Compactness in density Ramsey Theory

Seminar on topological semigroups and Ramsey Theory
Howard University

Based on joint work with Steve Jackson and William Mance

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Overview

Compactness in partition Ramsey Theory

The following two special cases of the compactness principle in Ramsey theory are useful when studying the partition regularity of polynomial equations.

Theorem (the compactness principle)

Let R be an integral domain, let $S \subseteq R$, and let $p \in R[x_1, \cdots, x_n]$.

1. *Given $r \in \mathbb{N}$, the equation $p(x_1, \cdots, x_n) = 0$ is (injectively) r-partition regular over S if and only if it also (injectively) r-partition regular over some finite set $F_r \subseteq S$.*
2. *The equation $p(x_1, \cdots, x_n) = 0$ is (injectively) partition regular over S if and only if for every $r \in \mathbb{N}$ it is (injectively) r-partition regular over some finite set $F_r \subseteq S$.*

For a more general treatment, the reader is referred to [2].

The group of right quotients of a semigroup

Let G be a group and let $S \subseteq G$ be a subsemigroup for which every $g \in G$ can be written in the form $g = rs^{-1}$ for some $r, s \in S$. G is called **a group of right quotients of** S. We see that $(\mathbb{Z}, +)$ is a group of right quotients of $(\mathbb{N}, +)$, and $(\mathbb{Q} \setminus \{0\}, \times)$ is a group of right quotients of $(\mathbb{Z} \setminus \{0\}, \times)$. If G_1 and G_2 are groups of right quotients of a semigroup S, then G_1 and G_2 are isomorphic as groups, so we speak of **the** group of right quotients of S. If G is a group of right quotients of S, then S is thick in G, i.e., for every $F \in \mathscr{P}_f(G)$, there exists $s \in S$ for which $Fs \subseteq S$. We observe that if S_2 is the free semigroup of two generators a and b, and F_2 is the free group on the generators a and b, then S_2 generates F_2 as a group. However, S_2 is not thick in F_2, and F_2 is not a group of right quotients of S_2.

left reversible semigroups

A semigroup S is **left reversible** if any two principle right ideals of S intersect. In other words, S is left reversible if for any $r, s \in S$, we have $rS \cap sS \neq \emptyset$, so there exists $r_1, s_1 \in S$ for which $rr_1 = ss_1$. It is well known that every left amenable semigroup is left reversible. It is also well known that every cancellative left reversible semigroup embeds in its group of right quotients. Consequently, every cancellative left amenable semigroup S embeds in its group of right quotients G, which is an amenable group. If a cancellative semigroup S embeds in a group of its right quotients, then S is left reversible. See [1, 3] for more about embedding semigroups into groups.

Invertible extensions of measure preserving systems

Theorem

Let S be a cancellative left-reversible semigroup, let G be the group of right quotients of S, and let $\mathcal{X} := (X, \mathscr{B}, \mu, (T_s)_{s\in S})$ be a S-system. There exists a G-system $\mathcal{Y} := (Y, \mathscr{A}, \nu, (R_s)_{s\in G})$, such that the S-system $\mathcal{Y}' := (Y, \mathscr{A}, \nu, (R_s)_{s\in S})$ is an extension of $\mathcal{X}$.

Example: We begin with the $\mathbb{N}$-system $([0,1], \mathscr{B}, m, \times 2)$, which is clearly not an invertible system. We the identify this system with $(\{0,1\}^{\mathbb{N}}, \mathscr{B}', \mu_2^{\mathbb{N}}, \sigma)$, where σ is the left-shift. The invertible extension of this system is seen to be $(\{0,1\}^{\mathbb{Z}}, \mathscr{B}'', \mu_2^{\mathbb{Z}}, \sigma)$, and the factor map $\pi : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{N}}$ is given by $\pi(x_n)_{n\in\mathbb{Z}} = (x_n)_{n\in\mathbb{N}}$.

Compactness in Density Ramsey Theory 1/2

Theorem (F., Jackson, Mance, 2024+)

Let S be a countably infinite cancellative left-amenable semigroup, let $\delta \in (0,1)$, and let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be right-translation invariant. The following are equivalent:

1. *If $B \subseteq S$ satisfies $d^*(B) \geq \delta$, then $\exists\ A \in \mathcal{A}$ with $A \subseteq B$.*
2. *There exists $K \in \mathscr{P}_f(S)$ and $\epsilon > 0$ such that for any (K,ϵ)-invariant set $F \in \mathscr{P}_f(S)$ and for all $B \subseteq F$ with $|B| \geq \delta|F|$, there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
3. *There exists a $H \in \mathscr{P}_f(S)$ such that for all $B \subseteq H$ with $|B| \geq \delta|H|$, there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
4. *There exists a $H \in \mathscr{P}_f(S)$ and an $\epsilon > 0$ such that for all $B \subseteq H$ with $|B| \geq (\delta - \epsilon)|H|$, there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
5. *There exists $\epsilon > 0$ such that if $B \subseteq S$ satisfies $d^*(B) > \delta - \epsilon$, then $\exists\ A \in \mathcal{A}$ with $A \subseteq B$.*

Compactness in Density Ramsey Theory 2/2

Corollary

Let S be a countably infinite cancellative left-amenable semigroup, let $\delta \in [0,1)$, and let $\mathcal{A} \subseteq \mathscr{P}_f(X)$ be right-translation invariant. The following are equivalent:

