The Ergodic Hierarchy of Mixing, van der Corput's difference theorem, and the ergodic theory of noncommuting operators

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Overview

Table of Contents

The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is **uniformly distributed** if for any open interval $(a,b) \subseteq [0,1]$ we have

$$\lim_{N \to \infty} \frac{1}{N} |\{1 \leq n \leq N \mid x_n \in (a,b)\}| = b - a. \tag{1}$$

Theorem (van der Corput, [27])

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha)_{n=1}^{\infty}$ is uniformly distributed.

Theorem (HvdCDT1, Bergelson [3, Theorem 1.4])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \qquad (2)$$

for every $h \in \mathbb{N}$, then

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0. \qquad (3)$$

Hilbertian van der Corput Difference Theorems 2/3

Theorem (HvdCDT2, Bergelson [3, Page 3])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h \to \infty} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \tag{4}$$

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0. \tag{5}$$

Hilbertian van der Corput Difference Theorems 3/3

Theorem (HvdCDT3, Bergelson [3, Theorem 1.5])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \ then \tag{6}$$

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n \right\| = 0. \tag{7}$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \tag{8}$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy)

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \tag{9}$$

Furstenberg's proof in [17, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^{\infty}$. See also [4, Theorem 2.1] for a proof using HvdCDT1 directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg, [17])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, any $A \in \mathscr{B}$ with $\mu(A) > 0$, and any $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A\right) > 0. \qquad (10)$$

The proof presented in [9] uses HvdCT3 as Theorem 7.11, and the proof in [18] uses a variation.

Theorem (Bergelson and Leibman, [6])

For any m.p.s. $(X, \mathscr{B}, \mu, \{T_i\}_{i=1}^{\ell})$ with the T_is commuting, any $A \in \mathscr{B}$ with $\mu(A) > 0$, and any $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$, there exists $n \in \mathbb{N}$ for which

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_{\ell}^{-p_{\ell}(n)}A\right) > 0. \qquad (11)$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

Some of the Ergodic Hierarchy of Mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If for every $f, g \in L_0^2(X, \mu)$

1. $\lim_{N \to \infty} \dfrac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, g \rangle = 0$, then $\mathcal{X}$ is ergodic.

2. $\lim_{N \to \infty} \dfrac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, g \rangle| = 0$, then $\mathcal{X}$ is weakly mixing,

3. $\lim_{n \to \infty} \langle U_T^n f, g \rangle = 0$, then $\mathcal{X}$ is strongly mixing,

4. and if $L_0^2(X, \mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n, m \in \mathbb{Z}}$, then $\mathcal{X}$ has Lebesgue spectrum.

5. which is the same as $(\langle U_T^n f, g \rangle)_{n=1}^{\infty}$ being Fourier coefficients of some $h \in L^1([0, 1], \mathcal{L})$, where $\mathcal{L}$ is the Lebesgue measure.

These definitions also apply to individual elements $f \in L_0^2(X, \mu)$.

The Symmetric Ergodic Hierarchy of Mixing

> **Theorem**
>
> Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If for every $f \in L_0^2(X, \mu)$
>
> 1. $\displaystyle \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, f \rangle = 0$, then $\mathcal{X}$ is *ergodic*,
>
> 2. $\displaystyle \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, f \rangle| = 0$, then $\mathcal{X}$ is *weakly mixing*,
>
> 3. $\displaystyle \lim_{n \to \infty} \langle U_T^n f, f \rangle = 0$, then $\mathcal{X}$ is *strongly mixing*,
>
> 4. $\mathcal{X}$ has *Lebesgue spectrum* if $(\langle U_T^n f, f \rangle)_{n=1}^{\infty}$ are the Fourier coefficients of some $h \in L^1([0,1], \mathcal{L})$ taking nonnegative real values.
>
> This theorem also applies to individual elements $f \in L_0^2(X, \mu)$.

