

Van der Corput's difference theorem and the left regular representation.

George Mason University
Geometry MMA: Metrics, Measures, and Algorithms Seminar

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Overview

- 1 Van der Corput's difference theorem and some applications
- 2 The Furstenberg correspondence principle between combinatorics and dynamics
- 3 Lebesgue spectrum, singular spectrum, and the left regular representation
- 4 Van der Corput's difference theorem and the left regular representation
- 5 Applications (Bonus)
 - Background on noncommutative ergodic theory
 - New results from mixing vdCs

Table of Contents

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The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ is **uniformly distributed** if for any open interval $(a, b) \subseteq [0, 1]$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\{1 \leq n \leq N \mid x_n \in (a, b)\}| = b - a. \quad (1)$$

Theorem (van der Corput, 1931 [34])

If $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha)_{n=1}^{\infty}$ is uniformly distributed.

Theorem (HvdCDT1, Bergelson, 1987 [2, Theorem 1.4])

If \mathcal{H} is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle = 0, \quad (2)$$

for every $h \in \mathbb{N}$, then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0. \quad (3)$$

Theorem (HvdCDT2, Bergelson, 1987 [2, Page 3])

If \mathcal{H} is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h \rightarrow \infty} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \quad (4)$$

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0. \quad (5)$$

Hilbertian van der Corput Difference Theorems 3/3

Theorem (HvdCDT3, Bergelson, 1987 [2, Theorem 1.5])

If \mathcal{H} is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bded seq. satisfying

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then } \lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0.$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

See [12, Chapter 2] for variations of vdCT related to the levels of mixing in the ergodic hierarchy of mixing properties, as well as similar variations in the context of uniform distribution. See also [33] and [9].

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) (X, \mathcal{B}, μ, T) , and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \quad (6)$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy [19],[28])

For any m.p.s. (X, \mathcal{B}, μ, T) , and any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \quad (7)$$

Furstenberg's proof in [19, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^\infty$. See also [3, Theorem 2.1] for a proof using HvdCDT1 directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg multiple recurrence, [19])

For any m.p.s. (X, \mathcal{B}, μ, T) , any $A \in \mathcal{B}$ with $\mu(A) > 0$, and any $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-\ell n}A) > 0. \quad (8)$$

The proof presented in [10] uses HvdCT3 as Theorem 7.11, and the proof in [20] uses a variation.

Theorem (Bergelson and Leibman, [5, Theorem A_0])

For any m.p.s. $(X, \mathcal{B}, \mu, \{T_i\}_{i=1}^{\ell})$ with the T_i s commuting, any $A \in \mathcal{B}$ with $\mu(A) > 0$, and any $\{p_i(x)\}_{i=1}^{\ell} \subseteq \mathbb{N}[x]$, there exists $n \in \mathbb{N}$ for which

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \dots \cap T_{\ell}^{-p_{\ell}(n)}A\right) > 0. \quad (9)$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

Table of Contents

- 1 Van der Corput's difference theorem and some applications
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Furstenberg's correspondence principle

Definition

For a set $E \subseteq \mathbb{N}$, the **natural upper density** of E is denoted by $\bar{d}(E)$ and is given by

$$\bar{d}(E) = \limsup_{N \rightarrow \infty} \frac{1}{N} |E \cap [1, N]|. \quad (10)$$

We see that $\bar{d}(2\mathbb{N}) = \bar{d}(2\mathbb{N} + 1) = \frac{1}{2}$, $\bar{d}(a\mathbb{N} + b) = \frac{1}{a}$ for any $a, b \in \mathbb{N}$, and $\bar{d}(\{n^2\}_{n \in \mathbb{N}}) = 0$.

Theorem (The correspondence principle, [19],[3, Theorem 1.8])

Given a set $E \subseteq \mathbb{N}$ for which $\bar{d}(E) > 0$, there exists a measure preserving system (X, \mathcal{B}, μ, T) and a set $A \in \mathcal{B}$ with $\mu(A) = \bar{d}(E)$, such that for any $\ell, n_1, n_2, \dots, n_\ell \in \mathbb{N}$ we have

$$\bar{d}(E \cap (E - n_1) \cap \dots \cap (E - n_\ell)) \geq \mu(A \cap T^{-n_1}A \cap \dots \cap T^{-n_\ell}A)$$

Szemerédi's theorem

Theorem (Szemerédi [31], 1975)

If $E \subseteq \mathbb{N}$ satisfies $\bar{d}(E) > 0$, then for any $\ell \in \mathbb{N}$, E contains an arithmetic progression of length ℓ .

This was conjectured by Erdős and Turán [11] in 1936. The case of length 3 arithmetic progressions was resolved by Roth [27] in 1952. The case of length 4 arithmetic progressions was resolved by Szemerédi [30] in 1969. Furstenberg [19] gave the second proof in 1977, and Gowers [21] the third proof in 2001.

