The Partition Regularity of some Homogeneous Equations

Seminar on Topological Semigroups and Ramsey Theory at Howard University

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Partition Regularity of Homogeneous Equations

Theorem

Suppose that R is an integral domain and $\mathbf{F}$ is a (multiplicatively) homogeneous system of equations. (For example, $x^3 + y^3 = z^3$ or $xy = z^2$). The following are equivalent:

(i) *$\mathbf{F}$ is partition regular over $R \setminus \{0\}$.*

(ii) *For any multiplicatively pieceiwse syndetic set $A \subseteq R$, $\mathbf{F}$ has a solution contained in A.*

(iii) *For any very strongly multiplicatively central set $A \subseteq R$, $\mathbf{F}$ has a solution in A.*

This result is due to Chapman [Cha20] if A is multilplicatively syndetic in (iii) instead of very strongly multiplicatively central.

Difference of squares generate mult. thick sets

Lemma (F., unpublished)

Let R be an infinite integral domain and $A \subseteq R$. If A is 'A.P.'-rich (which is implied by additive or multiplicative piecewise syndeticity), then $B := \{x^2 - y^2 \mid x, y \in A\}$ is multiplicatively thick.

Please keep in mind how much stronger this lemma could (and should) be when examining the applications.

Corollary

Let R be an infinite integral domain with field of fractions K. For any $c \in K \setminus \{0\}$, the equation

$$c = \frac{x^2 - y^2}{w^2 - z^2} \tag{1}$$

is partition regular over R.

First main theorem 1/4

Theorem 1: Let R be an integral domain with field of fractions K and $n, m \in \mathbb{N}$ arbitrary.

(i) For any $c_0, c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any 'A.P.'-rich set A.

$$\begin{aligned} \frac{c_1 + c_0}{c_1} &= \frac{x^2 - y_1^2}{w^2 - z_1^2} \\ &\vdots \\ \frac{c_m + c_0}{c_m} &= \frac{x^2 - y_m^2}{w^2 - z_m^2} \end{aligned} \tag{2}$$

First main theorem 2/4

(ii) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{w}{z}(x_1^2 - y_1^2) \\ &\vdots \\ c_m &= \frac{w}{z}(x_m^2 - y_m^2) \end{aligned} \tag{3}$$

(iii) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{N}$.

$$\begin{aligned} c_1 &= \frac{w_1}{z_1}(x^2 - y^2) \\ c_2 &= \frac{w_2}{z_2}(x^2 - y^2) \end{aligned} \tag{4}$$

First main theorem 3/4

(iv) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{x^2 - y_1^2}{4wz_1} \\ &\vdots \\ c_m &= \frac{x^2 - y_m^2}{4wz_m} \end{aligned} \tag{5}$$

(v) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{N}$.

$$\begin{aligned} c_1 &= \frac{x^2 - y^2}{w_1 z_1} \\ c_2 &= \frac{x^2 - y^2}{w_2 z_2} \end{aligned} \tag{6}$$

First main theorem 4/4

(vi) Suppose that $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 1. The system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} zf_1(t_1, \cdots, t_n) &= x_1^2 - y_1^2 \\ &\vdots \\ zf_m(t_1, \cdots, t_n) &= x_m^2 - y_m^2 \end{aligned} \tag{7}$$

(vii) And if $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 3, then the same is true of the following system of equations:

$$\begin{aligned} f_1(t_1, \cdots, t_n) &= z(x_1^2 - y_1^2) \\ &\vdots \\ f_m(t_1, \cdots, t_n) &= z(x_m^2 - y_m^2) \end{aligned} \tag{8}$$

Example 1/2

Corollary

The following system of equations is partition regular over $\mathbb{Z}$.

$$\begin{aligned} z(2r+3t) &= x_1^2 - y_1^2 \\ z(3r+2t) &= x_2^2 - y_2^2 \\ z\frac{r^2}{t} &= x_3^2 - y_3^2 \\ z\frac{t^2}{r} &= x_4^2 - y_4^2 \\ z\frac{5r^3 - 7t^3}{2r^2 + 5t^2} &= x_5^2 - y_5^2 \end{aligned} \tag{9}$$

Example 2/2

Corollary

The following system of equations is partition regular over $\mathbb{Z}$.

$$
\begin{aligned}
r^3 &= z(x_1^2 - y_1^2) \\
t^3 &= z(x_2^2 - y_2^2) \\
r^3 + t^3 &= z(x_3^2 - y_3^2) \\
2r^3 - 3r^2t + 7rt^2 - t^3 &= z(x_4^2 - y_4^2) \\
\left\lfloor 2^{\frac{t}{r}} \right\rfloor \left\lfloor \ln\left(\frac{r}{t}\right) \right\rfloor \frac{5r^4 + 7t^4}{9r - 17t} &= z(x_5^2 - y_5^2)
\end{aligned}
\tag{10}
$$

Future work 1/3

Conjecture (F., 2023)

Let R be an infinite integral domain and $A \subseteq R$. If A is multiplicatively syndetic, then $B := \{x^2 - y^2 \mid x, y \in A\}$ is multiplicatively thickly syndetic.

