Van der Corput's difference theorem, the Furstenberg Correspondence Principle, and The Ergodic Hierarchy of Mixing properties.

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Overview

1. Van der Corput's difference theorem and some applications
2. The Furstenberg correspondence principle between combinatorics and dynamics
3. Mixing properties
4. Mixing van der Corput difference theorems
5. Applications (Bonus)
   - Background on noncommutative ergodic theory
   - New results from mixing vdCs
   - Examples of systems with singular spectrum

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The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is **uniformly distributed** if for any open interval $(a,b) \subseteq [0,1]$ we have

$$\lim_{N\to\infty} \frac{1}{N} |\{1 \leq n \leq N \mid x_n \in (a,b)\}| = b - a. \tag{1}$$

Theorem (van der Corput, 1931 [34])

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha)_{n=1}^{\infty}$ is uniformly distributed.

Hilbertian van der Corput Difference Theorems 1/3

Theorem (HvdCDT1, Bergelson, 1987 [3, Theorem 1.4])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0, \tag{2}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} \left|\left|\frac{1}{N}\sum_{n=1}^{N} x_n\right|\right| = 0. \tag{3}$$

Theorem (HvdCDT2, Bergelson, 1987 [3, Page 3])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \tag{4}$$

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0. \tag{5}$$

Theorem (HvdCDT3, Bergelson, 1987 [3, Theorem 1.5])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \tag{6}$$

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0. \tag{7}$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \tag{8}$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy [19],[29])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \tag{9}$$

Furstenberg's proof in [19, Proposition 1.3] uses a form of vdCDT since it uses the uniform distribution of $(n^2\alpha)_{n=1}^{\infty}$. See also [4, Theorem 2.1] for a proof using HvdCDT1 directly.

Applications of HvdCDTs 2/2

Theorem (Furstenberg multiple recurrence, [19])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, *any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\ell \in \mathbb{N}$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A\right) > 0. \tag{10}$$

The proof presented in [10] uses HvdCT3 as Theorem 7.11, and the proof in [20] uses a variation.

Theorem (Bergelson and Leibman, [6, Theorem A_0])

For any m.p.s. $(X, \mathscr{B}, \mu, \{T_i\}_{i=1}^{\ell})$ *with the* T_i*s commuting, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu\left(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_\ell^{-p_\ell(n)}A\right) > 0. \tag{11}$$

Uses an equivalent form of HvdCT3 as Lemma 2.4.

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Furstenberg's correspondence principle

Definition

For a set $E \subseteq \mathbb{N}$, the **natural upper density** of E is denoted by $\overline{d}(E)$ and is given by

$$\overline{d}(E) = \limsup_{N\to\infty} \frac{1}{N} \left| E \cap [1, N] \right| . \tag{12}$$

We see that $\overline{d}(2\mathbb{N}) = \overline{d}(2\mathbb{N}+1) = \frac{1}{2}$, $\overline{d}(a\mathbb{N}+b) = \frac{1}{a}$ for any $a, b \in \mathbb{N}$, and $\overline{d}(\{n^2\}_{n\in\mathbb{N}}) = 0$.

Theorem (The correspondence principle, [19],[4, Theorem 1.8])

Given a set $E \subseteq \mathbb{N}$ for which $\overline{d}(E) > 0$, there exists a measure preserving system $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ with $\mu(A) = \overline{d}(E)$, such that for any $\ell, n_1, n_2, \cdots, n_\ell \in \mathbb{N}$ we have

$$\overline{d}\left(E \cap (E - n_1) \cap \cdots \cap (E - n_\ell)\right) \geq \mu\left(A \cap T^{-n_1} A \cap \cdots \cap T^{-n_\ell} A\right)$$

Szemerédi's theorem

Theorem (Szemerédi [31], 1975)

If $E \subseteq \mathbb{N}$ satisfies $\overline{d}(E) > 0$, then for any $\ell \in \mathbb{N}$, E contains an arithmetic progression of length ℓ.

