(Very strongly) Central sets and Ramsey Theory in $\mathbb{N}$

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Topological dynamics and Central sets

Definition

A **topological dynamical system** is a pair (X, T) where X is a compact Hausdorff space, and $T : X \to X$ is a continuous map. The system is **minimal** if for any $x \in X$, the sequence $(T^n x)_{n=1}^{\infty}$ is dense. A point $x \in X$ is **uniformly recurrent** if for any open set $x \in U \subseteq X$, the set $\{n \in \mathbb{N} \mid T^n x \in U\}$ is a syndetic set. If $X = (X, d)$ is a metric space, then points $x, y \in X$ are **proximal** if for all $\epsilon > 0$ there exists $n = n(\epsilon)$ for which $d(T^n x, T^n y) < \epsilon$.

Theorem

A topological dynamical system (X, T) is minimal if and only if every $x \in X$ is uniformly recurrent. Furthermore, (X, T) is minimal if and only if for any open set $\emptyset \neq U \subseteq X$, there exists $N \in \mathbb{N}$ such that $\bigcup_{i=1}^{N} T^{-i} U \supseteq X$. Moreover, this is equivalent to $\emptyset$ and X being the only closed T-invariant subsets of X.

Central Sets and $\beta\mathbb{N}$

Definition (Furstenberg, [Fur81],[FW78])

A set $C \subseteq \mathbb{N}$ is a **(additively) central set** if there exists a topological dynamical system (X, T) with X a metric space, an open set $\emptyset \neq U \subseteq X$, a uniformly recurrent point $y \in U$, and a point $x \in X$ proximal to y for which $C = \{n \in \mathbb{N} \mid T^n x \in U\}$.

Theorem (Bergelson and Hindman [BH90])

A set $C \subseteq \mathbb{N}$ is an additively central set if $c\ell(C) \subseteq \beta\mathbb{N}$ contains a minimal idempotent of $(\beta\mathbb{N}, +)$. Similarly, $C \subseteq \mathbb{N}$ is a multiplicatively central set if $c\ell(C) \subseteq \beta\mathbb{N}$ contains a minimal idempotent of $(\beta\mathbb{N}, \cdot)$.

Exercise: Give a dynamical formulation for $C \subseteq \mathbb{N}$ to be multiplicatively central.

strongly central sets

Definition

A set $C \subseteq \mathbb{N}$ is **additively (multiplicatively) strongly central** if for any minimal left ideal L of $(\beta\mathbb{N}, +)$ (of $(\beta\mathbb{N}, \cdot)$) we have $c\ell(C) \cap L$ contains an idempotent.

Theorem

A set $C \subseteq \mathbb{N}$ is an additively (multiplicatively) strongly central set if for any additively (multiplicatively) thick set $T \subseteq \mathbb{N}$, the set $C \cap T$ is central.

It is still an open problem as to whether or not the previous theorem is a characterization of strongly central sets. See [BHS12, Theorem 2.8] for a dynamical characterization of strongly central sets. We also point out that strongly central sets are syndetic, while central sets need not be.

very strongly central sets

Definition

A set $C \subseteq \mathbb{N}$ is **very strongly additively central** if and only if there exists a minimal dynamical system (X, T), and open set $\emptyset \neq U \subseteq X$, and a point $y \in c\ell(U)$ for which $\{n \in \mathbb{N} \mid T^n y \in U\} \subseteq C$.

Theorem

If $C \subseteq \mathbb{N}$ is a very strongly additively (multiplicatively) central, then there exists a minimal right ideal R of $(\beta\mathbb{N}, +)$ (of $(\beta\mathbb{N}, \cdot)$) such that $c\ell(C)$ contains all idempotents of R.

It is still an open problem as to whether or not the previous theorem is a characterization of very strongly central sets. See [BHS12, Theorem 2.9] for a list of equivalent definitions of very strongly central sets.

Central sets and partition regularity

Theorem

If $C \subseteq \mathbb{N}$ is an additively (multiplicatively) central set, and $C = \bigcup_{i=1}^{r} C_i$ is a partition, then C_{i_0} is a central set for some $1 \leq i_0 \leq r$. In particular, since $\mathbb{N}$ is a central set, for any finite partition of $\mathbb{N}$, one of the cells is a central set.

Theorem

Let $C \subseteq \mathbb{N}$ be an additively central set.

1. *There exist $x, y, z \in C$ with $x + y = z$.*
2. *For any $\ell \in \mathbb{N}$, there exists $a, d \in C$ with $a + jd \in C$ for $0 \leq j \leq \ell$.*
3. *If a homogeneous system of linear equations is partition regular over $\mathbb{N}$, then it will contain a solution in C. (See [BJM17] for polynomial analogues)*
4. *C contains an IP-set.*

The Central sets Theorem

Definition

A set $A \subseteq \mathbb{N}$ is an **IP-set**[a] if there exists some $(x_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ with $\mathrm{FS}(x_n)_{n=1}^{\infty} = A$, where $FS(x_n)_{n=1}^{\infty} := \{\sum_{n \in F} x_n \mid F \in \mathscr{P}_f(\mathbb{N})\}$.

