A generalization of van der Corput's difference theorem and applications to recurrence

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The van der Corput difference trick in Hilbert spaces

Theorem 1: Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a bounded sequence.

(i) If for every $h \in \mathbb{N}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0, \text{ then } \lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N}x_n\right|\right| = 0.$$

(ii) If $\lim_{h\to\infty}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0$, then $\lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N}x_n\right|\right| = 0$.

(iii) If $\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0$, then $\lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N}x_n\right|\right| = 0$.

The ergodic hierarchy of mixing properties

Theorem 2: Let $\mathcal{X} := (X, \mathscr{B}, \mu, T)$ be an invertible measure preserving system.

1. $\mathcal{X}$ is ergodic if for any $f \in L_0^2(X,\mu)$ we have $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle U_T^n f, f\rangle = 0$.
2. $\mathcal{X}$ is weak mixing if for any $f \in L_0^2(X,\mu)$ we have $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left|\langle U_T^n f, f\rangle\right| = 0$.
3. $\mathcal{X}$ is strong mixing if for any $f \in L_0^2(X,\mu)$ we have $\lim_{n\to\infty}\langle U_T^n f, f\rangle = 0$.
4. $\mathcal{X}$ has Lebesgue spectrum if $L_0^2(X,\mu)$ has a basis of the form $\{U_T^n f_m\}_{m,n\in\mathbb{Z}}$. This is equivalent to $(\langle U_T^n f, f\rangle)_{n=1}^{\infty}$ being the Fourier coefficients of a measure μ <<Lebesgue for every $f \in L_0^2(X,\mu)$.

Connecting vdCt to the ergodic hierarchy of mixing

Theorem 3: Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a bounded sequence. Suppose that $(c_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ are bounded sequences that are spectrally singular, anti-mixing (cf. Parreau factor), and compact respectively.

(i) If for every $h \in \mathbb{N}$ we have

$$\underbrace{\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0}_{\text{Lebesgue spectrum}}, \text{ then } \lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N}c_n x_n\right|\right| = 0.$$

(ii) If $\underbrace{\lim_{h\to\infty}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0}_{\text{strong mixing}}$, then $\lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N}c_n x_n\right|\right| = 0$.

(iii) If $\underbrace{\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0}_{\text{weak mixing}}$, then $\lim_{N\to\infty}\left|\left|\frac{1}{N}\sum_{n=1}^{N}c_n x_n\right|\right| = 0$.

Failure of ergodic theorems without commutativity

Theorem 4(Frantzikinakis, Lesigne, Wierdl): Let $a, b : \mathbb{N} \to \mathbb{Z}\setminus\{0\}$ be injective sequences and F be any subset of $\mathbb{N}$. Then there exist a probability space $(X, \mathscr{B}, \mu)$, measure preserving automorphisms $T, S : X \to X$, both of them Bernoulli, and $A \in \mathscr{B}$, such that

$$\mu\left(T^{-a(n)}A \cap S^{-b(n)}A\right) = \begin{cases} 0 & \text{if } n \in F, \\ \frac{1}{4} & \text{if } n \notin F. \end{cases} \quad (1)$$

Recurrence with noncommuting transformations

Theorem 5(Frantzikinakis 2022): Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{t\to\infty}\frac{a(t)}{t^{d+\epsilon}} = \lim_{t\to\infty}\frac{t^{d+1}}{a(t)} = \infty. \ (E.g.\ a(t) = t^{\beta}, \beta \in \mathbb{R}_{\geq 1}\setminus\mathbb{N})$$

Furthermore, let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations (not necessarily commuting). Suppose that the system $(X, \mathscr{B}, \mu, T)$ has zero entropy. Then

(i) For every $f, g \in L^{\infty}(X,\mu)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f \cdot S^{\lfloor a(n)\rfloor}g = \mathbb{E}[f|\mathcal{I}_T]\cdot\mathbb{E}[g|\mathcal{I}_S],$$

where the limit is taken in $L^2(X,\mu)$.