Using the correspondence principle

We will now deduce Szemerédi's Theorem from the Furstenberg Multiple Recurrence Theorem by using the correspondence principle. Let $E \subseteq \mathbb{N}$ be such that $\overline{d}(E) > 0$, let (X, \mathcal{B}, μ, T) and $A \in \mathcal{B}$ be given by the correspondence principle, and let $\ell \in \mathbb{N}$ be arbitrary. Furstenberg's Multiple Recurrence Theorem tells us that there exists $n \in \mathbb{N}$ for which

$$\gamma(n) := \mu(A \cap T^{-n}A \cap \cdots \cap T^{-n\ell}A) > 0.$$

The correspondence principle tells us that for

$E(n) := E \cap (E - n) \cap \cdots \cap (E - n\ell)$ we have $\overline{d}(E(n)) \geq \gamma(n) > 0$.

Since $E(n) \neq \emptyset$, we see that for $a \in E(n)$ we have $a, a + n, \dots, a + n\ell \in E$.

More results in density Ramsey theory

Theorem (Furstenberg-Sárközy [19],[28])

Let $E \subseteq \mathbb{N}$ be such that $\overline{d}(E) > 0$. Then there exists $n \in \mathbb{N}$ for which $\overline{d}(E \cap (E - n^2)) > 0$. In particular, there exists $x, y \in E$ with $x - y = n^2$.

Theorem (Polynomial Szemerédi, due to Bergelson and Leibman)

Let $E \subseteq \mathbb{N}$ be such that $\overline{d}(E) > 0$ and let $p_1, \dots, p_\ell \in x\mathbb{Z}[x]$ be arbitrary. Then there exists $n \in \mathbb{N}$ for which

$$\overline{d}(E \cap (E - p_1(n)) \cap \dots \cap (E - p_\ell(n))) > 0. \quad (11)$$

In particular, there exists $a, n \in \mathbb{N}$ for which $a, a + p_1(n), \dots, a + p_\ell(n) \in E$.

Note that this is only a special case of Theorems B and B' of [5].

Table of Contents

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- 3 Lebesgue spectrum, singular spectrum, and the left regular representation
- 4 Van der Corput's difference theorem and the left regular representation
- 5 Applications (Bonus)
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Lebesgue spectrum and singular spectrum

Definition

Let $\mathcal{X} = (X, \mathcal{B}, \mu, T)$ be an invertible m.p.s. and let $U_T : L^2(X, \mu) \rightarrow L^2(X, \mu)$ be the Koopman operator induced by T . If $L^2_0(X, \mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m \in \mathbb{Z}}$, then \mathcal{X} has **Lebesgue spectrum**. This implies that for all $f \in L^2_0(X, \mu)$, the sequence $(\langle U_T^n f, f \rangle)_{n=1}^\infty$ is the Fourier coefficients of some measure $\nu \ll \mathcal{L}$, where \mathcal{L} is the Lebesgue measure. On the other hand, if for every $f \in L^2(X, \mu)$, the sequence $(\langle U_T^n f, f \rangle)_{n=1}^\infty$ is the Fourier coefficients of some positive measure $\nu \perp \mathcal{L}$, then the system \mathcal{X} has **singular spectrum**.

Examples of systems with Lebesgue spectrum

Any K-mixing system has Lebesgue spectrum, hence all Bernoulli systems have Lebesgue spectrum. The Sinai factor theorem [29] tells us that a non-atomic ergodic m.p.s. with positive entropy has a Bernoulli shift as a factor, and consequently has a factor with Lebesgue spectrum. It follows that the original system does NOT have singular spectrum. The horocycle flow is an example of a system with Lebesgue spectrum [26] that also has 0-entropy [22].

Examples of systems with singular spectrum

In [23] and [4] it is shown that if (X, \mathcal{B}, μ) is a standard probability space, and $\text{Aut}(X, \mathcal{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism (such as a group rotation) has singular spectrum, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \text{Aut}(X, \mathcal{B}, \mu)$ denote the collection of strongly mixing transformation, and note that \mathcal{S} is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. Since \mathcal{S} is not a complete metric space with respect to the strong operator topology, a new topology was introduced in [32], with respect to which \mathcal{S} is a complete metric space. It is shown in the Corollary to Theorem 7 of [32] that a generic $T \in \mathcal{S}$ has singular spectrum, and such a T is mixing of all orders due a well known result of Host [24]. See [13],[25][1], [6], and [18] for more examples of T that have singular spectrum.

