Van der Corput's difference theorem and the left regular representation.

Operator Theoretic Aspects of Ergodic Theory
11th Miniworkshop

Based on https://arxiv.org/abs/2303.11832
and https://arxiv.org/abs/2308.05560

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November 24, 2023

Overview

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Hilbertian van der Corput Difference Theorem

Theorem (Bergelson, 1987 [Ber87, Theorem 1.4])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \tag{1}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0. \tag{2}$$

We see that if $\mathcal{H} = \mathbb{C}$ and $x_n = e^{2\pi i n\sqrt{2}}$, then $\lim_{N\to\infty} \frac{1}{N} x_n = 0$ even though $|\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle| = 1$ for all $h \in \mathbb{N}$. This naturally raises the question of what other properties a sequence $(x_n)_{n=1}^\infty$ must have if it satisfies the hypothesis of vdCDT?

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \tag{3}$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy [Fur77],[Sár78])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \tag{4}$$

Furstenberg's proof in [Fur77, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^{\infty}$. See also [Ber96, Theorem 2.1] for a proof using vdCDT directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg multiple recurrence, [Fur77])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, *any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\ell \in \mathbb{N}$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A\right) > 0. \tag{5}$$

The proof presented in [EW11] uses a form of vdCDT as Theorem 7.11, and the proof in [Fur81] uses another variation.

Theorem (Bergelson and Leibman, [BL96, Theorem A_0])

For any m.p.s. $(X, \mathscr{B}, \mu, \{T_i\}_{i=1}^{\ell})$ *with the* T_i*s commuting, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_\ell^{-p_\ell(n)}A\right) > 0. \tag{6}$$

Uses a variation of vdCDT as Lemma 2.4.

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Lebesgue spectrum and singular spectrum

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be an invertible m.p.s. and let $U_T : L^2(X,\mu) \to L^2(X,\mu)$ be the Koopman operator induced by T. If $L^2_0(X,\mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m\in\mathbb{Z}}$, then $\mathcal{X}$ has **Lebesgue spectrum**. This implies that for all $f \in L^2_0(X,\mu)$, the sequence $(\langle U_T^n f, f\rangle)_{n=1}^\infty$ is the Fourier coefficients of some measure $\nu << \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure. On the other hand, if for every $f \in L^2(X,\mu)$, the sequence $(\langle U_T^n f, f\rangle)_{n=1}^\infty$ is the Fourier coeficients of some positive measure $\nu \perp \mathcal{L}$, then the system $\mathcal{X}$ has **singular spectrum**.

Examples of systems with Lebesgue spectrum

Any K-mixing system has Lebesgue spectrum, hence all Bernoulli systems have Lebesgue spectrum. The Sinai factor theorem [Sin62] tells us that a non-atomic ergodic m.p.s. with positive entropy has a Bernoulli shift as a factor, and consequently has a factor with Lebesgue spectrum. It follows that the original system does NOT have singular spectrum. The horocycle flow is an example of a system with Lebesgue spectrum [Par53] that also has 0-entropy [Gur61].

Examples of systems with singular spectrum

In [Hal44] and [KS67] it is shown that if $(X, \mathscr{B}, \mu)$ is a standard probability space, and $\text{Aut}(X, \mathscr{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism (such as a group rotation) has singular spectrum, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \text{Aut}(X, \mathscr{B}, \mu)$ denote the collection of strongly mixing transformation, and note that $\mathcal{S}$ is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. The work of [Tik07] shows that there is a natural topology on $\mathcal{S}$ for which a generic T has singular spectrum.

The left regular representation

Let G is a locally compact Hausdorff group with left Haar measure λ. There is a unitary representation L of G on $L^2(G,\lambda)$ given by $(L_g f)(h) = f(g^{-1}h)$, which is known as the **left regular representation**. If $f \in L^2(G,\lambda)$ is a continuous positive definite function, then there exists a function $h \in L^2(G,\lambda)$ for which $f(g) = \langle L_g h, h\rangle$. In particular, consider a representation U of G on $\mathcal{H}$, and a cyclic vector $f \in \mathcal{H}$ such that

$$\int_G |\langle U_g f, f\rangle|^2 d\lambda(g) < \infty. \tag{7}$$

Then U is isomorphic to a subrepresentation of the left regular representation.

