Van der Corput's difference theorem, Lebesgue spectrum, and the left regular representation.

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Overview

1. Van der Corput's difference theorem and some applications
2. Lebesgue spectrum, singular spectrum, and the left regular representation
3. Van der Corput's difference theorem and Lebesgue spectrum
4. Applications

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The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is **uniformly distributed** if for any open interval $(a,b) \subseteq [0,1]$ we have

$$\lim_{N\to\infty} \frac{1}{N} |\{1 \leq n \leq N \mid x_n \in (a,b)\}| = b - a. \tag{1}$$

Theorem (van der Corput, 1931 [vdC31])

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha)_{n=1}^{\infty}$ is uniformly distributed.

Hilbertian van der Corput Difference Theorems 1/3

Theorem (HvdCDT1, Bergelson, 1987 [Ber87, Theorem 1.4])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \tag{2}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0. \tag{3}$$

Hilbertian van der Corput Difference Theorems 2/3

Theorem (HvdCDT2, Bergelson, 1987 [Ber87, Page 3])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0, \ \ \textit{then} \tag{4}$$

$$\lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N}x_n\right|\right| = 0. \tag{5}$$

Hilbertian van der Corput Difference Theorems 3/3

Theorem (HvdCDT3, Bergelson, 1987 [Ber87, Theorem 1.5])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bded seq. satisfying

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then } \lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0.$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

See [Far22, Chapter 2] for variations of vdCT related to the levels of mixing in the ergodic hierarchy of mixing properties, as well as similar variations in the context of uniform distribution. See also [Tse16] and [EKN22].

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \tag{6}$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy [Fur77],[Sár78])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \tag{7}$$

Furstenberg's proof in [Fur77, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^{\infty}$. See also [Ber96, Theorem 2.1] for a proof using HvdCDT1 directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg multiple recurrence, [Fur77])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, *any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\ell \in \mathbb{N}$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A\right) > 0. \quad (8)$$

The proof presented in [EW11] uses HvdCT3 as Theorem 7.11, and the proof in [Fur81] uses a variation.

Theorem (Bergelson and Leibman, [BL96, Theorem A_0])

For any m.p.s. $(X, \mathscr{B}, \mu, \{T_i\}_{i=1}^{\ell})$ *with the* T_i*s commuting, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_\ell^{-p_\ell(n)}A\right) > 0. \quad (9)$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

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Lebesgue spectrum and singular spectrum

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be an invertible m.p.s. and let $U_T : L^2(X, \mu) \to L^2(X, \mu)$ be the Koopman operator induced by T. If $L^2_0(X, \mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m\in\mathbb{Z}}$, then $\mathcal{X}$ has **Lebesgue spectrum**. This implies that for all $f \in L^2_0(X, \mu)$, the sequence $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ is the Fourier coefficients of some measure $\nu << \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure. On the other hand, if for every $f \in L^2(X, \mu)$, the sequence $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ is the Fourier coeficients of some positive measure $\nu \perp \mathcal{L}$, then the system $\mathcal{X}$ has **singular spectrum**.

Examples of systems with Lebesgue spectrum

Any K-mixing system has Lebesgue spectrum, hence all Bernoulli systems have Lebesgue spectrum. The Sinai factor theorem [Sin62] tells us that a non-atomic ergodic m.p.s. with positive entropy has a Bernoulli shift as a factor, and consequently has a factor with Lebesgue spectrum. It follows that the original system does NOT have singular spectrum. The horocycle flow is an example of a system with Lebesgue spectrum [Par53] that also has 0-entropy [Gur61]. These results generalize to measure preserving actions of amenable groups if the notion of Lebesgue spectrum is suitably replaced with the Left regular representation.

Examples of systems with singular spectrum

In [Hal44] and [KS67] it is shown that if $(X, \mathscr{B}, \mu)$ is a standard probability space, and $\text{Aut}(X, \mathscr{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism (such as a group rotation) has singular spectrum, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \text{Aut}(X, \mathscr{B}, \mu)$ denote the collection of strongly mixing transformation, and note that $\mathcal{S}$ is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. Since $\mathcal{S}$ is not a complete metric space with respect to the strong operator topology, a new topology was introduced in [Tik07], with respect to which $\mathcal{S}$ is a complete metric space. It is shown in the Corollary to Theorem 7 of [Tik07] that a generic $T \in \mathcal{S}$ has singular spectrum, and such a T is mixing of all orders due a well known result of Host [Hos91]. See [Fay06],[KR97][AH12], [BS22], and [FL06] for more examples of T that have singular spectrum.

