Van der Corput's difference theorem and the left regular representation.

Ergodic Theory Seminar
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Overview

1. Van der Corput's difference theorem and some applications
2. Lebesgue spectrum, singular spectrum, and the left regular representation
3. Van der Corput's difference theorem and the left regular representation
4. Applications
   - Background on noncommutative ergodic theory
   - New results from mixing vdCs

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The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is **uniformly distributed** if for any open interval $(a,b) \subseteq [0,1]$ we have

$$\lim_{N\to\infty} \frac{1}{N} |\{1 \leq n \leq N \mid x_n \in (a,b)\}| = b - a. \tag{1}$$

Theorem (van der Corput, 1931 [29])

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha)_{n=1}^{\infty}$ is uniformly distributed.

Hilbertian van der Corput Difference Theorems 1/3

Theorem (HvdCDT1, Bergelson, 1987 [2, Theorem 1.4])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \tag{2}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0. \tag{3}$$

Hilbertian van der Corput Difference Theorems 2/3

Theorem (HvdCDT2, Bergelson, 1987 [2, Page 3])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \ \textit{then} \tag{4}$$

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0. \tag{5}$$

Hilbertian van der Corput Difference Theorems 3/3

Theorem (HvdCDT3, Bergelson, 1987 [2, Theorem 1.5])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bded seq. satisfying

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then } \lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0.$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

See [11, Chapter 2] for variations of vdCT related to the levels of mixing in the ergodic hierarchy of mixing properties, as well as similar variations in the context of uniform distribution. See also [28] and [9].

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \tag{6}$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy [18],[25])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \tag{7}$$

Furstenberg's proof in [18, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^{\infty}$. See also [3, Theorem 2.1] for a proof using HvdCDT1 directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg multiple recurrence, [18])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$*, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$*, and any* $\ell \in \mathbb{N}$*, there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A\right) > 0. \quad (8)$$

The proof presented in [10] uses HvdCT3 as Theorem 7.11, and the proof in [19] uses a variation.

Theorem (Bergelson and Leibman, [5, Theorem A_0])

For any m.p.s. $(X, \mathscr{B}, \mu, \{T_i\}_{i=1}^{\ell})$ *with the* T_i*s commuting, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$*, and any* $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$*, there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_\ell^{-p_\ell(n)}A\right) > 0. \quad (9)$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

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Lebesgue spectrum and singular spectrum

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be an invertible m.p.s. and let $U_T : L^2(X,\mu) \to L^2(X,\mu)$ be the Koopman operator induced by T. If $L_0^2(X,\mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m\in\mathbb{Z}}$, then $\mathcal{X}$ has **Lebesgue spectrum**. This implies that for all $f \in L_0^2(X,\mu)$, the sequence $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ is the Fourier coefficients of some measure $\nu << \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure. On the other hand, if for every $f \in L^2(X,\mu)$, the sequence $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ is the Fourier coeficients of some positive measure $\nu \perp \mathcal{L}$, then the system $\mathcal{X}$ has **singular spectrum**.

Examples of systems with Lebesgue spectrum

Any K-mixing system has Lebesgue spectrum, hence all Bernoulli systems have Lebesgue spectrum. The Sinai factor theorem [26] tells us that a non-atomic ergodic m.p.s. with positive entropy has a Bernoulli shift as a factor, and consequently has a factor with Lebesgue spectrum. It follows that the original system does NOT have singular spectrum. The horocycle flow is an example of a system with Lebesgue spectrum [24] that also has 0-entropy [20].

Examples of systems with singular spectrum

In [21] and [4] it is shown that if $(X, \mathscr{B}, \mu)$ is a standard probability space, and $\text{Aut}(X, \mathscr{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism (such as a group rotation) has singular spectrum, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \text{Aut}(X, \mathscr{B}, \mu)$ denote the collection of strongly mixing transformation, and note that $\mathcal{S}$ is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. Since $\mathcal{S}$ is not a complete metric space with respect to the strong operator topology, a new topology was introduced in [27], with respect to which $\mathcal{S}$ is a complete metric space. It is shown in the Corollary to Theorem 7 of [27] that a generic $T \in \mathcal{S}$ has singular spectrum, and such a T is mixing of all orders due a well known result of Host [22]. See [12],[23][1], [6], and [17] for more examples of T that have singular spectrum.