Corollary (to the conjecture being true)

The following system of equations is partition regular over $\mathbb{N}$.

$$\begin{aligned} z^3 &= w(x_1^2 - y_1^2) \\ wz &= x_2^2 - y_2^2 \end{aligned} \tag{11}$$

In general, an affirmative answer to this Conecture would allow us to combine many of the previous P.R. systems of equations into even bigger P.R. systems of equations. What more can we conjecture if A is very strongly multiplicatively central?

A cubic form generating mult. thick sets

Lemma

Let R be an infinite integral domain containing a solution ζ to $x^2 + x + 1 = 0$ and $A \subseteq R$. If A is 'A.P.'-rich, then $B := \{x^3 + y^3 + z^3 - 3xyz \mid x, y, z \in A\}$ is multiplicatively thick.

Corollary

Let R be an infinite integral domain containing a solution ζ to $x^2 + x + 1 = 0$ and let K be its field of fractions. For any $c \in K \setminus \{0\}$, the equation

$$c = \frac{x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1}{x_2^3 + y_2^3 + z_2^3 - 3x_2y_2z_2} \tag{12}$$

is partition regular.

$0 = x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \zeta y + \zeta^2 z)(x + \zeta^2 y + \zeta z)$ is nontrivially partition regular over $\mathbb{Z}[\zeta]$ but not $\mathbb{Z}$.

Second main theorem 1/4

Theorem 2: Let R be an infinite integral domain containing a solution ζ to $x^2+x+1=0$ and let K be its field of fractions.

(i) For any $c_0, c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} \frac{c_1+c_0}{c_1} &= \frac{x^3+y_1^3+z_1^3-3xy_1z_1}{u^3+w_1^3+v_1^3-3uw_1v_1} \\ &\vdots \\ \frac{c_m+c_0}{c_m} &= \frac{x^3+y_m^3+z_m^3-3xy_mz_m}{u^3+w_m^3+v_m^3-3uw_mv_m} \end{aligned} \tag{13}$$

Second main theorem 2/4

(ii) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{w}{z}(x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1) \\ &\vdots \\ c_m &= \frac{w}{z}(x_m^3 + y_m^3 + z_m^3 - 3x_my_mz_m) \end{aligned} \tag{14}$$

(iii) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{Z}$.

$$\begin{aligned} c_1 &= \frac{w_1}{z_1}(x^3 + y^3 + z^3 - 3xyz) \\ c_2 &= \frac{w_2}{z_2}(x^3 + y^3 + z^3 - 3xyz) \end{aligned} \tag{15}$$

Second main theorem 3/4

(iv) For any $c_1, \cdots, c_m \in R \setminus \{0\}$, the system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} c_1 &= \frac{x^3 + y_1^3 + z_1^3 - 3xy_1z_1}{27uvw_1} \\ &\vdots \\ c_m &= \frac{x^3 + y_m^3 + z_m^3 - 3xy_mz_m}{27uvw_m} \end{aligned} \tag{16}$$

(v) For any distinct $c_1, c_2 \in \mathbb{Z} \setminus \{0\}$, the system of equations below is not partition regular over $\mathbb{Z}$.

$$\begin{aligned} c_1 &= \frac{x^3 + y^3 + z^3 - 3xyz}{u_1v_1w_1} \\ c_2 &= \frac{x^3 + y^3 + z^3 - 3xyz}{u_2v_2w_2} \end{aligned} \tag{17}$$

Second main theorem 4/4

(vi) Suppose that $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 2. The system of equations below is partition regular over R, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} zf_1(t_1, \cdots, t_n) &= x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1 \\ &\vdots \\ zf_m(t_1, \cdots, t_n) &= x_m^3 + y_m^3 + z_m^3 - 3x_my_mz_m \end{aligned} \tag{18}$$

(vii) And if $f_1, \cdots, f_m : K^n \to K$ are homogeneous of degree 4, then the same is true of the following system of equations:

$$\begin{aligned} f_1(t_1, \cdots, t_n) &= z(x_1^3 + y_1^3 + z_1^3 - 3x_1y_1z_1) \\ &\vdots \\ f_m(t_1, \cdots, t_n) &= z(x_m^3 + y_m^3 + z_m^3 - 3x_my_mz_m) \end{aligned} \tag{19}$$

Future work 2/3

Conjecture

Let R be an infinite integral domain containing a solution to $x^2+x+1=0$ and $A \subseteq R$. If A is multiplicatively syndetic, then $B := \{x^3+y^3+z^3-3xyz \mid x,y,z \in A\}$ is multiplicatively thickly syndetic.