Dual notions to various levels of mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If $f \in L^2(X, \mu)$ satisfies

1. $U_T f = f$, then f is invariant.
2. $f \in L^2(X, K, \mu)$ where (X, K, μ, T) is the Kronecker factor of $(X, \mathscr{B}, \mu, T)$, then f is compact.
3. $f \in L^2(X, \mathscr{B}_P, \mu)$, where $\mathscr{B}_p$ is the Parreau factor from [22, Theorem 11], then f is 'anti-mixing' (provisional term).
4. If $(\langle U_T^n f, f \rangle)_{n=1}^{\infty}$ are the Fourier coefficients of a measure $\mu_{f,T}$ that is mutually singular with the Lebesgue measure then f has singular spectrum.
5. T has singular spectrum if all $f \in L^2(X, \mu)$ have singular spectrum, i.e., the maximal spectral type of T is singular.

Disjointness and orthogonality

Theorem

For $f, g \in L_0^2(X, \mu)$, we have $\langle f, g \rangle = 0$ if

1. *f is invariant and g is ergodic.*
2. *f is compact and g is weakly mixing.*
3. *f is 'anti-mixing' and g is strongly mixing.*

*f has singular spectrum and g has Lebesgue spectrum.*

Table of Contents

A weak mixing van der Corput difference theorem

Theorem (F. 2022)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \qquad (12)$$

then $(x_n)_{n=1}^{\infty}$ is a *nearly weakly mixing sequence*. This means that for any other bounded sequence $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ we morally (*but not literally*) have that

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} \left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, y_n \rangle \right| = 0. \qquad (13)$$

Loosely speaking, this can be interpretted as a weak mixing in any ultrapower $\mathscr{H}$ of $\mathcal{H}$ with respect to a unitary operator induced by the left shift. Note that elements of $\mathscr{H}$ are sequences in $\mathcal{H}$.

A strong mixing van der Corput difference theorem

Theorem (F. 2022)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h \to \infty} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{14}$$

then $(x_n)_{n=1}^{\infty}$ is a *nearly strongly mixing sequence*. This means that for any other bounded sequence $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ we morally (but not literally) have that

$$\lim_{h \to \infty} \left| \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, y_n \rangle \right| = 0. \tag{15}$$

Loosely speaking, this can be interpreted as a strong mixing in any ultrapower $\mathscr{H}$ of $\mathcal{H}$ with respect to a unitary operator induced by the left shift. Note that elements of $\mathscr{H}$ are sequences in $\mathcal{H}$.

A Lebesgue spectrum vdCdt

Theorem (F. 2023)

If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying for all $h \in \mathbb{N}$

$$\sum_{h=1}^\infty \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \qquad (16)$$

then $(x_n)_{n=1}^\infty$ is a spectrally Lebesgue sequence. If $\mathcal{H} = L^2(X, \mu)$ and $(y_n)_{n=1}^\infty \subseteq L^\infty(X, \mu)$ is bounded and spectrally singular, then

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^N x_n y_n \right\| = 0. \qquad (\#)$$

Upgrading the weak convergence from # to the strong convergence in # necessitates a new proof of the classical vdCDT. See [10, Chapter 2] for variations of MvdCT related to other levels of mixing, as well as uniform distribution. See also [26].

Table of Contents

Noncommutative ergodic theorems 1/2

Theorem (Frantzikinakis [12, Corollary 1.7])

Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t \to \infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t \to \infty} \frac{t^{d+1}}{a(t)} = \infty. \quad \left(e.g. \ a(t) = t^{1.5}\right) \quad (17)$$

Furthermore, let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations. Suppose that the system $(X, \mathscr{B}, \mu, T)$ has zero entropy. Then

(i) *For every $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f | \mathcal{I}_T] \cdot \mathbb{E}[g | \mathcal{I}_S], \quad (18)$$

where the limit is taken in $L^2(X, \mu)$.

