generic point $y \in X$ for which $q_n = f(T^n y)$.

1. If $X = \{1\}$ and T is (necessarily) the identity, then we recover base-b, i.e., $q_n = b$ for all n.
2. If $X = \{0, 1\}$, $Tx = x + 1 \pmod 2$, and $f(i) = b_i$, then we get $q_n = b_0$ if n is even and $q_n = b_1$ if n is odd. (See [2])
3. If $X = [0, 1]$, $Tx = x + \alpha \pmod 1$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $f = 2\mathbb{1}_{[0,\frac{1}{2})} + 3\mathbb{1}_{[\frac{1}{2},1)}$, then $(q_n)_{n=1}^{\infty}$ will be almost periodic sequence of 2s and 3s. (There is a model in which f is continuous)
4. We may also consider $f(x) = \left\lfloor \frac{1}{x} \right\rfloor$ in the previous example.

Uniform normality of a Cantor series

Let $Q = (q_n)_{n=1}^{\infty}$ be a dynamically generated basic sequences. For each block $B = (b_1, \cdots, b_\ell) \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$, we have that

$$Q_B := \lim_{N\to\infty} \frac{1}{N} \underbrace{\#\{1 \leq n \leq N \mid (q_n, q_{n+1}, \cdots, q_{n+\ell-1}) = B\}}_{Q_B(N)}$$

exists, and it is 0 if and only if the block B never appears in Q. $x = 0.x_1x_2\cdots x_n\cdots_Q \in [0,1]$ is Q-**uniformly normal** if for any block of digits $D = (d_1, \cdots, d_\ell)$, and any block of bases $B = (b_1, \cdots, b_\ell)$ with $Q_B > 0$ and $b_j > d_j$ for all j, we have

$$\lim_{N\to\infty} \frac{1}{Q_B(N)} \#\{1 \leq n \leq N \mid (x_n, \cdots, x_{n+\ell-1}) = D \;\& \; (q_n, \cdots, q_{n+\ell-1}) = B\} = \prod_{j=1}^{\ell} \frac{1}{b_j},$$

and we denote the set of such x by $\mathcal{UN}(Q)$. I.e., $x \in \mathcal{UN}(Q)$ iff the pairs (D, B) of digits and bases occur with the correct frequency.

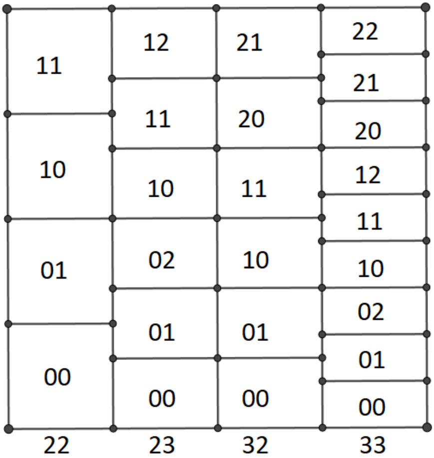
Uniform distribution normality of a Cantor series

Let $Q = (q_n)_{n=1}^{\infty}$ be a dynamically generated basic sequence, generated by the (continuous) m.p.s. $(X, \mathscr{B}, \mu, T)$ and the continuous function $f : X \to \mathbb{N}_{\geq 2}$. In particular, we have $q_n = f(T^n y)$ for some $y \in X$. Furthermore, let us assume that this representation is *minimal* in the sense that X is a polish space, f is continuous, and the topology generated by f and T is that of X. A number $x \in [0, 1]$ is Q-**uniformly distribution normal** if $(S^n(y, x))_{n=1}^{\infty}$ is uniformly distributed in $X \times [0, 1]$, where $S(y, x) = (Ty, f(y)x)$, and we denote the set of such x by $\mathcal{UDN}(Q)$.

A nice equivalence 1/2

Theorem

If $Q = (q_n)_{n=1}^{\infty}$ is a dynamically generated basic sequence, then $\mathcal{UN}(Q) = \mathcal{UDN}(Q)$.



