

# Van der Corput's difference theorem and the left regular representation.

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Geometry MMA: Metrics, Measures, and Algorithms Seminar

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- 1 Van der Corput's difference theorem and some applications
- 2 The Furstenberg correspondence principle between combinatorics and dynamics
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- 5 Applications (Bonus)
  - Background on noncommutative ergodic theory
  - New results from mixing vdCs

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# The Classical van der Corput Difference Theorem

## Definition

A sequence  $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$  is **uniformly distributed** if for any open interval  $(a, b) \subseteq [0, 1]$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} |\{1 \leq n \leq N \mid x_n \in (a, b)\}| = b - a. \quad (1)$$

## Theorem (van der Corput, 1931 [34])

If  $(x_n)_{n=1}^{\infty} \subseteq [0, 1]$  is such that  $(x_{n+h} - x_n)_{n=1}^{\infty}$  is uniformly distributed for every  $h \in \mathbb{N}$ , then  $(x_n)_{n=1}^{\infty}$  is itself uniformly distributed.

## Corollary

If  $\alpha \in \mathbb{R}$  is irrational, then  $(n^2\alpha)_{n=1}^{\infty}$  is uniformly distributed.

Theorem (HvdCDT1, Bergelson, 1987 [2, Theorem 1.4])

If  $\mathcal{H}$  is a Hilbert space and  $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$  is a bounded sequence satisfying

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle = 0, \quad (2)$$

for every  $h \in \mathbb{N}$ , then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0. \quad (3)$$

Theorem (HvdCDT2, Bergelson, 1987 [2, Page 3])

If  $\mathcal{H}$  is a Hilbert space and  $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$  is a bounded sequence satisfying

$$\lim_{h \rightarrow \infty} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \quad (4)$$

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0. \quad (5)$$

Theorem (HvdCDT3, Bergelson, 1987 [2, Theorem 1.5])

If  $\mathcal{H}$  is a Hilbert space and  $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$  is a bded seq. satisfying

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then } \lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n \right\| = 0.$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

See [12, Chapter 2] for variations of vdCT related to the levels of mixing in the ergodic hierarchy of mixing properties, as well as similar variations in the context of uniform distribution. See also [33] and [9].

# Applications of HvdCDTs 1/2

## Theorem (Poincaré)

For any measure preserving system (m.p.s.)  $(X, \mathcal{B}, \mu, T)$ , and any  $A \in \mathcal{B}$  with  $\mu(A) > 0$ , there exists  $n \in \mathbb{N}$  for which

$$\mu(A \cap T^{-n}A) > 0. \quad (6)$$

Does not need vdCDT.

## Theorem (Furstenberg-Sárközy [19],[28])

For any m.p.s.  $(X, \mathcal{B}, \mu, T)$ , and any  $A \in \mathcal{B}$  with  $\mu(A) > 0$ , there exists  $n \in \mathbb{N}$  for which

$$\mu(A \cap T^{-n^2}A) > 0. \quad (7)$$

Furstenberg's proof in [19, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of  $(n^2\alpha)_{n=1}^{\infty}$ . See also [3, Theorem 2.1] for a proof using HvdCDT1 directly.

# Applications of HvdCDTs 2/2

Theorem (Furstenberg multiple recurrence, [19])

For any m.p.s.  $(X, \mathcal{B}, \mu, T)$ , any  $A \in \mathcal{B}$  with  $\mu(A) > 0$ , and any  $\ell \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  for which

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A) > 0. \quad (8)$$

The proof presented in [10] uses HvdCT3 as Theorem 7.11, and the proof in [20] uses a variation.

Theorem (Bergelson and Leibman, [5, Theorem A<sub>0</sub>])

For any m.p.s.  $(X, \mathcal{B}, \mu, \{T_i\}_{i=1}^\ell)$  with the  $T_i$ s commuting, any  $A \in \mathcal{B}$  with  $\mu(A) > 0$ , and any  $\{p_i(x)\}_{i=1}^\ell \subseteq x\mathbb{N}[x]$ , there exists  $n \in \mathbb{N}$  for which

$$\mu(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_\ell^{-p_\ell(n)}A) > 0. \quad (9)$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

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# Furstenberg's correspondence principle

## Definition

For a set  $E \subseteq \mathbb{N}$ , the **natural upper density** of  $E$  is denoted by  $\overline{d}(E)$  and is given by

$$\overline{d}(E) = \limsup_{N \rightarrow \infty} \frac{1}{N} |E \cap [1, N]|. \quad (10)$$

We see that  $\overline{d}(2\mathbb{N}) = \overline{d}(2\mathbb{N} + 1) = \frac{1}{2}$ ,  $\overline{d}(a\mathbb{N} + b) = \frac{1}{a}$  for any  $a, b \in \mathbb{N}$ , and  $\overline{d}(\{n^2\}_{n \in \mathbb{N}}) = 0$ .