The left regular representation

Let G is a locally compact Hausdorff group with left Haar measure λ . There is a unitary representation L of G on $L^2(G, \nu)$ given by $(L_g f)(h) = f(g^{-1}h)$, which is known as the **left regular representation**. If $f \in L^2(G, \nu)$ is a positive definite function, then there exists a function $h \in L^2(G, \lambda)$ for which $f(g) = \langle L_g h, h \rangle$. In particular, consider a representation U of G on \mathcal{H} , and a cyclic vector $f \in \mathcal{H}$ such that

$$\int_G |\langle U_g f, f \rangle|^2 d\lambda(g) < \infty. \quad (12)$$

Then U is isomorphic to a subrepresentation of the left regular representation.

Spectrum and the left regular representation

Let G be an amenable group and $\mathcal{X} := (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ a measure preserving G -system, which we abbreviate to G -system. We let $U : L^2(X, \mu) \rightarrow L^2(X, \mu)$ denote the unitary representation of G induced by T , i.e., $U_g f = f \circ T_{g^{-1}}$. The system \mathcal{X} has **Lebesgue spectrum** if U decomposes into a direct sum of countably many copies of the left regular representation. The system \mathcal{X} has **singular spectrum** if the representation U is disjoint from the left regular representation. Dooley and Golodets [8] showed that if G is countable and \mathcal{X} has completely positive entropy (analogue of K -mixing) then it also has Lebesgue spectrum. Danilenko and Park [7] proved an analogue of Sinai's factor theorem when G is countable, from which we deduce that \mathcal{X} does not have singular spectrum when it is free, ergodic, and has positive entropy.

Table of Contents

- 1 Van der Corput's difference theorem and some applications
- 2 The Furstenberg correspondence principle between combinatorics and dynamics
- 3 Lebesgue spectrum, singular spectrum, and the left regular representation
- 4 Van der Corput's difference theorem and the left regular representation
- 5 Applications (Bonus)
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A Lebesgue spectrum vdCDT

Theorem (F. 2023)

If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\sum_{h=1}^{\infty} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \quad (13)$$

then $(x_n)_{n=1}^\infty$ is a spectrally Lebesgue sequence. If $\mathcal{H} = L^2(X, \mu)$ and $(y_n)_{n=1}^\infty \subseteq L^\infty(X, \mu)$ is bounded and spectrally singular, then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n y_n \right\| = 0. \quad (14)$$

A Lebesgue spectrum vdCDT for amenable groups

Theorem (F. 2023)

Let G be a countable amenable group and $(F_n)_{n=1}^\infty$ a left Følner sequence. If $(x_g)_{g \in G} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\sum_{h \in G} \limsup_{N \rightarrow \infty} \left| \frac{1}{|F_N|} \sum_{g \in F_N} \langle x_{gh}, x_g \rangle \right|^2 < \infty, \quad (15)$$

then $(x_g)_{g \in G}$ is a spectrally Lebesgue sequence. If $\mathcal{H} = L^2(X, \mu)$ and $(y_n)_{n=1}^\infty \subseteq L^\infty(X, \mu)$ is bounded and spectrally singular, then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{|F_N|} \sum_{g \in F_N} x_g y_g \right\| = 0. \quad (16)$$

Table of Contents

- 1 Van der Corput's difference theorem and some applications
- 2 The Furstenberg correspondence principle between combinatorics and dynamics
- 3 Lebesgue spectrum, singular spectrum, and the left regular representation
- 4 Van der Corput's difference theorem and the left regular representation
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Noncommutative ergodic theorems 1/2

Theorem (Frantzikinakis [14, Corollary 1.7])

Let $a : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t \rightarrow \infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t \rightarrow \infty} \frac{t^{d+1}}{a(t)} = \infty. \quad (\text{e.g. } a(t) = t^{1.5}) \quad (17)$$

Furthermore, let (X, \mathcal{B}, μ) be a probability space and $T, S : X \rightarrow X$ be measure preserving transformations. Suppose that the system (X, \mathcal{B}, μ, T) has zero entropy. Then

(i) For every $f, g \in L^\infty(X, \mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f | \mathcal{I}_T] \cdot \mathbb{E}[g | \mathcal{I}_S], \quad (18)$$

where the limit is taken in $L^2(X, \mu)$.

Theorem (Continued)

(ii) For every $A \in \mathcal{B}$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}A \cap S^{-\lfloor a(n) \rfloor}A) \geq \mu(A)^3. \quad (19)$$

Frantzikinakis and Host [15] proved a similar theorem for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2.

An Example

Theorem (Frantzikinakis, Lesigne, Wierdl [17, Lemma 4.1])

Let $a, b : \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$ be injective sequences and F be any subset of \mathbb{N} . Then there exist a probability space (X, \mathcal{B}, μ) , measure preserving automorphisms $T, S : X \rightarrow X$, both of them Bernoulli, and $A \in \mathcal{B}$, such that

$$\mu(T^{-a(n)}A \cap S^{-b(n)}A) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \quad (20)$$

In light of Sinai's Factor Theorem, we see that the assumption of 0-entropy in the last 2 slides cannot be weakened.