This was conjectured by Erdős and Turán [11] in 1936. The case of length 3 arithmetic progressions was resolved by Roth [28] in 1952. The case of length 4 arithmetic progressions was resolved by Szemerédi [30] in 1969. Furstenberg [19] gave the second proof in 1977, and Gowers [21] the third proof in 2001.

Using the correspondence principle

We will now deduce Szemerédi's Theorem from the Furstenberg Multiple Recurrence Theorem by using the correspondence principle. Let $E \subseteq \mathbb{N}$ be such that $\overline{d}(E) > 0$, let $(X, \mathscr{B}, \mu, T)$ and $A \in \mathscr{B}$ be given by the correspondence principle, and let $\ell \in \mathbb{N}$ be arbitrary. Furstenberg's Multiple Recurrence Theorem tells us that there exists $n \in \mathbb{N}$ for which

$$\gamma(n) := \mu\left(A \cap T^{-n}A \cap \cdots \cap T^{-n\ell}A\right) > 0.$$

The correspondence principle tells us that for

$E(n) := E \cap (E-n) \cap \cdots \cap (E-n\ell)$ we have $\overline{d}(E(n)) \geq \gamma(n) > 0$.

Since $E(n) \neq \emptyset$, we see that for $a \in E(n)$ we have $a, a+n, \cdots, a+n\ell \in E$.

More results in density Ramsey theory

Theorem (Furstenberg-Sárközy [19],[29])

Let $E \subseteq \mathbb{N}$ be such that $\overline{d}(E) > 0$. Then there exists $n \in \mathbb{N}$ for which $\overline{d}(E \cap (E - n^2)) > 0$. In particular, there exists $x, y \in E$ with $x - y = n^2$.

Theorem (Polynomial Szemerédi, due to Bergelson and Leibman)

Let $E \subseteq \mathbb{N}$ be such that $\overline{d}(E) > 0$ and let $p_1, \cdots, p_\ell \in x\mathbb{Z}[x]$ be arbitrary. Then there exists $n \in \mathbb{N}$ for which

$$\overline{d}\left(E \cap (E - p_1(n)) \cap \cdots \cap (E - p_\ell(n))\right) > 0. \tag{13}$$

In particular, there exists $a, n \in \mathbb{N}$ for which $a, a + p_1(n), \cdots, a + p_\ell(n) \in E$.

Note that this is only a special case of Theorems B and B' of [6].

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Some of the Ergodic Hierarchy of Mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If for every $f, g \in L^2_0(X, \mu)$

1. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, g \rangle = 0$, then $\mathcal{X}$ is ergodic.
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, g \rangle| = 0$, then $\mathcal{X}$ is weakly mixing,
3. $\lim_{n\to\infty} \langle U_T^n f, g \rangle = 0$, then $\mathcal{X}$ is strongly mixing,
4. and if $L^2_0(X, \mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m\in\mathbb{Z}}$, then $\mathcal{X}$ has Lebesgue spectrum.
5. which is the same as $(\langle U_T^n f, g \rangle)_{n=1}^{\infty}$ being Fourier coefficients of some $h \in L^1([0,1], \mathcal{L})$, where $\mathcal{L}$ is the Lebesgue measure.

These definitions also apply to individual elements $f \in L^2_0(X, \mu)$.

