

Notions of normality for dynamically generated basic sequences

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Basic Sequences, normality, and distribution normality

A sequence $Q = (q_n)_{n=1}^\infty \in \mathbb{N}_{\geq 2}^\mathbb{N}$ is called a **basic sequence**. We let $\tilde{Q} = \prod_{n=1}^\infty [0, q_n - 1]$ denote the space of Q -Cantor series of the elements of $[0, 1]$ under the correspondence given by

$$x = \sum_{n=1}^{\infty} \frac{a_n}{q_1 q_2 \cdots q_n}, \quad (1)$$

where we assume without loss of generality that $a_n \neq q_n - 1$ infinitely often. We may abbreviate this correspondence by writing $x = 0.a_1 a_2 \cdots a_n \cdots_Q$, and we let $\phi_Q : \tilde{Q} \rightarrow [0, 1]$ be the map given by $\phi_Q(a_n)_{n=1}^\infty = x$. We say that $(a_n)_{n=1}^\infty \in \tilde{Q}$ is **Q -distribution normal** if the sequence $(q_n q_{n-1} \cdots q_1 \phi_Q(a_m)_{m=1}^\infty)_{n=1}^\infty$ is uniformly distributed mod 1, and we let $\mathcal{DN}(Q)$ denote the collection of Q -distribution normal sequences. The sequence $(a_n)_{n=1}^\infty$ is **Q -normal** if all blocks of digits occur with the expected frequency (for which we omit the precise definition), and we denote the collection of Q -normal sequences by $\mathcal{N}(Q)$.

Dynamically generated basic sequences

A basic sequence $Q = (q_n)_{n=1}^\infty$ is **dynamically generated** if there exists a separable metric space X , a probability measure on (X, \mathcal{B}) with full topological support where \mathcal{B} being the Borel σ -algebra, a continuous μ -preserving transformation $T : X \rightarrow X$, a generic point $x \in X$ for (T, μ) , and a continuous $f : X \rightarrow \mathbb{N}_{\geq 2}$ for which $q_n = f(T^n x)$. If the system (X, \mathcal{B}, μ, T) has zero entropy, then Q is **deterministic**. We also have the following characterization of dynamically generated basic sequences.

Theorem 1: A basic sequence $Q = (q_n)_{n=1}^\infty$ is dynamically generated if and only if:

(i) For any $w := [w_1, w_2, \dots, w_k] \in \mathbb{N}_{\geq 2}^k$ the following limit exists

$$d(w) := \lim_{N \rightarrow \infty} \frac{1}{N} |\{1 \leq n \leq N \mid q_{n+i} = w_i \ \forall \ 0 \leq i < k\}|. \quad (2)$$

(ii) If $d(w) = 0$, then $w \neq [q_n, q_{n+1}, \dots, q_{n+k-1}]$ for any $n \in \mathbb{N}$.

(iii) For any $k \in \mathbb{N}$, we have $\sum_{|w|=k} d(w) = 1$.

(iv) Q is deterministic if and only if it has 'subexponential measurable word complexity'.

Examples of basic sequences

- If $b \in \mathbb{N}_{\geq 2}$ and $Q = (b)_{n=1}^\infty$, then $\mathcal{N}(Q) = \mathcal{DN}(Q)$, and $\mathcal{N}(Q)$ is precisely the set of numbers that are normal base- b .
- If $Q = (q_n)_{n=1}^\infty$ with $q_{2n} = 2$ and $q_{2n-1} = 3$, then $\mathcal{N}(Q) = \mathcal{DN}(Q)$, and $\mathcal{N}(Q)$ is precisely the set of numbers that are normal base-6. Furthermore, we see that Q is a dynamically generated basic sequence. In fact, any periodic Q is deterministic and dynamically generated.
- Let $Q = (q_n)_{n=1}^\infty$ be such that $\lim_{n \rightarrow \infty} q_n = \infty$ and $\sum_{n=1}^\infty \frac{1}{q_n} = \infty$. Then, $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ and $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ are $D_2(\Pi_3^0)$ -complete. A specific example of such a Q is given by $q_n = n + 1$.
- One representation of the Thue-Morse sequence $(d_n)_{n=1}^\infty$ is given by $d_n = (-1)^{c_n}$, where c_n is the number of 1s appearing in the base 2 expansion of n . For any $a, b \geq 2$, the sequence $Q = Q(a, b) = (q_n(a, b))_{n=1}^\infty$ given by $q_n(a, b) = a$ if $d_n = 1$ and $q_n(a, b) = b$ if $d_n = -1$ is a deterministic dynamically generated basic sequence.
- Let $(d_n)_{n=1}^\infty$ be the digits of the base 10 expansion of the Champernowne number. The sequence $Q_C = (q_n)_{n=1}^\infty$ given by $q_n = d_n + 2$ is a non-deterministic dynamically generated basic sequence.
- Let $(d_n)_{n=1}^\infty$ be the digits of the continued fraction expansion of the Adler-Keane-Smorodinsky number. The sequence $Q = (q_n)_{n=1}^\infty$ given by $q_n = d_n + 1$ is a non-deterministic dynamically generated basic sequence. We note that $\limsup_{n \rightarrow \infty} q_n = \infty$.