Corollary (to the conjecture being true)

The following system of equations is partition regular over $\mathbb{N}$.

$$\begin{aligned} t^4 &= s(x_1^3+y_1^3+z_1^3-3x_1y_1z_1) \\ st^2 &= x_2^3+y_2^3+z_2^3-3x_2y_2z_2 \end{aligned} \tag{20}$$

In general, an affirmative answer to this Conecture 10 would allow us to combine many of the previous P.R. systems of equations into even bigger P.R. systems of equations. What more can we conjecture if A is very strongly multiplicatively central?

Third main result 1/3

Theorem 3: Let R be an integral domain and let $\mathbf{A} = (a_{i,j})_{1 \leq i,j \leq n} \in M_{n \times n}(R)$ satisfy $\det(\mathbf{A}) \neq 0$ and $\sum_{j=1}^{n} a_{1,j} = 0$. Let

$$g_{\mathbf{A}}(x_1, \cdots, x_n) = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{i,j} x_i \right). \tag{21}$$

(i) If $A \subseteq R$ if 'A.P.'-rich, then $B := \{g_{\mathbf{A}}(x_1, \cdots, x_n) \mid x_1, \cdots, x_n \in A\}$ is multiplicatively thick.

Third main result 2/3

(ii) Suppose that $f_1, \cdots, f_m : R^n \to R$ are homogeneous of degree $n-1$. The system of equations below is partition regular, as it will contain a solution in any multiplicatively piecewise syndetic set A.

$$\begin{aligned} zf_1(t_1, \cdots, t_n) &= g_{\mathbf{A}}(x_{1,1}, \cdots, x_{1,n}) \\ &\vdots \\ zf_m(t_1, \cdots, t_n) &= g_{\mathbf{A}}(x_{m,1}, \cdots, x_{m,n}) \end{aligned} \tag{22}$$

(iii) And if $f_1, \cdots, f_m : R^n \to R$ are homogeneous of degree $n+1$, then the same is true of the following system:

$$\begin{aligned} f_1(t_1, \cdots, t_n) &= zg_{\mathbf{A}}(x_{1,1}, \cdots, x_{1,n}) \\ &\vdots \\ f_m(t_1, \cdots, t_n) &= zg_{\mathbf{A}}(x_{m,1}, \cdots, x_{m,n}) \end{aligned} \tag{23}$$

Third main result 3/3

Theorem 4: Let $\mathbf{A} = (a_{i,j})_{1 \leq i,j \leq n} \in M_{n \times n}(\mathbb{Z} \setminus \{0\})$ be such that for $1 \leq i \leq n$ and $I \subseteq [1, n]$, we have $\sum_{j \in I} a_{i,j} \neq 0$ unless $|I| < 2$ or $a_{i,j} = 0$ for some $j \in I$. For $\emptyset \neq I \subseteq [1, n]$ let $c_I = \prod_{i=1}^{n} (\sum_{j \in I} a_{i,j})$. If $c \in R \setminus \{c_I\}_{I \subseteq [1,n]}$, then

$$ct^{n+1} = zg_{\mathbf{A}}(x_1, \cdots, x_n) \tag{24}$$

is not partition regular over $\mathbb{N}$.

Future work 3/3

Question

(i) *For what* $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ *is the equation*

$$z^3 = w(ax + by)(cx + dy) \tag{25}$$

partition regular over $\mathbb{N}$*? What happens if* $a = b = c = 1$ *and* $d = 2$*?*

(ii) *How can Theorem 3(i) be improved if A is multiplicatively syndetic? Will B be multiplicatively thickly syndetic? What happens if A is very strongly multiplicatively central?*

The Equation 25 is partition regular if $a = b = c = d = 1$ or if $a = d = 1$ and $b = c = 0$, so the assumption that $\sum_{j=1}^{n} a_{i,j} = 0$ and the assumption that $\mathbf{A}$ has nonzero entries in Theorem 3 are not strictly necessary.

References I

[Cha20] J. Chapman.
Partition regularity and multiplicatively syndetic sets.
Acta Arith., 196:109–138, 2020.