(ii) For every $A \in \mathscr{B}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A\cap S^{-\lfloor a(n)\rfloor}A) \geq \mu(A)^3.$$

Whether a similar result holds for $a(n) = p(n) \in \mathbb{Z}[x]$ was asked by Frantzikinakis and later answered in the positive by Frantzikinakis and Host.

Theorem 6: Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving automorphisms. Suppose that the m.p.s. $(X, \mathscr{B}, \mu, T)$ has singular spectrum. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha \pmod 1)_{n=1}^{\infty}$ is uniformly distributed in $\overline{\{n\alpha \mid n \in \mathbb{N}\}}$ for all $\alpha \in \mathbb{R}$ and $h \in \mathbb{N}$.

(i) For any $f, g \in L^{\infty}(X,\mu)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\cdot S^{k_n}g = \mathbb{E}[f|\mathcal{I}_T]\cdot\mathbb{E}[g|\mathcal{I}_S],$$

with convergence taking place in $L^2(X,\mu)$.

(ii) If $A \in \mathscr{B}$ then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A\cap S^{-k_n}A) \geq \mu(A)^3.$$

(iii) If $((k_{n+h} - k_n)\alpha)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ then (i)-(ii) hold when $(X, \mathscr{B}, \mu, S)$ is totally ergodic.

Sets of K but not $K+1$ Recurrence

Theorem 7(Frantzikinakis, Lesigne, Wierdl): Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$.

(i) If $(X, \mathscr{B}, \mu)$ is a probability space and $T_1, T_2, \cdots, T_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A\cap T_1^{-n}A\cap T_2^{-n}A\cap\cdots\cap T_{k-1}^{-n}A) > 0.$$

(ii) There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R$ we have

$$\mu(A\cap T^{-n}A\cap T^{-2n}A\cap\cdots\cap T^{-kn}A) = 0.$$

Theorem 8: Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S, T_1, T_2, \cdots, T_{k-1} : X \to X$ invertible measure preserving transformations generating a nilpotent group, for which $(X, \mathscr{B}, \mu, S)$ has singular spectrum and $T_1, \cdots, T_{k-1}$ commute. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A\cap S^{-n}A\cap T_1^{-n}A\cap T_2^{-n}A\cap\cdots\cap T_{k-1}^{-n}A) > 0.$$

It is worth noting that the example of a m.p.s. $(X, \mathscr{B}, \mu, T)$ given satisfying Theorem 10(ii) has zero entropy and a Lebesgue component in its maximal spectral type.

Multiple ergodic averages with noncommutativity

Theorem 9: Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum and S is totally ergodic. Let $p_1, \cdots, p_K \in \mathbb{Q}[x]$ be integer polynomials for which $\deg(p_1) \geq 2$ and $\deg(p_i) \geq 2 + \deg(p_{i-1})$. For any $f, g_1, \cdots, g_K \in L^{\infty}(X,\mu)$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\prod_{i=1}^{K}S^{p_i(n)}g_i = \mathbb{E}[f|\mathcal{I}_T]\prod_{i=1}^{K}\int_X g_i d\mu, \quad (2)$$

with convergence taking place in $L^2(X,\mu)$.

Theorem 10: Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, R, S : X \to X$ be measure preserving automorphisms. Suppose that T has singular spectrum, R and S commute, and S is weakly mixing. Let $\ell \in \mathbb{N}$ and let $p_1, \cdots, p_\ell \in \mathbb{Q}[x]$ be pairwise essentially distinct integer polynomials, each having degree at least 2. For any $f, h, g_1, \cdots, g_\ell \in L^{\infty}(X,\mu)$ satisfying $\int_X g_j d\mu = 0$ for some $1 \leq j \leq \ell$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f R^n h\prod_{j=1}^{\ell}S^{p_j(n)}g_j = 0, \quad (3)$$

with convergence taking place in $L^2(X,\mu)$.