Van der Corput sets and a converse to the Furstenberg Correspondence Principle

Ergodic group actions and unitary representations
at IMPAN

Sohail Farhangi

Based on joint work with Robin Tucker-Drob as well as the work of Saúl Rodrígues Martínez

Slides available on sohailfarhangi.com

June 19, 2024

Van der Corput sets/sets of operatorial recurrence

Definition

A set $V \subseteq \mathbb{N}$ is a **van der Corput (vdC) set** if for any $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ for which $(x_{n+v} - x_n \pmod 1)_{n=1}^{\infty}$ is uniformly distributed in $[0,1]$ for all $v \in V$, we have that $(x_n)_{n=1}^{\infty}$ is also uniformly distribtued in $[0,1]$. The set V is a **set of operatorial recurrence** if for any unitary operator $U : \mathcal{H} \to \mathcal{H}$ and any $\xi \in \mathcal{H}$ satisfying $\langle U^v \xi, \xi \rangle = 0$ for all $v \in V$, we have $P\xi = 0$, where P is the orthogonal projection onto the space of U-invariant vectors.

Equivalent characterizations

Theorem (cf. [Ruz84, Per88, NRS12])

For a set $V \subseteq \mathbb{N}$, the following are equivalent:

- (i) *V is a vdC set.*
- (ii) *V is a set of operatorial recurrence.*
- (iii) *For any sequence of complex numbers $(c_n)_{n=1}^{\infty}$ of modulus 1 that satisfy*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{n+v}\overline{c_n} = 0 \ \forall \ v \in V \text{ we have } \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n = 0.$$

- (iv) *The implication above is satisfied for any sequence of complex $(c_n)_{n=1}^{\infty}$ satisfying*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |c_n|^2 < \infty.$$

Examples and nonexamples of vdC sets in $\mathbb{N}$

1. The set of squares is a vdC set. More generally, if $p : \mathbb{Z} \to \mathbb{Z}$ is a divisible polynomial, then $p(\mathbb{Z}) \cap \mathbb{N}$ is a vdC set.
2. Letting $\mathcal{P}$ denote the set of primes, $\mathcal{P} + 1$ and $\mathcal{P} - 1$ are both vdC sets.
3. For any increasing sequence $(a_n)_{n=1}^{\infty} \subseteq \mathbb{N}$, the set $\{a_n - a_m \mid n > m\}$ is a vdC set.
4. If $V \subseteq \mathbb{N}$ has natural density 1, then V is a vdC set.
5. For any $n \in \mathbb{N}$, the set $A_n = \mathbb{N} \setminus (n\mathbb{N})$ is NOT a vdC set.
6. More generally, if $V \subseteq \mathbb{N}$ is not a set of measurable recurrence, then V is NOT a vdC set.

$\mathcal{F}$-vdC sets and vdC sets

Definition ((cf. Rodríguez 2024+))

Let G be a countable amenable group and let $\mathcal{F} = (F_n)_{n=1}^{\infty}$ be a Følner sequence in G. A set $V \subseteq G$ is a **F-vdC set** if for any bounded sequence of complex numbers $(c_g)_{g \in G}$ satisfying

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{g \in F_N} c_{vg}\overline{c_g} = 0 \ \forall \ v \in V \text{ we have } \lim_{N\to\infty} \frac{1}{|F_N|} \sum_{g \in F_N} c_g = 0.$$

A set $V \subseteq G$ is a **vdC set** if for any measure preserving system $(X, \mathscr{B}, \mu, (T_g)_{g \in G})$ and any $f \in L^\infty(X, \mu)$ satisfying $\langle T_v f, f \rangle = 0$ for all $v \in V$, we have $\int_X f d\mu = 0$.

Theorem (Farhangi, Rodríguez, Tucker-Drob, 2024+)

For a countable amenable group G, $V \subseteq G$ is $\mathcal{F}$-vdC iff it is vdC.

This answers a question of Bergelson and Lesigne [BL08].