The left regular representation

Let G is a locally compact Hausdorff group with left Haar measure λ. There is a unitary representation L of G on $L^2(G,\nu)$ given by $(L_g f)(h) = f(g^{-1}h)$, which is known as the **left regular representation**. If $f \in L^2(G,\nu)$ is a positive definite function, then there exists a function $h \in L^2(G,\lambda)$ for which $f(g) = \langle L_g h, h\rangle$. In particular, consider a representation U of G on $\mathcal{H}$, and a cyclic vector $f \in \mathcal{H}$ such that

$$\int_G |\langle U_g f, f\rangle|^2 d\lambda(g) < \infty. \tag{10}$$

Then U is isomorphic to a subrepresentation of the left regular representation.

Spectrum and the left regular representation

Let G be an amenable group and $\mathcal{X} := (X, \mathscr{B}, \mu, \{T_g\}_{g\in g})$ a measure preserving G-system, which we abbreciate to G-system. We let $U : L^2(X,\mu) \to L^2(X,\mu)$ denote the unitary representation of G induced by T, i.e., $U_g f = f \circ T_{g^{-1}}$. The system $\mathcal{X}$ has **Lebesgue spectrum** if U decomposes into a direct sum of countably many copies of the left regular representation. The system $\mathcal{X}$ has **singular spectrum** if the representation U is disjoint from the left regular representation. Dooley and Golodets [8] showed that if G is countable and $\mathcal{X}$ has completely positive entropy (analogue of K-mixing) then it also has Lebesgue spectrum. Danilenko and Park [7] proved an analogue of Sinai's factor theorem when G is countable, from which we deduce that $\mathcal{X}$ does not have singular spectrum when it is free, ergodic, and has positive entropy.

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A Lebesgue spectrum vdCDT

Theorem (F. 2023)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\sum_{h=1}^{\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \tag{11}$$

then $(x_n)_{n=1}^{\infty}$ is a spectrally Lebesgue sequence. If $\mathcal{H} = L^2(X,\mu)$ and $(y_n)_{n=1}^{\infty} \subseteq L^{\infty}(X,\mu)$ is bounded and spectrally singular, then

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n y_n \right|\right| = 0. \tag{12}$$

A Lebesgue spectrum vdCDT for amenable groups

Theorem (F. 2023)

Let G be a countable amenable group and $(F_n)_{n=1}^{\infty}$ a left Følner sequence. If $(x_g)_{g\in G} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\sum_{h\in G} \limsup_{N\to\infty} \left| \frac{1}{|F_N|} \sum_{g\in F_N} \langle x_{gh}, x_g \rangle \right|^2 < \infty, \tag{13}$$

then $(x_g)_{g\in G}$ is a spectrally Lebesgue sequence. If $\mathcal{H} = L^2(X,\mu)$ and $(y_n)_{n=1}^{\infty} \subseteq L^{\infty}(X,\mu)$ is bounded and spectrally singular, then

$$\lim_{N\to\infty} \left\| \frac{1}{|F_N|} \sum_{g\in F_N} x_g y_g \right\| = 0. \tag{14}$$

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Noncommutative ergodic theorems 1/2

Theorem (Frantzikinakis [13, Corollary 1.7])

Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t\to\infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t\to\infty} \frac{t^{d+1}}{a(t)} = \infty. \quad (e.g.\ a(t) = t^{1.5}) \tag{15}$$

Furthermore, let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations. Suppose that the system $(X, \mathscr{B}, \mu, T)$ has zero entropy. Then

(i) *For every $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f|\mathcal{I}_T] \cdot \mathbb{E}[g|\mathcal{I}_S], \tag{16}$$

where the limit is taken in $L^2(X, \mu)$.

Noncommutative ergodic theorems 2/2

Theorem (Continued)

(ii) *For every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu\left(A\cap T^{-n}A\cap S^{-\lfloor a(n)\rfloor}A\right)\geq\mu(A)^3. \tag{17}$$

Frantzikinakis and Host [14] proved a similar theorem for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2.