Theorem (Continued)

(ii) *For every $A \in \mathscr{B}$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu \left(A \cap T^{-n}A \cap S^{-\lfloor a(n) \rfloor}A \right) \geq \mu(A)^3. \qquad (19)$$

Frantzikinakis and Host [13] proved a similar theorem for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2. The zero entropy assumption on T cannot be removed as seen by [18, Page 40] or [2, Example 7.1]. Rohlin [24] showed that every T with singular spectrum must also have zero entropy. Note that the Horocycle flow has zero entropy [19] and Lebesgue spectrum [23]

There is no Roth Theorem for solvable groups

Theorem (Bergelson-Leibman [7, Theorem 1.2])

Let G be a finitely generated solvable group of exponential growth. For any partition $R \bigcup P = \mathbb{Z} \setminus \{0\}$, there exist an action $\{T_g\}_{g \in G}$ of G on a probability space $(X, \mathscr{B}, \mu)$, $g_1, g_2 \in G$, and set $A \in \mathscr{B}$ with $\mu(A) > 0$ such that

$$\mu \left(T_{g_1^n} A \cap T_{g_2^n} A \right) = 0 \ \text{ if } n \in R \text{ and}$$
$$\mu \left(T_{g_1^n} A \cap T_{g_2^n} A \right) \geq \frac{1}{6} \ \text{ if } n \in P.$$

Note that the group used in [2, Example 7.1] is non-solvable.

Another Example

Theorem (Frantzikinakis, Lesigne, Wierdl [15, Lemma 4.1])

Let $a, b : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ be injective sequences and F be any subset of $\mathbb{N}$. Then there exist a probability space $(X, \mathscr{B}, \mu)$, measure preserving automorphisms $T, S : X \to X$, both of them Bernoulli, and $A \in \mathscr{B}$, such that

$$\mu \left(T^{-a(n)} A \cap S^{-b(n)} A \right) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \tag{20}$$

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving automorphisms for which T has singular spectrum. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed in the orbit closure of α for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

① For any $f, g \in L^{\infty}(X, \mu)$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}\left[f | \mathcal{I}_T\right] \mathbb{E}[g | \mathcal{I}_S], \qquad (21)$$

with convergence taking place in $L^2(X, \mu)$.

Theorem (Continued)

(ii) If $A \in \mathscr{B}$ then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu \left(A \cap T^{-n} A \cap S^{-k_n} A \right) \geq \mu(A)^3. \quad (22)$$

(iii) If we only assume that $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$, then (i) and (ii) hold when S is totally ergodic.

Examples include $k_n = \lfloor a(n) \rfloor$ with $a(n)$ being as in frame 19, $k_n = \lfloor n^2 \log^2(n) \rfloor$, and for part (iii) we may take $k_n = p(n)$ for $p(x) \in x\mathbb{Z}[x]$ with degree at least 2.

Sets of K but not $K+1$ recurrence?

Theorem (Frantzikinakis, Lesigne, Wierdl [14])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{ n \in \mathbb{N} \mid n^k \alpha \in \left[\frac{1}{4}, \frac{3}{4} \right] \right\}$.

(i) *If $(X, \mathscr{B}, \mu)$ is a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which*

$$\mu \left(A \cap S_1^{-n} A \cap S_2^{-n} A \cap \cdots \cap S_{k-1}^{-n} A \right) > 0. \qquad (23)$$

(ii) *There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have*

$$\mu \left(A \cap T^{-n} A \cap T^{-2n} A \cap \cdots \cap T^{-kn} A \right) = 0. \qquad (24)$$

Theorem (F., 2023)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{ n \in \mathbb{N} \mid n^k \alpha \in \left[\frac{1}{4}, \frac{3}{4} \right] \right\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ commuting measure preserving automorphisms. Let $T : X \to X$ be an measure preserving automorphism with singular spectrum, and for which $\{T, S_1, S_2, \cdots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu \left(A \cap T^{-n} A \cap S_1^{-n} A \cap S_2^{-n} A \cap \cdots \cap S_{k-1}^{-n} A \right) > 0. \tag{25}$$

Since the system $(\mathbb{T}^2, \mathscr{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \sum_{n \in \mathbb{Z}} \delta_{n\alpha}$.