Modeling L^∞ on ℓ^∞

Theorem (Farhangi, Tucker-Drob, 2024+)

Let G be a countable amenable group, let $(\nu_n)_{n=1}^\infty$ be a sequence of asymptotically invariant probability measures, and let $(X, \mathscr{B}, \mu, (T_g)_{g\in G})$ be a measure preserving system. Given $f \in L^2(X,\mu)$, there exists a sequence of complex numbers $(c_g)_{g\in G}$ taking values in range(f) satisfying

$$\lim_{N\to\infty}\int_G |c_g|^2 d\nu_n(g) = ||f||_2^2,\ \lim_{N\to\infty}\int_G c_g d\nu_n(g) = \int_X f d\mu,\ \text{and}$$

$$\lim_{N\to\infty}\int_G c_{hg}\overline{c_g} d\nu_n(g) = \langle T_h f, f\rangle \text{ for all } h \in G.$$

Furthermore, if $f \in L^\infty(X,\mu)$, then the map $f \mapsto (c_g)_{g\in G}$ extends to a trace preserving $*$-algebra isomorphism between the closed G-equivariant algebras generated by f and $(c_g)_{g\in G}$ in $L^\infty(X,\mu)$ and $\ell^\infty(G,(\nu_n)_{n=1}^\infty)$ respectively.

Modeling L^∞ on ℓ^∞ (continued)

Theorem (Rodríguez, 2024+)

Let G be a countable amenable group, $(F_n)_{n=1}^\infty$ a Følner sequence, and $D \subseteq \mathbb{C}$ compact. For any m.p.s. $(X, \mathscr{B}, \mu, (T_g)_{g\in G})$ and any $f : X \to D$, there exists a sequence of complex numbers $(z_g)_{g\in G} \subseteq D$ such that for all $j \in \mathbb{N}$, all $h_1, \cdots, h_j \in G$, and all continuous $\rho : D^j \to \mathbb{C}$,

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{g\in F_N} \rho(z_{h_1 g}, \cdots, z_{h_j g}) = \int_X \rho(T_{h_1} f, \cdots, T_{h_j} f) d\mu.$$

The converse holds provided that the limit on the left hand side exists.

Converse to Furstenberg's Correspondence Principle

Theorem (Rodríguez 2024+)

Let G be a countable amenable group, $\mathcal{F} = (F_n)_{n=1}^{\infty}$ a Følner sequence, and $(X, \mathscr{B}, \mu, (T_g)_{g\in G})$ a measure preserving system. For any $A \in \mathscr{B}$, there exists a $B \subseteq G$ such that for all $\ell \in \mathbb{N}$ and $h_1, \cdots, h_\ell$ we have

$$\mu(T_{h_1}A^{\star_1} \cap \cdots T_{h_\ell}A^{\star_\ell}) = d_{\mathcal{F}}(h_1 B^{\star_1} \cap \cdots h_\ell B^{\star_\ell}), \tag{1}$$

where each $A^{\star_i}$ denotes A or A^c, and $B^{\star_i}$ agrees with $A^{\star_i}$.

Sets of operatorial recurrence in countable groups

Definition

Let G be a coutable group. A set $V \subseteq G$ is a set of **operatorial recurrence** if for any unitary representation U of G and any $\xi \in \mathcal{H}$ satisfying $\langle U_v\xi, \xi\rangle = 0$ for all $v \in V$, we have $P\xi = 0$, where P is the orthogonal projection onto the space of U-invariant vectors.

Theorem (Farhangi, Tucker-Drob, 2024+)

For a countable group G and a set $V \subseteq G$, TFAE:

1. *V is a set of operatorial recurrence.*
2. *For any unitary representation U of G, and any $\xi \in \mathcal{H}$ satisfying $\sum_{v\in V} |\langle U_v\xi, \xi\rangle|^p < \infty$ for some $p \geq 1$, we have ξ is orthogonal to all finite dimensional U-invariant subspaces.*
3. *If ϕ is a positive definite sequence on G satisfying $\phi(v) = 0$ for all $v \in V$, then $M(\phi) = 0$, where M is the unique invariant mean on the space of weakly almost periodic functions on G.*

Sets of operatorial recurrence in amenable groups

Theorem (Farhangi, Tucker-Drob (see also Rodríguez), 2024+)

Let G be a countably infinite amenable group, let $(F_n)_{n=1}^{\infty}$ be a left Følner sequence. For $V \subseteq G$ TFAE:

1. (i) *For any sequence $(u_g)_{g\in G}$ of complex numbers satisfying*

$$\limsup_{N\to\infty}\frac{1}{|F_N|}\sum_{g\in F_N}|u_g|^2<\infty \text{ and } \lim_{N\to\infty}\frac{1}{|F_N|}\sum_{g\in F_N}u_{vg}\overline{u_g}=0,$$

$$\text{for all } v\in V, \text{ we have } \lim_{N\to\infty}\frac{1}{|F_N|}\sum_{g\in F_N}u_g=0.$$

