Decidability in Ramsey theory

Seminar on topological semigroups and Ramsey Theory
Howard University

Based on joint work with Steve Jackson and William Mance

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Sohail Farhangi
Slides available on sohailfarhangi.com

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Overview

Computable sets

Definition

A set $A \subseteq \mathbb{N}$ is **computable** if there exists an algorithm (Turing machine) that halts on every input, and outputs 1 if and only if the input is an element of A. If S is a countably infinite set, then $A \subseteq S$ is **computable** if there exists a a computable bijection $\phi : S \to \mathbb{N}$ for which the set $\phi(A)$ is computable.

Examples of computable subsets of $\mathbb{N}$ include the set of squares, the set of primes, the set of powers of 2, and the set of square free numbers. If $S \subseteq \mathbb{Z}[x]$ denotes the collection of polynomials of degree at most 2, and $A \subseteq S$ is those polynomials that have an integer root, then A is a computable set. Rado's Theorem [5] shows us that if S is the set of finite systems of linear equations with coefficients in $\mathbb{Z}$, and $A \subseteq S$ consists of those systems that are partition regular over $\mathbb{N}$, then A is computable.

Computably enumerable sets

Definition

A set $A \subseteq \mathbb{N}$ is **computably enumerable** if there exists an algorithm (Turing machine) such that the set of inputs for which the algorithm halts is exactly A. Equivalently, A is computably enumerable if there exists an algorithm that enumerates the members of A.

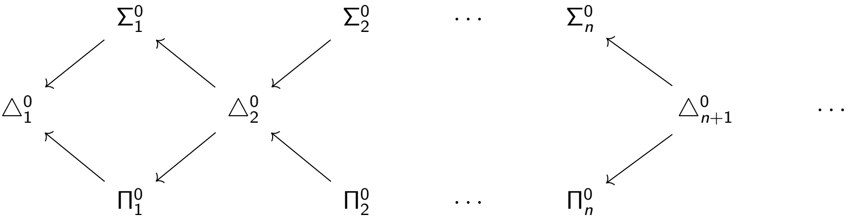
Given a polynomial $p \in \mathbb{Z}[x_1, \cdots, x_n]$, the set Z_p of integer roots of p is seen to be computably enumerable from the second of the given definitions. If $S = \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$, and $A \subseteq S$ is those polynomials that possess an integer root, then A is seen to be computably enumerable from the first of the given definitions.

Lemma

A set $A \subseteq \mathbb{N}$ is computable if and only if A and A^c are both computably enumerable.

The lightface hierarchy

We denote the collection of computable subsets of $\mathbb{N}$ by $\triangle_1^0$, the collection of computably enumerable subsets of $\mathbb{N}$ by Σ_1^0, and we let Π_1^0 denote the collection of sets whose complement is in Σ_1^0. We inductively define Σ_{n+1}^0 to be the collection of sets that are reducible to a set in Π_n^0 in a "computably enumerable fashion", Π_{n+1}^0 is the complement of Σ_{n+1}^0, and $\triangle_{n+1}^0 = \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$.



Hilbert's 10th problem (HTP)

At the International Congress of Math in 1900, David Hilbert presented 10 important problems in mathematics, and 2 years later published a completed list of 23 problems now known as Hilbert's problems. The 10^{th} of the 23 problems (published but not presented) asked if the set $A \subseteq S := \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ of polynomials that have an integer root is computable.

Theorem (Matiyasevič, 1971)

The set A is not computable.

See [2] for an exposition of the proof of this result and the history.

Open Problem: Is the set $A_{\mathbb{Q}} \subseteq S$ of polynomials that have a root in $\mathbb{Q}$ computable?
The latter problem is referred to as Hilbert's 10th problem over $\mathbb{Q}$.
It is generally believed that $A_{\mathbb{Q}}$ is not computable.

Variations of Hilbert's 10th problem

Given a **computable** integral domain R, we let $HTP(R)$ refer to the following statement:

HTP(R): The set $A \subseteq S := \bigcup_{n=1}^{\infty} R[x_1, \cdots, x_n]$ of those polynomials that have a root in R is not computable.