A nice equivalence 2/2

Theorem

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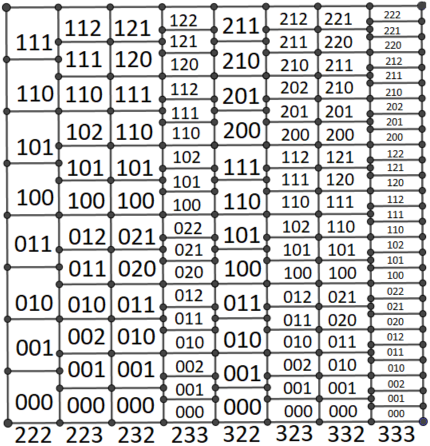

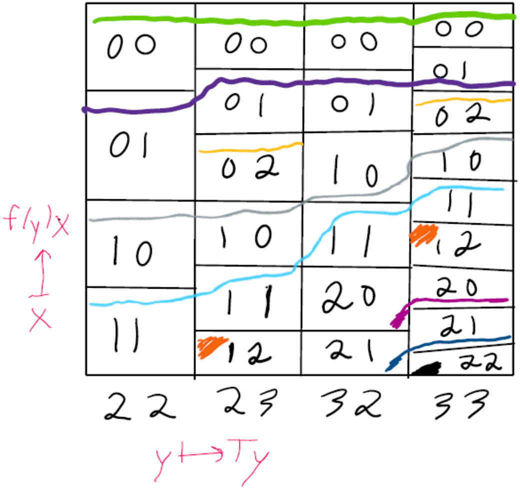

Q-normality means $S^n(y,x)$ visits the colored regions with the correct frequency, where $S(y,x) = (Ty, f(y)x)$

When $\mathcal{N}(Q) = \mathcal{DN}(Q)$

Theorem

Suppose that $Q = (q_n)_{n=1}^{\infty}$ is a basic sequence (dynamically) generated by $(X, \mathscr{B}, \mu, T, y, f)$ with $(X, \mathscr{B}, \mu, T)$ being ergodic and having zero (measurable) entropy, and $\int_X \ln(f) d\mu < \infty$. Then $\mathcal{N}(Q) = \mathcal{UN}(Q) = \mathcal{UDN}(Q) = \mathcal{DN}(Q)$. Furthermore, if $f : X \to \{b^n\}_{n=1}^{\infty}$ for some $b \geq 2$, then $\mathcal{N}(Q) = \mathcal{N}_b$.

Examples of zero entropy systems to consider.

1. $Tx = x + \alpha \pmod 1$ with $\alpha \in \mathbb{R}$.
2. T is any finite interval exchange transformation.
3. T is the Horocycle flow.
4. T is a rank 1 transformation.
5. $(q_n)_{n=1}^{\infty}$ "is" the Thue-Morse sequence with 2s and 4s.

When $\mathcal{DN}(Q) \not\to \mathcal{N}(Q)$

Theorem

There exists a dynamically generated sequences $Q = (q_n)_{n=1}^{\infty}$ and a sequence of digits $(E_n)_{n=1}^{\infty}$ for which $x = x_1 x_2 \cdots x_n \cdots_Q$ is distribution normal but not normal.

Sketch: Let $x \in [0, 1]$ be normal base-4 (which is the same as normal base-2). We will now construct a sequence $(q_n)_{n=1}^{\infty} \in \{2, 4\}^{\mathbb{N}}$ in which the 2s always appear in blocks of even size (groups of 2,4,6,...). We let $q_1 = q_2 = 2$ if $x \in [\frac{1}{2}, 1)$ and $q_1 = 4$ otherwise. We now replace x with $4x$ and repeat this procedure inductively to construct the rest of the q_n. The number x is Q-distribution normal by construction, but it is not Q-normal since the digits 2 and 3 never appear.

When $\mathcal{N}(Q) \not\to \mathcal{DN}(Q)$

Theorem

There exists a dynamically generated sequences $Q = (q_n)_{n=1}^{\infty}$ and a sequence of digits $(E_n)_{n=1}^{\infty}$ for which $x = E_1 E_2 \cdots E_n \cdots_Q$ is normal but not distribution normal.

Proof: Let Ω be a probability space and $(q_n(\omega_1))_{n=1}^{\infty}$ a sequences of i.i.d. random variables taking values in $\{2, 4\}$ (can also be done for {2,6}) with $\mathbb{P}(X_n = 2) = \mathbb{P}(X_n = 4) = \frac{1}{2}$. Consider

$$E_n(\omega_1)(\omega_2) = \begin{cases} 0 & \text{with probability } \frac{1}{2} + \epsilon \text{ if } q_n(\omega_1) = 2 \\ 1 & \text{with probability } \frac{1}{2} - \epsilon \text{ if } q_n(\omega_1) = 2 \\ 0 & \text{with probability } \frac{1}{4} - \epsilon \text{ if } q_n(\omega_1) = 4 \\ 1 & \text{with probability } \frac{1}{4} + \epsilon \text{ if } q_n(\omega_1) = 4 \\ 2 & \text{with probability } \frac{1}{4} \qquad \text{if } q_n(\omega_1) = 4 \\ 3 & \text{with probability } \frac{1}{4} \qquad \text{if } q_n(\omega_1) = 4. \end{cases}$$

Hot Spot Theorems

Pyatetskii-Shapiro [11] introduced what is now commonly known as the Hot Spot Theorem.