The Symmetric Ergodic Hierarchy of Mixing

Theorem

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ *be a m.p.s. If for every* $f \in L^2_0(X, \mu)$

1. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, f\rangle = 0$, *then* $\mathcal{X}$ *is ergodic,*
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, f\rangle| = 0$, *then* $\mathcal{X}$ *is weakly mixing,*
3. $\lim_{n\to\infty} \langle U_T^n f, f\rangle = 0$, *then* $\mathcal{X}$ *is strongly mixing,*
4. $\mathcal{X}$ *has Lebesgue spectrum if* $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ *are the Fourier coefficients of some* $h \in L^1([0,1], \mathcal{L})$ *taking nonnegative real values.*

This theorem also applies to individual elements $f \in L^2_0(X, \mu)$*.*

Dual notions to various levels of mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If $f \in L^2(X, \mu)$ satisfies

1. $U_T f = f$, then f is invariant.
2. $f \in L^2(X, K, \mu)$ where (X, K, μ, T) is the Kronecker factor of $(X, \mathscr{B}, \mu, T)$, then f is compact.
3. $f \in L^2(X, \mathscr{B}_P, \mu)$, where $\mathscr{B}_p$ is the Parreau factor from [25, Theorem 11], then f is 'anti-mixing' (provisional term).
4. If $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ are the Fourier coefficients of a measure $\mu_{f,T}$ that is mutually singular with the Lebesgue measure then f has singular spectrum.
5. T has singular spectrum if all $f \in L^2(X, \mu)$ have singular spectrum, i.e., the maximal spectral type of T is singular.

Disjointness and orthogonality

Theorem

For $f, g \in L_0^2(X, \mu)$, we have $\langle f, g \rangle = 0$ if

1. *f is invariant and g is ergodic.*
2. *f is compact and g is weakly mixing.*
3. *f is 'anti-mixing' and g is strongly mixing.*

\# *f has singular spectrum and g has Lebesgue spectrum.*

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A weak mixing van der Corput difference theorem

Theorem (F. 2022)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ *is a bounded sequence satisfying*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0, \qquad (14)$$

then $(x_n)_{n=1}^{\infty}$ *is a nearly weakly mixing sequence. This means that for any other bounded sequence* $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ *we morally (but not literally) have that*

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\left|\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, y_n\rangle\right| = 0. \qquad (15)$$

Loosely speaking, this can be interpretted as a weak mixing in any ultrapower $\mathscr{H}$ *of* $\mathcal{H}$ *with respect to a unitary operator induced by the left shift. Note that elements of* $\mathscr{H}$ *are sequences in* $\mathcal{H}$*.*

A strong mixing van der Corput difference theorem

Theorem (F. 2022)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{16}$$

then $(x_n)_{n=1}^{\infty}$ is a nearly strongly mixing sequence. This means that for any other bounded sequence $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ we morally (but not literally) have that

$$\lim_{h\to\infty} \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, y_n \rangle \right| = 0. \tag{17}$$

Loosely speaking, this can be interpretted as a strong mixing in any ultrapower $\mathscr{H}$ of $\mathcal{H}$ with respect to a unitary operator induced by the left shift. Note that elements of $\mathscr{H}$ are sequences in $\mathcal{H}$.

A Lebesgue spectrum vdCdt

Theorem (F. 2023)

If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying for all $h \in \mathbb{N}$

$$\sum_{h=1}^{\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right|^2 < \infty, \tag{18}$$

then $(x_n)_{n=1}^\infty$ is a spectrally Lebesgue sequence. If $\mathcal{H} = L^2(X, \mu)$ and $(y_n)_{n=1}^\infty \subseteq L^\infty(X, \mu)$ is bounded and spectrally singular, then

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} x_n y_n \right\| = 0. \tag{#}$$

Upgrading the weak convergence from # to the strong convergence in # necessitates a new proof of the classical vdCDT. See [12, Chapter 2] for variations of MvdCT related to other levels of mixing, as well as uniform distribution. See also [33] and [9].

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Noncommutative ergodic theorems 1/2

Theorem (Frantzikinakis [14, Corollary 1.7])

Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t\to\infty} \frac{a(t)}{t^{d+\epsilon}} = \lim_{t\to\infty} \frac{t^{d+1}}{a(t)} = \infty. \quad \left(e.g.\ a(t) = t^{1.5}\right) \tag{19}$$

Furthermore, let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations. Suppose that the system $(X, \mathscr{B}, \mu, T)$ has zero entropy. Then

(i) *For every $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f|\mathcal{I}_T] \cdot \mathbb{E}[g|\mathcal{I}_S], \tag{20}$$

where the limit is taken in $L^2(X, \mu)$.