Application 3/4 (A special case)

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum and S is totally ergodic. Let $p_1, \cdots, p_K \in \mathbb{Q}[x]$ be integer polynomials for which $\deg(p_1) \geq 2$ and $\deg(p_i) \geq 2 + \deg(p_{i-1})$. For any $f, g_1, \cdots, g_K \in L^\infty(X, \mu)$, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \prod_{i=1}^{K} S^{p_i(n)} g_i = \mathbb{E}[f|\mathcal{I}_T] \prod_{i=1}^{K} \int_X g_i d\mu, \qquad (26)$$

with convergence taking place in $L^2(X, \mu)$.

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, R, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum, R and S commute, and S is weakly mixing. Let $\ell \in \mathbb{N}$ and let $p_1, \cdots, p_\ell \in \mathbb{Q}[x]$ be pairwise essentially distinct integer polynomials, each having degree at least 2. For any $f, h, g_1, \cdots, g_\ell \in L^\infty(X, \mu)$ satisfying $\int_X g_j d\mu = 0$ for some $1 \le j \le \ell$, we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot R^n h \cdot \prod_{j=1}^{\ell} S^{p_j(n)} g_j = 0, \qquad (27)$$

with convergence taking place in $L^2(X, \mu)$.

An example to justify our assumptions

Consider the m.p.s. $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T, S)$ with $S(x, y) = (x + 2\alpha, y + x)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $T(x, y) = (x, y + x)$. We see that $([0,1]^2, \mathscr{B}, \mathcal{L}^2, S)$ and $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T)$ are both zero entropy systems that are not weakly mixing, and the former is totally ergodic. Furthermore, T and S generate a 2-step nilpotent group. For $f_0(x, y) = e^{2\pi i(x-y)}, f_1(x, y) = e^{2\pi iy}$, and $f_2(x, y) = e^{-2\pi ix}$, we see that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_0(x, y) S^n f_1(x, y) S^{\frac{1}{2}(n^2 - n)} f_2(x, y)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i\left((1-n)x - y + y + nx + (n^2 - n)\alpha - x - (n^2 - n)\alpha\right)} = 1 \neq 0.$$

Table of Contents

Examples of systems with singular spectrum

In [5, Proposition 2.9] it is shown that if $(X, \mathscr{B}, \mu)$ is a standard probability space, and $\text{Aut}(X, \mathscr{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism has singular spectrum, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \text{Aut}(X, \mathscr{B}, \mu)$ denote the collection of strongly mixing transformation, and note that $\mathcal{S}$ is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. Since $\mathcal{S}$ is not a complete metric space with respect to the topology induced by the strong operator topology, a new topology was introduced in [25], with respect to which $\mathcal{S}$ is a complete metric space. It is shown in the Corollary to Theorem 7 of [25] that a generic $T \in \mathcal{S}$ has singular spectrum, and such a T is mixing of all orders due a well known result of Host [20]. See [11] and [21] for concrete examples of $T \in \mathcal{S}$ that have singular spectrum. See also [1], [8], and [16].

[1] C. Aistleitner and M. Hofer.
On the maximal spectral type of a class of rank one
transformations.
Dyn. Syst., 27(4):515–523, 2012.

[2] D. Berend.
Joint ergodicity and mixing.
J. Analyse Math., 45:255–284, 1985.

[3] V. Bergelson.
Weakly mixing PET.
Ergodic Theory Dynam. Systems, 7(3):337–349, 1987.

References II

[4] V. Bergelson.
Ergodic Ramsey theory—an update.
In *Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994)*,
volume 228 of *London Math. Soc. Lecture Note Ser.*, pages
1–61. Cambridge Univ. Press, Cambridge, 1996.