## Theorem (The correspondence principle, [19],[3, Theorem 1.8])

Given a set  $E \subseteq \mathbb{N}$  for which  $\overline{d}(E) > 0$ , there exists a measure preserving system  $(X, \mathcal{B}, \mu, T)$  and a set  $A \in \mathcal{B}$  with  $\mu(A) = \overline{d}(E)$ , such that for any  $\ell, n_1, n_2, \dots, n_\ell \in \mathbb{N}$  we have

$$\overline{d}(E \cap (E - n_1) \cap \dots \cap (E - n_\ell)) \geq \mu(A \cap T^{-n_1}A \cap \dots \cap T^{-n_\ell}A)$$

# Szemerédi's theorem

## Theorem (Szemerédi [31], 1975)

*If  $E \subseteq \mathbb{N}$  satisfies  $\overline{d}(E) > 0$ , then for any  $\ell \in \mathbb{N}$ ,  $E$  contains an arithmetic progression of length  $\ell$ .*

This was conjectured by Erdős and Turán [11] in 1936. The case of length 3 arithmetic progressions was resolved by Roth [27] in 1952. The case of length 4 arithmetic progressions was resolved by Szemerédi [30] in 1969. Furstenberg [19] gave the second proof in 1977, and Gowers [21] the third proof in 2001.

# Using the correspondence principle

We will now deduce Szemerédi's Theorem from the Furstenberg Multiple Recurrence Theorem by using the correspondence principle. Let  $E \subseteq \mathbb{N}$  be such that  $\overline{d}(E) > 0$ , let  $(X, \mathcal{B}, \mu, T)$  and  $A \in \mathcal{B}$  be given by the correspondence principle, and let  $\ell \in \mathbb{N}$  be arbitrary. Furstenberg's Multiple Recurrence Theorem tells us that there exists  $n \in \mathbb{N}$  for which

$$\gamma(n) := \mu(A \cap T^{-n}A \cap \cdots \cap T^{-n\ell}A) > 0.$$

The correspondence principle tells us that for

$$E(n) := E \cap (E - n) \cap \cdots \cap (E - n\ell) \text{ we have } \overline{d}(E(n)) \geq \gamma(n) > 0.$$

Since  $E(n) \neq \emptyset$ , we see that for  $a \in E(n)$  we have  $a, a + n, \dots, a + n\ell \in E$ .

# More results in density Ramsey theory

Theorem (Furstenberg-Sárközy [19],[28])

Let  $E \subseteq \mathbb{N}$  be such that  $\overline{d}(E) > 0$ . Then there exists  $n \in \mathbb{N}$  for which  $\overline{d}(E \cap (E - n^2)) > 0$ . In particular, there exists  $x, y \in E$  with  $x - y = n^2$ .

Theorem (Polynomial Szemerédi, due to Bergelson and Leibman)

Let  $E \subseteq \mathbb{N}$  be such that  $\overline{d}(E) > 0$  and let  $p_1, \dots, p_\ell \in x\mathbb{Z}[x]$  be arbitrary. Then there exists  $n \in \mathbb{N}$  for which

$$\overline{d}(E \cap (E - p_1(n)) \cap \dots \cap (E - p_\ell(n))) > 0. \quad (11)$$

In particular, there exists  $a, n \in \mathbb{N}$  for which  $a, a + p_1(n), \dots, a + p_\ell(n) \in E$ .

Note that this is only a special case of Theorems B and B' of [5].

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# Lebesgue spectrum and singular spectrum

## Definition

Let  $\mathcal{X} = (X, \mathcal{B}, \mu, T)$  be an invertible m.p.s. and let  $U_T : L^2(X, \mu) \rightarrow L^2(X, \mu)$  be the Koopman operator induced by  $T$ . If  $L_0^2(X, \mu)$  has an orthogonal basis of the form  $\{U_T^n f_m\}_{n, m \in \mathbb{Z}}$ , then  $\mathcal{X}$  has **Lebesgue spectrum**. This implies that for all  $f \in L_0^2(X, \mu)$ , the sequence  $(\langle U_T^n f, f \rangle)_{n=1}^\infty$  is the Fourier coefficients of some measure  $\nu \ll \mathcal{L}$ , where  $\mathcal{L}$  is the Lebesgue measure. On the other hand, if for every  $f \in L^2(X, \mu)$ , the sequence  $(\langle U_T^n f, f \rangle)_{n=1}^\infty$  is the Fourier coefficients of some positive measure  $\nu \perp \mathcal{L}$ , then the system  $\mathcal{X}$  has **singular spectrum**.