Noncommutative ergodic theorems 2/2

Theorem (Continued)

(ii) *For every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap S^{-\lfloor a(n)\rfloor}A\right) \geq \mu(A)^3. \tag{21}$$

Frantzikinakis and Host [15] proved a similar theorem for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2. The zero entropy assumption on T cannot be removed as seen by [20, Page 40] or [2, Example 7.1]. Rohlin [27] showed that every T with singular spectrum must also have zero entropy. Note that the Horocycle flow has zero entropy [22] and Lebesgue spectrum [26]

There is no Roth Theorem for solvable groups

Theorem (Bergelson-Leibman [7, Theorem 1.2])

Let G be a finitely generated solvable group of exponential growth. For any partition $R \bigcup P = \mathbb{Z} \setminus \{0\}$, there exist an action $\{T_g\}_{g \in G}$ of G on a probability space $(X, \mathscr{B}, \mu)$, $g_1, g_2 \in G$, and set $A \in \mathscr{B}$ with $\mu(A) > 0$ such that

$$\mu\left(T_{g_1^n} A \cap T_{g_2^n} A\right) = 0 \ \ \text{if } n \in R \text{ and}$$
$$\mu\left(T_{g_1^n} A \cap T_{g_2^n} A\right) \geq \frac{1}{6} \ \text{if } n \in P.$$

Note that the group used in [2, Example 7.1] is non-solvable.

Another Example

Theorem (Frantzikinakis, Lesigne, Wierdl [17, Lemma 4.1])

Let $a, b : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ be injective sequences and F be any subset of $\mathbb{N}$. Then there exist a probability space $(X, \mathscr{B}, \mu)$, measure preserving automorphisms $T, S : X \to X$, both of them Bernoulli, and $A \in \mathscr{B}$, such that

$$\mu\left(T^{-a(n)}A \cap S^{-b(n)}A\right) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \tag{22}$$

Application 1/4

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving automorphisms for which T has singular spectrum. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed in the orbit closure of α for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

(i) *For any $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}[f|\mathcal{I}_T]\,\mathbb{E}[g|\mathcal{I}_S], \qquad (23)$$

with convergence taking place in $L^2(X, \mu)$.

Application 1/4 continued

Theorem (Continued)

(ii) *If $A \in \mathscr{B}$ then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap S^{-k_n}A\right) \geq \mu(A)^3. \qquad (24)$$

(iii) *If we only assume that $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$, then (i) and (ii) hold when S is totally ergodic.*

Examples include $k_n = \lfloor a(n) \rfloor$ with $a(n)$ being as in frame 19, $k_n = \lfloor n^2 \log^2(n) \rfloor$, and for part (iii) we may take $k_n = p(n)$ for $p(x) \in x\mathbb{Z}[x]$ with degree at least 2.

Sets of K but not $K+1$ recurrence?

Theorem (Frantzikinakis, Lesigne, Wierdl [16])

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$.

(i) *If $(X, \mathscr{B}, \mu)$ is a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which*

$$\mu\left(A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{25}$$

(ii) *There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have*

$$\mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A\right) = 0. \tag{26}$$

Application 2/4

Theorem (F., 2023)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \left\{n \in \mathbb{N} \mid n^k\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S_1, S_2, \cdots, S_{k-1} : X \to X$ commuting measure preserving automorphisms. Let $T : X \to X$ be an measure preserving automorphism with singular spectrum, and for which $\{T, S_1, S_2, \cdots, S_{k-1}\}$ generate a nilpotent group. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu\left(A \cap T^{-n}A \cap S_1^{-n}A \cap S_2^{-n}A \cap \cdots \cap S_{k-1}^{-n}A\right) > 0. \tag{27}$$

Since the system $(\mathbb{T}^2, \mathscr{B}^2, \mathcal{L}^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ can be used in item (ii) of the last slide when $k = 2$, the current theorem does not hold for a general T with 0 entropy. Also note that the maximal spectral type of T is $\mathcal{L} + \sum_{n \in \mathbb{Z}} \delta_{n\alpha}$.