Uniform normality and uniform distribution normality

Let $Q = (q_n)_{n=1}^\infty$ be a dynamically generated basic sequence. $x \in [0, 1]$ is **uniformly Q -normal** if the pairs of bases and digits that appear in the base Q expansion of x appear with the correct frequency. To be more precise, let us fix a block $D = (d_1, \dots, d_K)$ of potential digits and a block $B = (b_1, \dots, b_K)$ of potential bases. Let $S_B = \{j \in \mathbb{N} \mid (q_j, q_{j+1}, \dots, q_{j+K-1}) = (b_1, b_2, \dots, b_K)\}$, and define

$$Q_n(D, B) = \sum_{j \in S_B \cap [1, n]} \frac{\mathcal{I}_{Q,j}(D)}{b_1 b_2 \cdots b_K}, \text{ and} \quad (3)$$

$$N_n^D(Q, z, B) = \#\{i \in S_B \cap [1, n] : d_j = E_{i+j}(z), \ \forall \ 1 \leq j \leq k\}. \quad (4)$$

We say that $z = 0.E_1 E_2 \cdots E_n \cdots_Q$ is **uniformly Q -normal** if for all blocks D and B with $D < B$ and $\lim_{n \rightarrow \infty} Q_n(D, B) = \infty$ we have

$$\lim_{n \rightarrow \infty} \frac{N_n^D(Q, z, B)}{Q_n(D, B)} = 1, \quad (5)$$

and we denote the collection of such sequences by $\mathcal{UN}(Q)$. Now let (X, \mathcal{B}, μ, T) be a system corresponding to Q and $f \in C(X)$ a function corresponding to Q . We say that $(E_n)_{n=1}^\infty$ is **uniformly Q -distribution normal** if (x, y) has a uniformly distributed orbit in the skew product system $(X \times [0, 1], \mathcal{B} \times \mathcal{L}, \mu \times m, T \rtimes M)$, where $M_n z = n z \pmod{1}$, $T \rtimes M(x, z) = (Tx, M_{f(x)} z)$, \mathcal{L} is the Lebesgue σ -algebra on $[0, 1]$ and m is the Lebesgue measure. We denote the collection of such sequences by $\mathcal{UDN}(Q)$. It is worth noting that a priori, the definition of uniformly Q -distribution normal sequences depends on the system corresponding to Q and the function corresponding to Q . Our next result shows that this is not the case.

Theorem 2: If $Q = (q_n)_{n=1}^\infty$ is a dynamically generated basic sequence, then $\mathcal{UN}(Q) = \mathcal{UDN}(Q)$.

Results for deterministic dynamically generated basic sequences

Theorem 3: Let $g \in \mathbb{N}_{\geq 2}$ and let $Q = (q_n)_{n=1}^\infty \in (\{g^n\}_{n=1}^\infty)^\mathbb{N}$ be a deterministic dynamically generated basic sequence satisfying

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log(q_n) < \infty. \quad (6)$$

For $y \in [0, 1]$ the following are equivalent:

(i) y is normal base g . (ii) y is uniformly Q -normal. (iii) y is Q -distribution normal (iv) y is Q -normal.

Theorem 4: If Q is a deterministic dynamically generated basic sequence, then $\mathcal{N}(Q) = \mathcal{DN}(Q) = \mathcal{UN}(Q)$.

Theorem 5 (Hot spot): Let $Q = (q_n)_{n=1}^\infty$ be a deterministic dynamically generated basic sequence. Let $x \in [0, 1]$ be such that for any $\sigma \in (0, 1)$, there exists a subset $\mathcal{N}_\sigma \subseteq \mathbb{N}$ of upper density at most σ , and a constant $C > 0$ (not dependent on σ) such that the following holds: For every $0 \leq a < b \leq 1$ we have

$$\limsup_{N \rightarrow \infty} \frac{\nu_{(a,b)}(x, N; \mathcal{N}_\sigma)}{N} < C(b - a)^\sigma, \quad (7)$$

where $\nu_{(a,b)}(x, N; \mathcal{N}_\sigma) = \#\{n \in [0, N] \setminus \mathcal{N}_\sigma : \prod_{i=1}^n q_i x \in (a, b)\}$. Then $x \in \mathcal{DN}(Q)$.