An Example

Theorem (Frantzikinakis, Lesigne, Wierdl [16, Lemma 4.1])

Let $a, b : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ *be injective sequences and* F *be any subset of* $\mathbb{N}$*. Then there exist a probability space* $(X, \mathscr{B}, \mu)$*, measure preserving automorphisms* $T, S : X \to X$*, both of them Bernoulli, and* $A \in \mathscr{B}$*, such that*

$$\mu\left(T^{-a(n)}A \cap S^{-b(n)}A\right) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \tag{18}$$

In light of Sinai's Factor Theorem, we see that the assumption of 0-entropy in the last 2 slides cannot be weakened.

Application 1/3

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving automorphisms for which T has singular spectrum. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed in the orbit closure of α for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

(i) *For any $f, g \in L^{\infty}(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}[f|\mathcal{I}_T]\,\mathbb{E}[g|\mathcal{I}_S], \tag{19}$$

with convergence taking place in $L^2(X, \mu)$.

Application $1/3$ continued

Theorem (Continued)

(ii) *If $A \in \mathscr{B}$ then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap S^{-k_n}A\right) \geq \mu(A)^3. \qquad (20)$$

(iii) *If we only assume that $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$, then (i) and (ii) hold when S is totally ergodic.*

Examples include $k_n = \lfloor a(n) \rfloor$ with $a(n)$ being as in frame 19, $k_n = \lfloor n^2 \log^2(n) \rfloor$, and for part (iii) we may take $k_n = p(n)$ for $p(x) \in x\mathbb{Z}[x]$ with degree at least 2. An analogous result is now known for countable abelian groups.

Sets of K but not $K+1$ recurrence?

Theorem (Frantzikinakis, Lesigne, Wierdl [15])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$.

(i) *If $(X, \mathscr{B}, \mu)$ is a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which*

$$\mu\left(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{21}$$

(ii) *There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A\right) = 0. \tag{22}$$

Application 2/3

Theorem (F., 2023)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ commuting measure preserving automorphisms. Let $T : X \to X$ be an measure preserving automorphism with singular spectrum, and for which $\{T, S_1, S_2, \cdots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu\left(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{23}$$

Since the system $(\mathbb{T}^2, \mathscr{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \sum_{n\in\mathbb{Z}} \delta_{n\alpha}$.

Application 3/3

Theorem (F., 2023)

Let K be a countable field with characteristic 0. Let $(X, \mathscr{B}, \nu)$ be a probability space and $T_g, S_g : X \to X$ measure preserving actions of $(K, +)$ for which the action $(T_g)_{g \in K}$ has singular spectrum and the action $(S_g)_{g \in K}$ is ergodic. Let $(F_n)_{n=1}^{\infty}$ be a Følner sequence in $(K, +)$ and $\ell \in \mathbb{N}$. Let $p_1, \cdots, p_\ell \in K[x]$ be polynomials for which $deg(p_{i+1}) \geq 2 + deg(p_i)$ for $1 \leq i < \ell$. Then for any $f_0, f_1, \cdots, f_\ell \in L^\infty(X, \mu)$ we have

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{n \in F_N} T_n f_0 \prod_{j=1}^{\ell} S_{p_j(n)} f_j = \mathbb{E}[f_0 | \mathcal{I}_T] \prod_{j=1}^{\ell} \int_X f_j d\nu \qquad (24)$$

with convergence taking place in $L^2(X, \nu)$.

This is a corollary of a more general result involving joint ergodicity.

An example

Consider the m.p.s. $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T, S)$ with $S(x,y) = (x+2\alpha, y+x)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $T(x,y) = (x, y+x)$. We see that $([0,1]^2, \mathscr{B}, \mathcal{L}^2, S)$ and $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T)$ are both zero entropy systems that are not weakly mixing, and the former is totally ergodic. Furthermore, T and S generate a 2-step nilpotent group. For $f_0(x,y) = e^{2\pi i(x-y)}$, $f_1(x,y) = e^{2\pi i y}$, and $f_2(x,y) = e^{-2\pi i x}$, we see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_0(x,y) S^n f_1(x,y) S^{\frac{1}{2}(n^2-n)} f_2(x,y)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \left((1-n)x - y + y + nx + (n^2-n)\alpha - x - (n^2-n)\alpha\right)} = 1 \neq 0.$$

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