Application 3/4 (A special case)

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum and S is totally ergodic. Let $p_1, \cdots, p_K \in \mathbb{Q}[x]$ be integer polynomials for which $deg(p_1) \geq 2$ and $deg(p_i) \geq 2 + deg(p_{i-1})$. For any $f, g_1, \cdots, g_K \in L^\infty(X, \mu)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \prod_{i=1}^{K} S^{p_i(n)} g_i = \mathbb{E}[f | \mathcal{I}_T] \prod_{i=1}^{K} \int_X g_i d\mu, \tag{28}$$

with convergence taking place in $L^2(X, \mu)$.

Application 4/4 (A special case)

Theorem (F., 2023)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, R, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum, R and S commute, and S is weakly mixing. Let $\ell \in \mathbb{N}$ and let $p_1, \cdots, p_\ell \in \mathbb{Q}[x]$ be pairwise essentially distinct integer polynomials, each having degree at least 2. For any $f, h, g_1, \cdots, g_\ell \in L^\infty(X, \mu)$ satisfying $\int_X g_j d\mu = 0$ for some $1 \leq j \leq \ell$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot R^n h \cdot \prod_{j=1}^{\ell} S^{p_j(n)} g_j = 0, \tag{29}$$

with convergence taking place in $L^2(X, \mu)$.

An example to justify our assumptions

Consider the m.p.s. $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T, S)$ with $S(x,y) = (x+2\alpha, y+x)$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $T(x,y) = (x, y+x)$. We see that $([0,1]^2, \mathscr{B}, \mathcal{L}^2, S)$ and $([0,1]^2, \mathscr{B}, \mathcal{L}^2, T)$ are both zero entropy systems that are not weakly mixing, and the former is totally ergodic. Furthermore, T and S generate a 2-step nilpotent group. For $f_0(x,y) = e^{2\pi i(x-y)}$, $f_1(x,y) = e^{2\pi iy}$, and $f_2(x,y) = e^{-2\pi ix}$, we see that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^n f_0(x,y) S^n f_1(x,y) S^{\frac{1}{2}(n^2-n)} f_2(x,y)$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\left((1-n)x-y+y+nx+(n^2-n)\alpha-x-(n^2-n)\alpha\right)} = 1 \neq 0.$$

Examples of systems with singular spectrum

In [5, Proposition 2.9] it is shown that if $(X, \mathscr{B}, \mu)$ is a standard probability space, and $\mathrm{Aut}(X, \mathscr{B}, \mu)$ is endowed with the strong operator topology, then the set of transformations that are weakly mixing and rigid is a generic set. Since any rigid automorphism has singular spectrum, we see that the set of singular automorphisms is generic. Now let $\mathcal{S} \subseteq \mathrm{Aut}(X, \mathscr{B}, \mu)$ denote the collection of strongly mixing transformation, and note that $\mathcal{S}$ is a meager set since an automorphism cannot simultaneously be rigid and strongly mixing. Since $\mathcal{S}$ is not a complete metric space with respect to the topology induced by the strong operator topology, a new topology was introduced in [32], with respect to which $\mathcal{S}$ is a complete metric space. It is shown in the Corollary to Theorem 7 of [32] that a generic $T \in \mathcal{S}$ has singular spectrum, and such a T is mixing of all orders due a well known result of Host [23]. See [13] and [24] for concrete examples of $T \in \mathcal{S}$ that have singular spectrum. See also [1], [8], and [18].

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