# Examples of systems with Lebesgue spectrum

Any K-mixing system has Lebesgue spectrum, hence all Bernoulli systems have Lebesgue spectrum. The Sinai factor theorem [29] tells us that a non-atomic ergodic m.p.s. with positive entropy has a Bernoulli shift as a factor, and consequently has a factor with Lebesgue spectrum. It follows that the original system does NOT have singular spectrum. The horocycle flow is an example of a system with Lebesgue spectrum [26] that also has 0-entropy [22].

## Examples of systems with singular spectrum

In [23] and [4] it is shown that if  $(X, \mathcal{B}, \mu)$  is a standard probability space, and  $\text{Aut}(X, \mathcal{B}, \mu)$  is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism (such as a group rotation) has singular spectrum, we see that the set of singular automorphisms is generic. Now let

$\mathcal{S} \subseteq \text{Aut}(X, \mathcal{B}, \mu)$  denote the collection of strongly mixing transformation, and note that  $\mathcal{S}$  is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. Since  $\mathcal{S}$  is not a complete metric space with respect to the strong operator topology, a new topology was introduced in [32], with respect to which  $\mathcal{S}$  is a complete metric space. It is shown in the Corollary to Theorem 7 of [32] that a generic  $T \in \mathcal{S}$  has singular spectrum, and such a  $T$  is mixing of all orders due a well known result of Host [24]. See [13], [25][1], [6], and [18] for more examples of  $T$  that have singular spectrum.

# The left regular representation

Let  $G$  is a locally compact Hausdorff group with left Haar measure  $\lambda$ . There is a unitary representation  $L$  of  $G$  on  $L^2(G, \nu)$  given by  $(L_g f)(h) = f(g^{-1}h)$ , which is known as the **left regular representation**. If  $f \in L^2(G, \nu)$  is a positive definite function, then there exists a function  $h \in L^2(G, \lambda)$  for which  $f(g) = \langle L_g h, h \rangle$ . In particular, consider a representation  $U$  of  $G$  on  $\mathcal{H}$ , and a cyclic vector  $f \in \mathcal{H}$  such that

$$\int_G |\langle U_g f, f \rangle|^2 d\lambda(g) < \infty. \quad (12)$$

Then  $U$  is isomorphic to a subrepresentation of the left regular representation.

# Spectrum and the left regular representation

Let  $G$  be an amenable group and  $\mathcal{X} := (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$  a measure preserving  $G$ -system, which we abbreviate to  $G$ -system. We let  $U : L^2(X, \mu) \rightarrow L^2(X, \mu)$  denote the unitary representation of  $G$  induced by  $T$ , i.e.,  $U_g f = f \circ T_{g^{-1}}$ . The system  $\mathcal{X}$  has **Lebesgue spectrum** if  $U$  decomposes into a direct sum of countably many copies of the left regular representation. The system  $\mathcal{X}$  has **singular spectrum** if the representation  $U$  is disjoint from the left regular representation. Dooley and Golodets [8] showed that if  $G$  is countable and  $\mathcal{X}$  has completely positive entropy (analogue of  $K$ -mixing) then it also has Lebesgue spectrum. Danilenko and Park [7] proved an analogue of Sinai's factor theorem when  $G$  is countable, from which we deduce that  $\mathcal{X}$  does not have singular spectrum when it is free, ergodic, and has positive entropy.

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# A Lebesgue spectrum vdCDT

Theorem (F. 2023)

If  $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$  is a bounded sequence satisfying

$$\sum_{h=1}^{\infty} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \quad (13)$$

then  $(x_n)_{n=1}^{\infty}$  is a spectrally Lebesgue sequence. If  $\mathcal{H} = L^2(X, \mu)$  and  $(y_n)_{n=1}^{\infty} \subseteq L^{\infty}(X, \mu)$  is bounded and spectrally singular, then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n y_n \right\| = 0. \quad (14)$$

# A Lebesgue spectrum vdCDT for amenable groups

Theorem (F. 2023)