The statement HTP(R) can be true, or false depending on the integral domain R.

Theorem ([6, 4, 3])

Suppose that $R = \overline{\mathbb{F}_p}$ for some prime p, $R = \mathbb{Z}$, or that $R = R'[t]$ for some integral domain R'.

1. *$HTP(R)$ is true.*
2. *The set $A' \subseteq S$ of polynomials possessing a root $(z_1, \cdots, z_n) \in R^n$ with $z_i \neq z_j$ when $i \neq j$ is not computable.*

Reducing partition regularity to HTP

Lemma (cf. Krawczyk,Byszewski, 2021 [1])

Let R be an integral domain with field of fractions K. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K$, the system of equations

$$\frac{z_{3i-2} - z_{3i-1}}{z_{3i}} = k_i \text{ for all } 1 \leq i \leq m, \tag{1}$$

is partition regular over $R \setminus \{0\}$.

Corollary

Given an integral domain R, and a polynomial $p \in R[x_1, \cdots, x_n]$, p has a root in K if and only if the equation $p'(x_1, \cdots, x_{3n}) = 0$ with

$$p'(x_1, \cdots, x_{3n}) := p\left(\frac{x_1 - x_2}{x_3}, \cdots, \frac{x_{3n-2} - x_{3n-1}}{x_{3n}}\right)\left(\prod_{i=1}^{n} x_{3i}\right)^{deg(p)}$$

is partition regular over $R \setminus \{0\}$.

First main result

Theorem (F., Jackson, Mance, 2024+)

1. *Let us assume that HTP($\mathbb{Q}$) is true. For $\ell \in \mathbb{N}$, the set $A_\ell \subseteq \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ is ℓ-partition regular over $\mathbb{Z} \setminus \{0\}$ is computably enumerable but not computable, so it is Σ_1^0-complete. The set $A := \bigcap_{\ell=1}^{\infty} A_\ell$ is Π_2^0-complete.*
2. *Suppose that R is the ring of integers of an algebraic function field over a finite field of constants. For $\ell \in \mathbb{N}$, the set $A_\ell \subseteq \bigcup_{n=1}^{\infty} R[x_1, \cdots, x_n]$ of (homogeneous) polynomials p for which the equation $p(x_1, \cdots, x_n) = 0$ is ℓ-partition regular over $R \setminus \{0\}$ is Σ_1^0-complete. The set $A := \bigcap_{\ell=1}^{\infty} A_\ell$ is Π_2^0-complete.*

Reduction to HTP for density Ramsey theory 1/2

Lemma

Let R be a countably infinite integral domain with field of fractions K. For any $m \in \mathbb{N}$ and any $k_1, \cdots, k_m \in K^\times$ we have the following:

1. *If $A \subseteq R$ is such that $d^*(A) > 0$, then A contains a solution to the system of equations*

$$\frac{z_{4i-3} - z_{4i-2}}{z_{4i-1} - z_{4i}} = k_i \text{ for all } 1 \leq i \leq m. \tag{2}$$

   Furthermore, the solution can be taken such that $z_i \neq z_j$ when $i \neq j$.

2. *If $A \subseteq R \setminus \{0\}$ is such that $d^*_\times(A) > 0$, then A contains a solution $(z_1, \cdots, z_{4m})$ to the system (2), such that $z_i \neq z_j$ for $i \neq j$.*

Reduction to HTP for density Ramsey theory 2/2

Corollary

Let R be a countably infinite integral domain with field of fractions K, and let $p \in R[x_1, \cdots, x_n]$.