Spectrum and the left regular representation

Let G be a locally compact second countable (l.c.s.c.) amenable group and $\mathcal{X} := (X, \mathscr{B}, \mu, \{T_g\}_{g \in g})$ a measure preserving G-system, which we abbreviate to G-system. We let $U : L^2(X, \lambda) \to L^2(X, \lambda)$ denote the unitary representation of G induced by T, i.e., $U_g f = f \circ T_{g^{-1}}$. The system $\mathcal{X}$ has **Lebesgue spectrum** if U decomposes into a direct sum of countably many copies of the left regular representation. The system $\mathcal{X}$ has **singular spectrum** if the representation U is disjoint from the left regular representation.

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A Lebesgue spectrum vdCDT

Theorem (F. 2023)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\sum_{h=1}^{\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \tag{8}$$

then $(x_n)_{n=1}^{\infty}$ is a spectrally Lebesgue sequence. In particular, if $(c_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ is bounded and spectrally singular, then

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} c_n x_n \right|\right| = 0. \tag{9}$$

Furthermore, if $\mathcal{H} = L^2(X,\mu)$ and $(c_n)_{n=1}^{\infty} \subseteq L^{\infty}(X,\mu)$ is bounded and spectrally singular, then we again have Equation (9).

A Lebesgue spectrum vdCDT for amenable groups

Theorem (F. 2023)

Let G be a l.c.s.c. amenable group and $(F_n)_{n=1}^{\infty}$ a left Følner sequence. If $(x_g)_{g\in G} \subseteq \mathcal{H}$ is a bounded measurable sequence satisfying

$$\int_{h\in G} \limsup_{N\to\infty} \left|\frac{1}{\nu(F_N)} \int_{g\in F_N} \langle x_{gh}, x_g\rangle\right|^2 < \infty, \tag{10}$$

then $(x_g)_{g\in G}$ is a spectrally Lebesgue sequence. In particular, if $(c_g)_{g\in G} \subseteq \mathbb{C}$ is bounded, measurable, and spectrally singular, then

$$\lim_{N\to\infty} \left|\left|\frac{1}{\nu(F_N)} \int_{g\in F_N} c_g x_g\right|\right| = 0. \tag{11}$$

Furthermore, if $\mathcal{H} = L^2(X,\mu)$ and $(c_g)_{g\in G} \subseteq L^{\infty}(X,\mu)$ is bounded and spectrally singular, then we again have Equation (11).

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Sets of K but not $K+1$ recurrence?

Theorem (Frantzikinakis, Lesigne, Wierdl [FLW06])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$.

(i) *If $(X, \mathscr{B}, \mu)$ is a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which*

$$\mu\left(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \qquad (12)$$

(ii) *There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A\right) = 0. \qquad (13)$$

Application 1/3

Theorem (F., 2023)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{ n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right] \right\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ commuting measure preserving automorphisms. Let $T : X \to X$ be an measure preserving automorphism with singular spectrum, and for which $\{T, S_1, S_2, \cdots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu\left(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{14}$$

Since the system $(\mathbb{T}^2, \mathscr{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \sum_{n \in \mathbb{Z}} \delta_{n\alpha}$.

Noncommutative ergodic theorems 2/2

Theorem (Frantzikinakis [Fra22, Corollary 1.7])

Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t\to\infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t\to\infty} \frac{t^{d+1}}{a(t)} = \infty. \quad \left(e.g.\ a(t) = t^{1.5}\right) \tag{15}$$

Furthermore, let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations. Suppose that the system $(X, \mathscr{B}, \mu, T)$ has zero entropy. Then

(i) *For every $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f|\mathcal{I}_T] \cdot \mathbb{E}[g|\mathcal{I}_S], \tag{16}$$

where the limit is taken in $L^2(X, \mu)$.