Let  $G$  be a countable amenable group and  $(F_n)_{n=1}^\infty$  a left Følner sequence. If  $(x_g)_{g \in G} \subseteq \mathcal{H}$  is a bounded sequence satisfying

$$\sum_{h \in G} \limsup_{N \rightarrow \infty} \left| \frac{1}{|F_N|} \sum_{g \in F_N} \langle x_{gh}, x_g \rangle \right|^2 < \infty, \quad (15)$$

then  $(x_g)_{g \in G}$  is a spectrally Lebesgue sequence. If  $\mathcal{H} = L^2(X, \mu)$  and  $(y_n)_{n=1}^\infty \subseteq L^\infty(X, \mu)$  is bounded and spectrally singular, then

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{|F_N|} \sum_{g \in F_N} x_g y_g \right\| = 0. \quad (16)$$

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# Noncommutative ergodic theorems 1/2

Theorem (Frantzkinakis [14, Corollary 1.7])

Let  $a : \mathbb{R}_+ \rightarrow \mathbb{R}$  be a Hardy field function for which there exist some  $\epsilon > 0$  and  $d \in \mathbb{Z}_+$  satisfying

$$\lim_{t \rightarrow \infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t \rightarrow \infty} \frac{t^{d+1}}{a(t)} = \infty. \quad (\text{e.g. } a(t) = t^{1.5}) \quad (17)$$

Furthermore, let  $(X, \mathcal{B}, \mu)$  be a probability space and  $T, S : X \rightarrow X$  be measure preserving transformations. Suppose that the system  $(X, \mathcal{B}, \mu, T)$  has zero entropy. Then

(i) For every  $f, g \in L^\infty(X, \mu)$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f|\mathcal{I}_T] \cdot \mathbb{E}[g|\mathcal{I}_S], \quad (18)$$

where the limit is taken in  $L^2(X, \mu)$ .

## Theorem (Continued)

(ii) For every  $A \in \mathcal{B}$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}A \cap S^{-\lfloor a(n) \rfloor}A) \geq \mu(A)^3. \quad (19)$$

Frantzkinakis and Host [15] proved a similar theorem for  $a(n) = p(n)$  with  $p(x) \in \mathbb{Z}[x]$  of degree at least 2.

# An Example

Theorem (Frantzkinakis, Lesigne, Wierdl [17, Lemma 4.1])

Let  $a, b : \mathbb{N} \rightarrow \mathbb{Z} \setminus \{0\}$  be injective sequences and  $F$  be any subset of  $\mathbb{N}$ . Then there exist a probability space  $(X, \mathcal{B}, \mu)$ , measure preserving automorphisms  $T, S : X \rightarrow X$ , both of them Bernoulli, and  $A \in \mathcal{B}$ , such that

$$\mu(T^{-a(n)}A \cap S^{-b(n)}A) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \quad (20)$$

In light of Sinai's Factor Theorem, we see that the assumption of 0-entropy in the last 2 slides cannot be weakened.

## Theorem (F., 2023)

Let  $(X, \mathcal{B}, \mu)$  be a probability space and let  $T, S : X \rightarrow X$  be measure preserving automorphisms for which  $T$  has singular spectrum. Let  $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$  be a sequence for which  $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$  is uniformly distributed in the orbit closure of  $\alpha$  for all  $\alpha \in \mathbb{R}$  and  $h \in \mathbb{N}$ .

① For any  $f, g \in L^{\infty}(X, \mu)$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{k_n} g = \mathbb{E}[f | \mathcal{I}_T] \mathbb{E}[g | \mathcal{I}_S], \quad (21)$$

with convergence taking place in  $L^2(X, \mu)$ .

## Theorem (Continued)

(ii) *If  $A \in \mathcal{B}$  then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n}A \cap S^{-k_n}A) \geq \mu(A)^3. \quad (22)$$

(iii) *If we only assume that  $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$  is uniformly distributed for all  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and  $h \in \mathbb{N}$ , then (i) and (ii) hold when  $S$  is totally ergodic.*

Examples include  $k_n = \lfloor a(n) \rfloor$  with  $a(n)$  being as in frame 19,  $k_n = \lfloor n^2 \log^2(n) \rfloor$ , and for part (iii) we may take  $k_n = p(n)$  for  $p(x) \in x\mathbb{Z}[x]$  with degree at least 2. An analogous result is now known for countable abelian groups.