1. *p has a root in K if and only if for any $A \subseteq R$ with $d^*(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$, where*

$$p'(z_1, \cdots, z_{4n}) = p\left(\frac{z_1 - z_2}{z_3 - z_4}, \cdots, \frac{z_{4n-3} - z_{4n-2}}{z_{4n-1} - z_{4n}}\right)\left(\prod_{i=1}^{n}(z_{4n-1} - z_{4n})\right)^{deg(p)}.$$

2. *p has a root in K if and only if for any $A \subseteq R \setminus \{0\}$ with $d^*_\times(A) > 0$, there exist distinct $z_1, \cdots, z_{4n} \in A$ for which $p'(z_1, \cdots, z_{4n}) = 0$.*

Second main result

Theorem (F., Jackson, Mance, 2024+)

1. *Let us assume that HTP*$(\mathbb{Q})$ *is true. For* $\delta \in (0,1)$*, the set* $A_\delta \subseteq \bigcup_{n=1}^{\infty} \mathbb{Z}[x_1, \cdots, x_n]$ *of (homogeneous) polynomials* p *for which the equation* $p(x_1, \cdots, x_n) = 0$ *has an injective solution in any set* $B \subseteq \mathbb{Z}$ *with* $\overline{d}(B) \geq \delta$ *is* Σ_1^0*-complete. The set* $A := \bigcap_{\delta > 0} A_\delta$ *is* Π_2^0*-complete. Analogous results hold when* $\overline{d}$ *is replaced with* d^* *or* $d_\times^*$*.*
2. *Suppose that* R *is the ring of integers of an algebraic function field over a finite field of constants. For* $\delta > 0$*, the set* $A_\delta \subseteq \bigcup_{n=1}^{\infty} R[x_1, \cdots, x_n]$ *of (homogeneous) polynomials* p *for which the equation* $p(x_1, \cdots, x_n) = 0$ *has an injective solution in any set* $B \subseteq R$ *with* $d^*(B) \geq \delta$ *is* Σ_1^0*-complete. The set* $A := \bigcap_{\delta > 0} A_\delta$ *is* Π_2^0*-complete. Analogous results hold when* d^* *is replaced with* $\overline{d}_{\mathcal{F}}$ *or* $d_\times^*$*.*

Implications 1/2

Observe that if R is a countably infinite integral domain, then there are only countably many polynomials p with coefficients in R. Consequently, the set B of such polynomials that are not partition regular over $R \setminus \{0\}$ is countable, so for each $p \in B$ there exists a partition P_p of $R \setminus \{0\}$ that does not contain a root of p. Consequently, to determine whether or not a polynomial p is partition regular over $R \setminus \{0\}$, it suffices check whether or not p has a root in some cell of each partition in the family $\{P_b\}_{b \in B}$. The fact that the set of "partition regular polynomials" is Π^0_2-complete, means that there does not exist a simpler method of determining whether or not the equation $p(x_1, \cdots, x_n) = 0$ is partition regular over $R \setminus \{0\}$.

Implications 2/2

The following statement is false since it is describing a Σ_1^0 but the set of "partition regular polynomials" is (conditionally) a Π_2^0-complete set.

False Statement: Let R be a countably infinite integral domain. For each $p \in R[x_1, \cdots, x_n]$, there exists $q \in R[x_1, \cdots, x_m]$ that is a computable function of p such that $p(x_1, \cdots, x_n) = 0$ is partition regular over $R \setminus \{0\}$ if and only if q has a root in K.

However, the following conjecture has a chance of being true.

Conjecture

For each $p \in R[x_1, \cdots, x_n]$ and each $r \in \mathbb{N}$, there exists $q_r \in R[x_1, \cdots, x_m]$ that is a computable function of p and r such that $p(x_1, \cdots, x_n) = 0$ is r-partition regular over $R \setminus \{0\}$ if and only if q_r has a root in K.

Future work

Question

Can we prove a version of the results on the previous slide without the assumption that the solutions are injective?

Question

Given a $\ell \in \mathbb{N}$ and a finite system of linear equations, is there a computable condition to determine whether or not the system is ℓ-partition regular over $\mathbb{Z}$ (or over some integral domain R)?

Question

Given a $\delta \in (0, 1)$ and a finite system of linear equations, is there a computable condition to determine whether or not the system has a solution in every set $A \subseteq \mathbb{Z}$ with $d^*(A) > \delta$? How about $d^*_\times(A) > \delta$? What if we replace $\mathbb{Z}$ with an integral domain R?

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