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A Lebesgue spectrum vdCDT

Theorem (F. 2023)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\sum_{h=1}^{\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \tag{10}$$

then $(x_n)_{n=1}^{\infty}$ is a spectrally Lebesgue sequence. In particular, if $(c_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ is bounded and spectrally singular, then

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} c_n x_n \right\| = 0. \tag{11}$$

Furthermore, if $\mathcal{H} = L^2(X, \mu)$ and $(c_n)_{n=1}^{\infty} \subseteq L^{\infty}(X, \mu)$ is bounded and spectrally singular, then we again have Equation (11).

Discrepancy

Definition

For a sequence $(x_n)_{n=1}^N \subseteq [0,1]$, we define the discrepancy to be

$$D_N(x_n)_{n=1}^N := \sup_{0 \leq a < b \leq 1} \left| \frac{1}{N} \# \{1 \leq n \leq N \mid x_n \in (a,b)\} - (b-a) \right|,$$

and we define the discrepancy of $(x_n)_{n=1}^\infty \subseteq [0,1]$ to be

$$\overline{D}(x_n)_{n=1}^\infty := \limsup_{N \to \infty} D_N(x_n)_{n=1}^N.$$

A sequence $(x_n)_{n=1}^\infty \subseteq [0,1]$ is uniformly distributed (u.d.) if and only if $\overline{D}(x_n)_{n=1}^\infty = 0$. We also see that if $(x_n)_{n=1}^\infty$ is u.d. in some subinterval $(0,a)$, then $\overline{D}(x_n)_{n=1}^\infty = 1 - a$.
We also point out that $(\lfloor \sqrt{n} \rfloor \alpha + n\beta)_{n=1}^\infty$ is u.d. for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \in \mathbb{R}$ even though $\overline{D}(x_{n+h} - x_n)_{n=1}^\infty = 1$ for all $h \geq 1$.

Lebesgue spectrum vdCDT for uniform distribution

Theorem (F. 2024+)

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that

$$\sum_{h=1}^{\infty} \left(\overline{D}(x_{n+h} - x_n)_{n=1}^{\infty}\right)^2 < \infty, \tag{12}$$

then $(x_n)_{n=1}^{\infty}$ is a spectrally Lebesgue sequence. In particular, if $(y_n)_{n=1}^{\infty} \subseteq [0,1]$ is a spectrally singular uniformly distributed sequence, then $(x_n, y_n)_{n=1}^{\infty}$ is uniformly distributed in $[0,1]^2$. Furthermore, if $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ is spectrally singular and has positive lower density, then $(x_{n_k})_{k=1}^{\infty}$ is uniformly distributed.

Corollary

If $(n_k)_{k=1}^{\infty}$ is an enumeration of the locations of 1s in the Thue-Morse sequence, then for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the sequence $(n_k^2 \alpha)_{k=1}^{\infty}$ is uniformly distributed.

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An Example

Theorem (Frantzikinakis, Lesigne, Wierdl [FLW12, Lemma 4.1])

Let $a, b : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ *be injective sequences and* F *be any subset of* $\mathbb{N}$. *Then there exist a probability space* $(X, \mathscr{B}, \mu)$, *measure preserving automorphisms* $T, S : X \to X$, *both of them Bernoulli, and* $A \in \mathscr{B}$, *such that*

$$\mu\left(T^{-a(n)}A \cap S^{-b(n)}A\right) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \tag{13}$$

Question: Can a similar result be found for 0-entropy systems? A partial result is given in [HSY23].

Sets of K but not $K+1$ recurrence?

Theorem (Frantzikinakis, Lesigne, Wierdl [FLW06])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$.

(i) *If $(X, \mathscr{B}, \mu)$ is a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which*

$$\mu\left(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{14}$$

(ii) *There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A\right) = 0. \tag{15}$$

A strengthening

Theorem (F., 2023)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ commuting measure preserving automorphisms. Let $T : X \to X$ be an measure preserving automorphism with singular spectrum, and for which $\{T, S_1, S_2, \cdots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu\left(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{16}$$

Since the system $(\mathbb{T}^2, \mathscr{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \sum_{n\in\mathbb{Z}} \delta_{n\alpha}$.

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