# Sets of $K$ but not $K + 1$ recurrence?

Theorem (Frantzkinakis, Lesigne, Wierdl [16])

Let  $k \geq 2$  be an integer and  $\alpha \in \mathbb{R}$  be irrational. Let

$$R_k = \left\{ n \in \mathbb{N} \mid n^k \alpha \in \left[ \frac{1}{4}, \frac{3}{4} \right] \right\}.$$

(i) If  $(X, \mathcal{B}, \mu)$  is a probability space and

$S_1, S_2, \dots, S_{k-1} : X \rightarrow X$  are commuting measure preserving transformations, then for any  $A \in \mathcal{B}$  with  $\mu(A) > 0$ , there exists  $n \in R_k$  for which

$$\mu(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \dots \cap S_{k-1}^{-n}A) > 0. \quad (23)$$

(ii) There exists a m.p.s.  $(X, \mathcal{B}, \mu, T)$  and a set  $A \in \mathcal{B}$  satisfying  $\mu(A) > 0$  such that for all  $n \in R_k$  we have

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-kn}A) = 0. \quad (24)$$

# Application 2/3

## Theorem (F., 2023)

Let  $k \geq 2$  be an integer and  $\alpha \in \mathbb{R}$  be irrational. Let  $R_k = \{n \in \mathbb{N} \mid n^k \alpha \in [\frac{1}{4}, \frac{3}{4}]\}$ . Let  $(X, \mathcal{B}, \mu)$  be a probability space and  $S_1, S_2, \dots, S_{k-1} : X \rightarrow X$  commuting measure preserving automorphisms. Let  $T : X \rightarrow X$  be a measure preserving automorphism with singular spectrum, and for which  $\{T, S_1, S_2, \dots, S_{k-1}\}$  generate a nilpotent group. For any  $A \in \mathcal{B}$  with  $\mu(A) > 0$ , there exists  $n \in R$  for which

$$\mu(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \dots \cap S_{k-1}^{-n}A) > 0. \quad (25)$$

Since the system  $(\mathbb{T}^2, \mathcal{B}^2, \mathcal{L}^2, T)$  with  $T(x, y) = (x + \alpha, y + x)$  can be used in item (ii) of the last slide when  $k = 2$ , the current theorem does not hold for a general  $T$  with 0 entropy. Also note that the maximal spectral type of  $T$  is  $\mathcal{L} + \sum_{n \in \mathbb{Z}} \delta_{n\alpha}$ .

# Application 3/3

## Theorem (F., 2023)

Let  $K$  be a countable field with characteristic 0. Let  $(X, \mathcal{B}, \nu)$  be a probability space and  $T_g, S_g : X \rightarrow X$  measure preserving actions of  $(K, +)$  for which the action  $(T_g)_{g \in K}$  has singular spectrum and the action  $(S_g)_{g \in K}$  is ergodic. Let  $(F_n)_{n=1}^{\infty}$  be a Følner sequence in  $(K, +)$  and  $\ell \in \mathbb{N}$ . Let  $p_1, \dots, p_{\ell} \in K[x]$  be polynomials for which  $\deg(p_{i+1}) \geq 2 + \deg(p_i)$  for  $1 \leq i < \ell$ . Then for any  $f_0, f_1, \dots, f_{\ell} \in L^{\infty}(X, \mu)$  we have

$$\lim_{N \rightarrow \infty} \frac{1}{|F_N|} \sum_{n \in F_N} T_n f_0 \prod_{j=1}^{\ell} S_{p_j(n)} f_j = \mathbb{E}[f_0 | \mathcal{I}_T] \prod_{j=1}^{\ell} \int_X f_j d\nu \quad (26)$$

with convergence taking place in  $L^2(X, \nu)$ .

This is a corollary of a more general result involving joint ergodicity.

# An example

Consider the m.p.s.  $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, T, S)$  with  $S(x, y) = (x + 2\alpha, y + x)$  for some  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , and  $T(x, y) = (x, y + x)$ . We see that  $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, S)$  and  $([0, 1]^2, \mathcal{B}, \mathcal{L}^2, T)$  are both zero entropy systems that are not weakly mixing, and the former is totally ergodic. Furthermore,  $T$  and  $S$  generate a 2-step nilpotent group. For  $f_0(x, y) = e^{2\pi i(x-y)}$ ,  $f_1(x, y) = e^{2\pi iy}$ , and  $f_2(x, y) = e^{-2\pi ix}$ , we see that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f_0(x, y) S^n f_1(x, y) S^{\frac{1}{2}(n^2-n)} f_2(x, y) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e^{2\pi i \left( (1-n)x - y + y + nx + (n^2-n)\alpha - x - (n^2-n)\alpha \right)} = 1 \neq 0. \end{aligned}$$

# References I

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