[5] V. Bergelson, A. del Junco, M. Lemańczyk, and
J. Rosenblatt.
Rigidity and non-recurrence along sequences.
Ergodic Theory Dynam. Systems, 34(5):1464–1502, 2014.

[6] V. Bergelson and A. Leibman.
Polynomial extensions of van der Waerden's and Szemerédi's
theorems.
J. Amer. Math. Soc., 9(3):725–753, 1996.

[7] V. Bergelson and A. Leibman.
Failure of the Roth theorem for solvable groups of exponential growth.
Ergod. Th. and Dynam. Sys., 24:45–53, 2004.

[8] A. I. Bufetov and B. Solomyak.
On substitution automorphisms with pure singular spectrum.
Math. Z., 301(2):1315–1331, 2022.

[9] M. Einsiedler and T. Ward.
Ergodic theory with a view towards number theory, volume 259 of *Graduate Texts in Mathematics*.
Springer-Verlag London, Ltd., London, 2011.

References IV

[10] S. Farhangi.
Topics in ergodic theory and ramsey theory.
PhD dissertation, the Ohio State University, 2022.

[11] B. Fayad.
Smooth mixing flows with purely singular spectra.
Duke Math. J., 132(2):371–391, 2006.

[12] N. Frantzikinakis.
Furstenberg systems of Hardy field sequences and applications.
J. Anal. Math., 147(1):333–372, 2022.

[13] N. Frantzikinakis and B. Host.
Multiple recurrence and convergence without commutativity.
arXiv:2111.01518v2 **[math.DS]**, 2021.

References V

[14] N. Frantzikinakis, E. Lesigne, and M. Wierdl.
Sets of k-recurrence but not $(k + 1)$-recurrence.
Ann. Inst. Fourier (Grenoble), 56(4):839–849, 2006.

[15] N. Frantzikinakis, E. Lesigne, and M. Wierdl.
Random sequences and pointwise convergence of multiple
ergodic averages.
Indiana Univ. Math. J., 61(2):585–617, 2012.

[16] K. Frączek and M. Lemańczyk.
On mild mixing of special flows over irrational rotations under
piecewise smooth functions.
Ergodic Theory Dynam. Systems, 26(3):719–738, 2006.

[17] H. Furstenberg.
Ergodic behavior of diagonal measures and a theorem of
Szemerédi on arithmetic progressions.
J. Analyse Math., 31:204–256, 1977.

[18] H. Furstenberg.
*Recurrence in ergodic theory and combinatorial number
theory.*
Princeton University Press, Princeton, N.J., 1981.
M. B. Porter Lectures.

[19] B. M. Gurevič.
The entropy of horocycle flows.
Dokl. Akad. Nauk SSSR, 136:768–770, 1961.

[20] B. Host.
Mixing of all orders and pairwise independent joinings of
systems with singular spectrum.
Israel J. Math., 76(3):289–298, 1991.

[21] I. Klemes and K. Reinhold.
Rank one transformations with singular spectral type.
Israel J. Math., 98:1–14, 1997.

[22] M. Lemańczyk.
Spectral theory of dynamical systems.
In *Mathematics of complexity and dynamical systems. Vols.
1–3*, pages 1618–1638. Springer, New York, 2012.

[23] O. S. Parasyuk.
Flows of horocycles on surfaces of constant negative
curvature.
Uspehi Matem. Nauk (N.S.), 8(3(55)):125–126, 1953.

[24] V. A. Rohlin.
Lectures on the entropy theory of transformations with
invariant measure.
Uspehi Mat. Nauk, 22(5 (137)):3–56, 1967.

[25] S. V. Tikhonov.
A complete metric on the set of mixing transformations.
Mat. Sb., 198(4):135–158, 2007.

[26] A. Tserunyan.
A Ramsey theorem on semigroups and a general van der
Corput lemma.
J. Symb. Log., 81(2):718–741, 2016.

[27] J. G. van der Corput.
Diophantische Ungleichungen. I. Zur Gleichverteilung Modulo
Eins.
Acta Math., 56(1):373–456, 1931.