Theorem

Fix $b \in \mathbb{N}_{\geq 2}$ and $x \in [0,1]$. If there exists $C > 0$ such that for all $0 \leq a < c \leq 1$ we have

$$\limsup_{N\to\infty} \frac{1}{N}|\{1 \leq n \leq N \mid b^n x \pmod 1 \in (a,c)\}| \leq C(c-a), \quad (4)$$

then x is normal base b.

This result was generalized in [4, 9, 8, 5]. We have analogues of the strongest version of the Hot Spot Theorem proven in [5] for a large class of deterministic dynamically genereated basic sequences.

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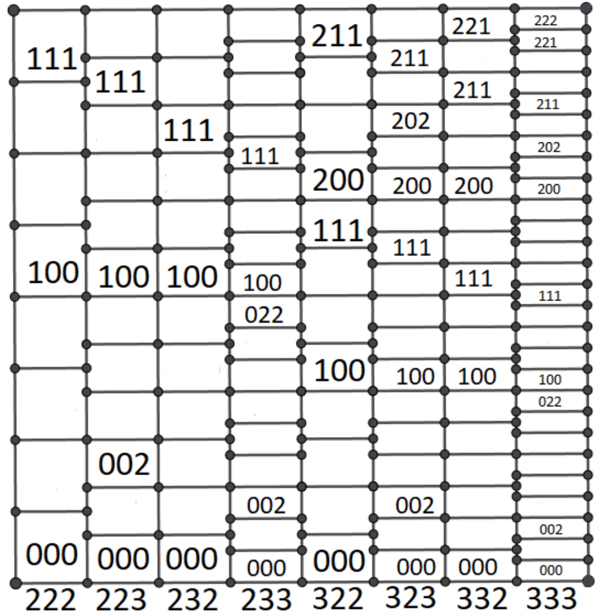
Dynamics and normality?



We saw on slide 14 that $x \in \mathcal{N}(Q)$ if and only if $(S^n(y,x))_{n=1}^{\infty}$ equidistributes in a particular S-invariant collection of open sets in $X \times [0,1]$.

Question

What is the factor of $(X, \mathscr{B}, \mu, T)$ that is generated by this collection of open sets?

Selection rules

Kamae and Weiss [7, 6] proved the following selection rule:

Let $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ be an increasing sequence with positive lower density.

(i) If $(n_k)_{k=1}^{\infty}$ is deterministic, then for any $x = 0.x_1x_2\cdots x_n\cdots_b$ that is normal base b, $x' := 0.x_{n_1}x_{n_2}\cdots x_{n_k}\cdots_b$ will be normal base b.

(ii) If $(n_k)_{k=1}^{\infty}$ is not deterministic, then there exists a $x = 0.x_1x_2\cdots x_n\cdots_b$ that is normal base b for which $x' := 0.x_{n_1}x_{n_2}\cdots x_{n_k}\cdots_b$ is not normal base b.

Question: What are the selection rules for dynamically generated basic sequences?

We have some partial results, but not a complete classification.

Normality preservation through addition

Rauzy [10] characterized those $y \in [0,1]$ for which $y + \mathcal{N}_b \pmod 1 = \mathcal{N}_b$. Given a dynamically generated basic sequence Q, can we characterize those $y \in [0,1]$ for which $y + \mathcal{N}(Q) \pmod 1 = \mathcal{N}(Q)$? How about the same question for $\mathcal{DN}(Q)$ or $\mathcal{UN}(Q)$?

Difference Sets

Question: Suppose that Q is a dynamically generated basic sequence for which $\mathcal{N}(Q) \neq \mathcal{DN}(Q)$. What can be said about the descriptive complexity and the Hausdorff dimensions of the difference sets $\mathcal{DN}(Q) \setminus \mathcal{N}(Q), \mathcal{N}(Q) \setminus \mathcal{DN}(Q), \mathcal{N}(Q) \setminus \mathcal{UN}(Q)$, and $\mathcal{DN}(Q) \setminus \mathcal{UN}(Q)$, and $(\mathcal{N}(Q) \cap \mathcal{DN}(Q)) \setminus \mathcal{UN}(Q)$?

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