Examples involving dynamically generated basic sequences

Example 1: Let Q_C be as in Example 5, which is generated by a Bernoulli system. We have

- $\dim_H(\mathcal{N}(Q_C) \setminus \mathcal{DN}(Q_C)) = \dim_H(\mathcal{DN}(Q_C) \setminus \mathcal{N}(Q_C)) = 1$.
- $\mathcal{DN}(Q_C) \setminus \mathcal{N}(Q_C)$ and $\mathcal{N}(Q_C) \setminus \mathcal{DN}(Q_C)$ are both $D_2(\Pi_3^0)$ -complete.
- There exists $x \in \mathcal{N}(Q_C)$ such that for $M_n := \prod_{j=1}^n q_j$, the sequence $\{\frac{1}{N} \sum_{n=1}^N \delta_{M_N x}\}_{N=1}^\infty$ converges in the weak* topology to a measure μ satisfying

$$\mu([a, b]) = \begin{cases} \frac{5}{6}(b - a) & \text{if } 0 \leq a < b \leq \frac{1}{2} \\ \frac{7}{6}(b - a) & \text{if } \frac{1}{2} \leq a < b \leq 1. \end{cases} \quad (8)$$

In particular, Q_C does not admit a Hot Spot Theorem.

Example 2: Let $g \in \mathbb{N}_{\geq 2}$ and let $Q = (q_n)_{n=1}^\infty \in (\{g^n\}_{n=1}^\infty)^\mathbb{N}$ be a dynamically generated basic sequence satisfying

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log(q_n) = \infty. \quad (9)$$

- There exists $y \in [0, 1] \setminus (\mathcal{N}(Q) \cup \mathcal{DN}(Q))$ that is normal base g .
- There exists $y \in \mathcal{UN}(Q)$ that is not normal base g .

Questions and conjectures for future work

Question 1: Is $\{Q \in \mathbb{N}_{\geq 2}^\mathbb{N} \mid \mathcal{N}(Q) = \mathcal{DN}(Q)\}$ a Borel set?

Question 2: Are there selection rules for deterministic dynamically generated basic sequences Q ? If Q is not deterministic, then this question can be asked separately for $\mathcal{N}(Q)$, $\mathcal{DN}(Q)$, and $\mathcal{UN}(Q)$.

Question 3: If Q is a non-deterministic dynamically generated basic sequence, do we have $\mathcal{UN}(Q) = \mathcal{N}(Q) \cap \mathcal{DN}(Q)$? If not, what can be said about the Hausdorff dimension and descriptive complexity of $(\mathcal{N}(Q) \cap \mathcal{DN}(Q)) \setminus \mathcal{UN}(Q)$?

Question 4: Given a deterministic dynamically generated basic sequence Q , what is the set of $y \in [0, 1]$ for which $y + \mathcal{N}(Q) \subseteq \mathcal{N}(Q)$? If Q is non-deterministic, then this question can be asked separately for $\mathcal{N}(Q)$, $\mathcal{DN}(Q)$, and $\mathcal{UN}(Q)$.

Question 5: Given a deterministic dynamically generated basic sequence Q , what is the set of $q \in \mathbb{Q}$ for which $q\mathcal{N}(Q) \subseteq \mathcal{N}(Q)$? If Q is non-deterministic, then this question can be asked separately for $\mathcal{N}(Q)$, $\mathcal{DN}(Q)$, and $\mathcal{UN}(Q)$. We observe that $n\mathcal{DN}(Q) \subseteq \mathcal{DN}(Q)$ for any Q and any $n \in \mathbb{N}$.

Conjecture 1: If Q is a non-deterministic dynamically generated basic sequence, then

- $\dim_H(\mathcal{N}(Q) \setminus \mathcal{DN}(Q)) = \dim_H(\mathcal{DN}(Q) \setminus \mathcal{N}(Q)) = 1$
- $\mathcal{DN}(Q) \setminus \mathcal{N}(Q)$ and $\mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ are both $D_2(\Pi_3^0)$ -complete.

Conjecture 2: If Q is a non-deterministic dynamically generated basic sequence, then there exists $x \in \mathcal{N}(Q) \setminus \mathcal{DN}(Q)$ and $C > 0$ for which

$$\limsup_{N \rightarrow \infty} \frac{1}{N} |\{1 \leq n \leq N \mid \prod_{i=1}^n q_i x \in (a, b)\}| \leq C(b - a), \quad (10)$$

for all $0 \leq a < b \leq 1$.