Application 1/3

Theorem (F., 2023)

Let (X, \mathcal{B}, μ) be a probability space and let $T, S : X \rightarrow X$ be measure preserving automorphisms for which T has singular spectrum. Let $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha)_{n=1}^\infty$ is uniformly distributed in the orbit closure of α for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

❶ For any $f, g \in L^\infty(X, \mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{k_n} g = \mathbb{E}[f | \mathcal{I}_T] \mathbb{E}[g | \mathcal{I}_S], \quad (21)$$

with convergence taking place in $L^2(X, \mu)$.

Application 1/3 continued

Theorem (Continued)

(ii) If $A \in \mathcal{B}$ then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}A \cap S^{-k_n}A) \geq \mu(A)^3. \quad (22)$$

(iii) If we only assume that $((k_{n+h} - k_n)\alpha)_{n=1}^\infty$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$, then (i) and (ii) hold when S is totally ergodic.

Examples include $k_n = \lfloor a(n) \rfloor$ with $a(n)$ being as in frame 19, $k_n = \lfloor n^2 \log^2(n) \rfloor$, and for part (iii) we may take $k_n = p(n)$ for $p(x) \in x\mathbb{Z}[x]$ with degree at least 2. An analogous result is now known for countable abelian groups.

Sets of K but not $K + 1$ recurrence?

Theorem (Frantzikinakis, Lesigne, Wierdl [16])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \{n \in \mathbb{N} \mid n^k \alpha \in [\frac{1}{4}, \frac{3}{4}]\}$.

- (i) If (X, \mathcal{B}, μ) is a probability space and $S_1, S_2, \dots, S_{k-1} : X \rightarrow X$ are commuting measure preserving transformations, then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which

$$\mu(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \dots \cap S_{k-1}^{-n}A) > 0. \quad (23)$$

- (ii) There exists a m.p.s. (X, \mathcal{B}, μ, T) and a set $A \in \mathcal{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-kn}A) = 0. \quad (24)$$

Application 2/3

Theorem (F., 2023)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \{n \in \mathbb{N} \mid n^k \alpha \in [\frac{1}{4}, \frac{3}{4}]\}$. Let (X, \mathcal{B}, μ) be a probability space and $S_1, S_2, \dots, S_{k-1} : X \rightarrow X$ commuting measure preserving automorphisms. Let $T : X \rightarrow X$ be an measure preserving automorphism with singular spectrum, and for which $\{T, S_1, S_2, \dots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \dots \cap S_{k-1}^{-n}A) > 0. \quad (25)$$

Since the system $(\mathbb{T}^2, \mathcal{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \sum_{n \in \mathbb{Z}} \delta_{n\alpha}$.

Application 3/3

Theorem (F., 2023)

Let K be a countable field with characteristic 0. Let (X, \mathcal{B}, ν) be a probability space and $T_g, S_g : X \rightarrow X$ measure preserving actions of $(K, +)$ for which the action $(T_g)_{g \in K}$ has singular spectrum and the action $(S_g)_{g \in K}$ is ergodic. Let $(F_n)_{n=1}^\infty$ be a Følner sequence in $(K, +)$ and $\ell \in \mathbb{N}$. Let $p_1, \dots, p_\ell \in K[x]$ be polynomials for which $\deg(p_{i+1}) \geq 2 + \deg(p_i)$ for $1 \leq i < \ell$. Then for any $f_0, f_1, \dots, f_\ell \in L^\infty(X, \mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{|F_N|} \sum_{n \in F_N} T_n f_0 \prod_{j=1}^{\ell} S_{p_j(n)} f_j = \mathbb{E}[f_0 | \mathcal{I}_T] \prod_{j=1}^{\ell} \int_X f_j d\nu \quad (26)$$

with convergence taking place in $L^2(X, \nu)$.

This is a corollary of a more general result involving joint ergodicity.

An example

Consider the m.p.s. $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, T, S)$ with $S(x, y) = (x + 2\alpha, y + x)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $T(x, y) = (x, y + x)$. We see that $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, S)$ and $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, T)$ are both zero entropy systems that are not weakly mixing, and the former is totally ergodic. Furthermore, T and S generate a 2-step nilpotent group. For $f_0(x, y) = e^{2\pi i(x-y)}$, $f_1(x, y) = e^{2\pi i y}$, and $f_2(x, y) = e^{-2\pi i x}$, we see that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f_0(x, y) S^n f_1(x, y) S^{\frac{1}{2}(n^2-n)} f_2(x, y) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i((1-n)x - y + y + nx + (n^2-n)\alpha - x - (n^2-n)\alpha)} = 1 \neq 0. \end{aligned}$$

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