A noncommutative ergodic theorem

Theorem (Continued)

(ii) *For every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap S^{-\lfloor a(n) \rfloor}A\right) \geq \mu(A)^3. \tag{17}$$

Frantzikinakis and Host [FH21] proved a similar theorem for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2.

An Example

Theorem (Frantzikinakis, Lesigne, Wierdl [FLW12, Lemma 4.1])

Let $a, b : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ be injective sequences and F be any subset of $\mathbb{N}$. Then there exist a probability space $(X, \mathscr{B}, \mu)$, measure preserving automorphisms $T, S : X \to X$, both of them Bernoulli, and $A \in \mathscr{B}$, such that

$$\mu\left(T^{-a(n)}A \cap S^{-b(n)}A\right) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \tag{18}$$

In light of Sinai's Factor Theorem, we see that the assumption of 0-entropy in the last 2 slides cannot be weakened.

Application 2/3

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving automorphisms for which T has singular spectrum. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed in the orbit closure of α for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

1. *For any $f, g \in L^{\infty}(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}[f|\mathcal{I}_T]\,\mathbb{E}[g|\mathcal{I}_S], \qquad (19)$$

with convergence taking place in $L^2(X, \mu)$.

Application 2/3 continued

Theorem (Continued)

(ii) *If $A \in \mathscr{B}$ then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap S^{-k_n}A\right) \geq \mu(A)^3. \qquad (20)$$

(iii) *If we only assume that $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $h \in \mathbb{N}$, then (i) and (ii) hold when S is totally ergodic.*

Examples include $k_n = \lfloor a(n) \rfloor$ with $a(n)$ being as in frame 19, $k_n = \lfloor n^2 \log^2(n) \rfloor$, and for part (iii) we may take $k_n = p(n)$ for $p(x) \in x\mathbb{Z}[x]$ with degree at least 2. An analogous result is now known for countable abelian groups.

Application 3/3

Theorem (F., 2023)

Let K be a countable field with characteristic 0. Let $(X, \mathscr{B}, \nu)$ be a probability space and $T_g, S_g : X \to X$ measure preserving actions of $(K, +)$ for which the action $(T_g)_{g\in K}$ has singular spectrum and the action $(S_g)_{g\in K}$ is ergodic. Let $(F_n)_{n=1}^{\infty}$ be a Følner sequence in $(K, +)$ and $\ell \in \mathbb{N}$. Let $p_1, \cdots, p_\ell \in K[x]$ be polynomials for which $deg(p_{i+1}) \geq 2 + deg(p_i)$ for $1 \leq i < \ell$. Then for any $f_0, f_1, \cdots, f_\ell \in L^\infty(X, \mu)$ we have

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{n\in F_N} T_n f_0 \prod_{j=1}^{\ell} S_{p_j(n)} f_j = \mathbb{E}[f_0|\mathcal{I}_T] \prod_{j=1}^{\ell} \int_X f_j d\nu \qquad (21)$$

with convergence taking place in $L^2(X, \nu)$.

This is a corollary of a more general result involving joint ergodicity.

An example

Consider the m.p.s. $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T, S)$ with $S(x,y) = (x+2\alpha, y+x)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $T(x,y) = (x, y+x)$. We see that $([0,1]^2, \mathscr{B}, \mathcal{L}^2, S)$ and $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T)$ are both zero entropy systems that are not weakly mixing, and the former is totally ergodic. Furthermore, T and S generate a 2-step nilpotent group. For $f_0(x,y) = e^{2\pi i(x-y)}$, $f_1(x,y) = e^{2\pi i y}$, and $f_2(x,y) = e^{-2\pi i x}$, we see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_0(x,y) S^n f_1(x,y) S^{\frac{1}{2}(n^2-n)} f_2(x,y)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i\left((1-n)x - y + y + nx + (n^2-n)\alpha - x - (n^2-n)\alpha\right)} = 1 \neq 0.$$

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