[a]Some authors say that A is an IP-set if A contains $\mathrm{FS}(x_n)_{n=1}^{\infty}$.

Theorem

Let $C \subseteq \mathbb{N}$ be an additive central set and let $(x_{n,i})_{n=1}^{\infty} \subseteq \mathbb{Z}$ be arbitrary for $1 \leq i \leq \ell$. There exists a sequence $(a_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ and a homomorphism $\alpha : \mathscr{P}_f(\mathbb{N}) \to \mathscr{P}_f(\mathbb{N})$ satisfying $min(\alpha(\{n+1\})) > max(\alpha(\{n\}))$ for which we have

$$FS(a_n + \sum_{m \in \alpha(\{n\})} x_{m,i}) \subseteq C \text{ for } 1 \leq i \leq \ell. \tag{1}$$

See [Phu15], [GLBP23] for generalizations and [Hin20] for history.

C-sets

Definition

A set $A \subseteq \mathbb{N}$ is a C-**set** if it satisfies the conclusion of the central sets theorem.

While a central set must be piecewise syndetic, a C-set need not even have positive upper Banach density as shown in [Hin09]. Interestingly, all of the current linear generalizations of the central sets theorem also hold for C-sets, and it is unkown as to whether or not the polynomial generalizations hold for C-sets.

Question: Is there a strengthening of the central sets theorem that does not hold for C-sets?

Central sets, homogeneity, and partition regularity

Definition

A function $f : \mathbb{N}^n \to \mathbb{Z}$ is **homogeneous of degree** d if for any $c, x_1, \cdots, x_n$ we have $f(cx_1, \cdots, cx_n) = c^d f(x_1, \cdots, x_n)$

Theorem (Farhangi, unpublished)

Let $f_1, \cdots, f_m : \mathbb{N}^n \to \mathbb{Z}$ be such that each f_i is homogeneous of degree d_i. If any very strongly multiplicatively central set C contains $x_1, \cdots, x_n$ for which $f_i(x_1, \cdots, x_n) = 0$ for all $1 \leq i \leq m$, then the system of equations

$$\begin{aligned} f_1(x_1, \cdots, x_n) &= 0 \\ &\vdots \\ f_m(x_1, \cdots, x_n) &= 0 \end{aligned} \tag{2}$$

is partition regular over $\mathbb{N}$.

See [Cha20] for a related result.

An open question

Question: Is there a strengthening of the central sets theorem (and its current generalizations) that can be proven for very strongly central sets?

There are many natural examples of degree d homogeneous diophantine equations whose partition regularity remains unknown. A few examples are $x^2+y^2=z^2$, $x^2=2wy+w^2$, $x^2+y^2+z^2=w^2$, $x^3+y^3+z^3=w^3$, and $x^4+y^4+z^4=2w^4$.

One way in which we could try to show that one of these equations is partition regular, is to show that it contains a solution in any very strongly multiplicatively central set. This provides some motivation for finding strengthenings of the central sets theorem that only apply to very strongly central sets.

Rado's Theorem revisited 1/2

Rado's Theorem gives a computable condition known as the columns condition, which can be used to quickly determine whether or not a given homogeneous system of linear equations is partition regular over $\mathbb{N}$. We will now give an alternative formulation of Rado's Theorem after defining the Rado partitions.

Definition

Given a prime p and some $k \in \mathbb{N}$, we let $C_{p,k}$ denote the partition of $\mathbb{N}$ into p^k cells, in which x and y are in the same cell if and only if they have the same first k digits in their base p expansion starting at the first nonzero digit.

Theorem (Rado,[Rad33])

A homogeneous system of linear equations is partition regular over $\mathbb{N}$ if and only if it contains a solution in some cell of the partition $C_{p,k}$ for any prime p and $k \in \mathbb{N}$.

Rado's Theorem revisited 2/2

Since a homogeneous system of linear equations, is (as the name implies) homogeneous, we can check that the system has a solution in some cell of $C_{p,k}$ if and only if it has a solution in every cell of $C_{p,k}$. However, not every cell of $C_{p,k}$ will be very strongly multiplicatively central, because not every cell of $C_{p,k}$ contains an IP-set. Since very strongly central sets are syndetic, the situation there is similar.

Question: Are there classes of equations in which we can determine whether or not they are partition regular by only checking their partition regularity along some countably infinite collection of partitions? In particular, can the partition regularity of a degree d homogeneous equation be determined by only checking whether or not it has a solution in some countable collection of very strongly multiplicatively central sets?

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