1. (i) *If $B \subseteq S$ satisfies $d^*(B) > \delta$, then there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
2. (ii) *For all $\gamma > \delta$, there exists $K \in \mathscr{P}_f(S)$ and $\epsilon > 0$ such that for any (K, ϵ)-invariant set $F \in \mathscr{P}_f(S)$ and for all $B \subseteq F$ with $|B| \geq \gamma |F|$, there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
3. (iii) *For all $\gamma > \delta$ there exists $F \in \mathscr{P}_f(S)$ such that for all $B \subseteq F$ with $|B| \geq \gamma |F|$, there exists $A \in \mathcal{A}$ with $A \subseteq B$.*

A fun corollary

Corollary (F., Jackson, Mance, 2024)

Let S be a countably infinite cancellative left-amenable semigroup and let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be right-translation invariant. There exists $\delta \in [0, 1)$ for which the following hold:

1. *If $B \subseteq S$ satisfies $d^*(B) > \delta$, then there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
2. *There exists a $B \subseteq S$ with $d^*(B) = \delta$, such that B does not contain any member of $\mathcal{A}$.*

Equivalences for Density Regularity 1/1

Theorem (Farhangi, Jackson, Mance, 2024+)

Let S be a countably infinite cancellative left-amenable semigroup, let G be the group of right quotients of S, let $\mathcal{F} = (F_n)_{n=1}^{\infty}$ be a left-Følner sequence in S, let $\delta \in [0,1)$ be arbitrary, and let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be arbitrary. We have (i)→(ii)→(iii)→(iv)→(v)→(vi) and (v)→(vii). If $\mathcal{A}$ is right-translation invariant, then (i)-(vii) are equivalent.

1. *If $B \subseteq S$ satisfies $d^*(B) > \delta$, then there exists $A \in \mathcal{A}$ with $A \subseteq B$ i.e., $\mathcal{A}$ is* **weakly δ-density regular**.
2. *If $B \subseteq S$ satisfies $\overline{d}_{\mathcal{F}}(B) > \delta$, then there exists $A \in \mathcal{A}$ with $A \subseteq B$.*
3. *For any ergodic G-system $(X, \mathscr{B}, \mu, (T_s)_{s \in G})$ and any $B \in \mathscr{B}$ with $\mu(B) > \delta$, there exists $A \in \mathcal{A}$ for which $\mu\left(\bigcap_{s \in A} T_s^{-1} B\right) > 0$.*

Equivalences for Density Regularity 2/2

Theorem (Farhangi, Jackson, Mance, 2024+)

(iv) *For any G-system $(X, \mathscr{B}, \mu, (T_s)_{s\in G})$ and any $B \in \mathscr{B}$ with $\mu(B) > \delta$, there exists $A \in \mathcal{A}$ for which $\mu\left(\bigcap_{s\in A} T_s^{-1}B\right) > 0$.*

(v) *For any S-system $(X, \mathscr{B}, \mu, (T_s)_{s\in S})$ and any $B \in \mathscr{B}$ with $\mu(B) > \delta$, there exists $A \in \mathcal{A}$ for which $\mu\left(\bigcap_{s\in A} T_s^{-1}B\right) > 0$.*

(vi) *If $h : G \to G$ is a homomorphism and $B \subseteq S$ satisfies $d^*(B) > \delta$, then $\exists\ A \in \mathcal{A}$ with $d^*\left(\bigcap_{a\in A} h(a)^{-1}B\right) > 0$.*

(vii) *If $h : G \to G$ is a homomorphism and $B \subseteq S$ satisfies $\overline{d}_{\mathcal{F}}(B) > \delta$, then $\exists\ A \in \mathcal{A}$ with $\overline{d}_{\mathcal{F}}\left(\bigcap_{a\in A} h(a)^{-1}B\right) > 0$.*

Uniformity of Density regularity 1/2

Theorem (F., Jackson, Mance, 2024+)

*Let S be a countably infinite cancellative left-amenable semigroup, let G be the group of right quotients of S. Let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be δ-**density regular**, i.e., for any measure preserving system $(X, \mathscr{B}, \mu, (T_s)_{s\in S})$ and any $B \in \mathscr{B}$ with $\mu(B) \geq \delta$, there exists $A \in \mathcal{A}$ for which*

$$\mu\left(\bigcap_{a\in A} T_a^{-1}B\right) > 0.$$

Then there exists a finite collection $\mathcal{A}_\delta \subseteq \mathcal{A}$ that is also δ-density regular. In fact, we can say even more.

Uniformity of Density regularity 2/2

Theorem (Uniformity of Density Regularity)

Let $\mathcal{A} \subseteq \mathscr{P}_f(S)$ be right-translation invariant and δ-density regular. Then there exists a finite subcollection $\mathcal{A}' \subseteq \mathcal{A}$ and a $\gamma > 0$ such that the following hold:

1. *There exists $K \in \mathscr{P}_f(S)$ and $\epsilon > 0$ such that for any (K, ϵ)-invariant set $F \in \mathscr{P}_f(S)$, and any $B \subseteq F$ with $|B| \geq (\delta - \epsilon)|F|$, there exists $A \in \mathcal{A}'$ for which $|F \cap \bigcap_{a \in A} a^{-1}B| > \gamma|F|$.*
2. *If $\mathcal{F}$ is a left-Følner sequence in S, and $B \subseteq S$ satisfies $\overline{d}_{\mathcal{F}}(B) \geq \delta$, then there exists $A \in \mathcal{A}'$ for which $\overline{d}_{\mathcal{F}}(\bigcap_{a \in A} a^{-1}B) \geq \gamma$.*
3. *For any S-system $\mathcal{X} := (X, \mathscr{B}, \mu, (T_s)_{s \in S})$ and any $B \in \mathscr{B}$ with $\mu(B) \geq \delta$, there exists $A \in \mathcal{A}'$ for which $\mu\left(\bigcap_{a \in A} T_a^{-1}B\right) \geq \gamma$.*

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