2. (ii) *Condition (i) with $(u_g)_{g\in G}$ being vectors in a Hilbert space $\mathcal{H}$.*
3. (iii) *For any m.p.s. $(X,\mathscr{B},\mu,(T_g)_{g\in G})$ and any $f\in L^2(X,\mu)$ satisfying $\langle T_v f, f\rangle = 0$ for all $v\in V$, we have $\int_X f d\mu = 0$.*
4. (iv) *V is a set of operatorial recurrence.*

Sets of operatorial recurrence in abelian groups

Theorem (Farhangi, Tucker-Drob (see also Rodríguez), 2024+)

Let G be a countably infinite abelian group, let $(F_n)_{n=1}^{\infty}$ be a left Følner sequence. For $V \subseteq G$ TFAE:

1. *For any sequence $(u_g)_{g\in G}$ of complex numbers of modulus 1,*
$$\text{if } \lim_{N\to\infty} \frac{1}{|F_N|}\sum_{g\in F_N} u_{vg}\overline{u_g} = 0 \text{ for all } v \in V, \text{ then}$$
$$\lim_{N\to\infty} \frac{1}{|F_N|}\sum_{g\in F_N} u_g = 0.$$
2. *Condition (i) with $(u_g)_{g\in G}$ being a bounded sequence.*
3. *For any m.p.s. $(X, \mathscr{B}, \mu, (T_g)_{g\in G})$ and any $f : X \to \mathbb{S}^1$ satisfying $\langle T_v f, f\rangle = 0$ for all $v \in V$, we have $\int_X f d\mu = 0$.*
4. *Condition (ii) with $f \in L^\infty(X, \mu)$.*
5. *V is a set of operatorial recurrence.*

Positive definite sequence in abelian groups

Theorem (Folklore)

Let G be a countable abelian group and let ν be a probability measure on $\widehat{G}$. There exists a m.p.s. $(X, \mathscr{B}, \mu, (T_g)_{g \in G})$ and a measurable $f : X \to \mathbb{S}^1$ for which $\langle T_g f, f \rangle = \hat{\nu}(g)$ and $\int_X f d\mu = \nu(\{0\})$.

Proof.

Let $X = \widehat{G} \times \mathbb{S}^1$, let $\mu = \nu \times m$, let $f(\chi, x) = x$ if $\chi \neq e_{\widehat{G}}$ and let $f(e_{\widehat{G}}, x) = 1$, and let $T_g(\chi, x) = (\chi, \chi(g)x)$. □

Remark

The motivation for this construction is to take the multiplication operators $U_g : L^2(\widehat{G}, \nu) \to L^2(\widehat{G}, \nu)$ that arise in the Spectral Theorem and then convert them into Koopman operators. This insight was motivated by the work of Ruzsa [Ruz84].

A conjecture

Conjecture (Farhangi, Tucker-Drob, 2024+)

Let G be a countable group and ϕ a positive definite sequence on G satisfying $\phi(e) = 1$. Then there exists a measure preserving system $(X, \mathscr{B}, \mu, (T_g)_{g \in G})$ and a measurable $f : X \to \mathbb{S}^1$ for which $\langle T_g f, f \rangle = \phi(g)$, and $\int_X f d\mu = M(\phi)$, where M is the unique mean on the set of weakly almost periodic functions on G.

Corollary

A set $V \subseteq G$ is a set of operatorial recurrence if and only if for any m.p.s. $(X, \mathscr{B}, \mu, (T_g)_{g \in G})$ and any measurable $f : X \to \mathbb{S}^1$ satisfying $\langle T_v f, f \rangle = 0$ for all $v \in V$, we have $\int_X f d\mu = 0$.

Remark

The above conjecture is currently not known even under the weaker assumption that $f \in L^\infty(X, \mu)$.

References I

[BL08] Vitaly Bergelson and Emmanuel Lesigne.
Van der Corput sets in $\mathbb{Z}^d$.
Colloq. Math., 110(1):1–49, 2008.

[NRS12] Marina Ninčević, Braslav Rabar, and Siniša Slijepčević.
Ergodic characterization of van der Corput sets.
Arch. Math. (Basel), 98(4):355–360, 2012.

[Per88] Yuval Peres.
Application of Banach limits to the study of sets of integers.
Israel J. Math., 62(1):17–31, 1988.

References II

[Ruz84] I. Z. Ruzsa.
Connections between the uniform distribution of a sequence and its differences.
In *Topics in classical number theory, Vol. I, II (Budapest, 1981)*, volume 34 of *Colloq. Math. Soc. János Bolyai*, pages 1419–1443. North-Holland, Amsterdam, 1984.