Katai's orthogonality criterion and the ergodic hierarchy of mixing properties

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1 Introduction

Theorem 1.1 (Katai's criterion)**.** Let $(a_n)_{n=1}^\infty \subseteq \mathbb{C}$ be a bounded sequence for which

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N a_{pn}\overline{a_{qn}} = 0, \tag{1}$$

whenever p and q are distinct primes. If $f : \mathbb{N} \to \mathbb{C}$ is a bounded multiplicative function, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N a_n f(n) = 0. \tag{2}$$

The purpose of this paper is to examine Theorem 1.1 from a more abstract perspective as is done with van der Corput's difference theorem in Chapter 2 of [4]. The main idea is to view a sequences of vectors $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ coming from a a Hilbert space $\mathcal{H}$ as vectors in a new Hilbert space $\mathscr{H}$ endowed with the inner product

$$\langle (x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty\rangle_{\mathscr{H}} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N \langle x_n, y_n\rangle_{\mathcal{H}}. \tag{3}$$

While the limit in Equation (3) may not exist, we can always pass to a subsequence for which it does exist. Section 2 is dedicated to a rigorous construction of the Hilbert space $\mathscr{H}$ and Section 3 is dedicated to proving variants of Katai's orthogonality criterion through the use of $\mathscr{H}$. For the sake of the current discussion, we will assume for the rest of this section that the limit defining $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ always exists.

Recalling that $\mathcal{H} = \mathbb{C}$ is a Hilbert space, we may now view Katai's criterion as follows. We have sequences $(a_n)_{n=1}^\infty$ and $(f(n))_{n=1}^\infty$ uniformly bounded by M that we will now view as vectors in a Hilbert space $\mathscr{H}$. For each prime p, we have a (not necessarily bounded) linear operator $U_p : \mathscr{H} \to \mathscr{H}$ defined by $U_p(x_n)_{n=1}^\infty = (x_{pn})_{n=1}^\infty$. The boundedness of $(a_n)_{n=1}^\infty$ and the assumption of Equation (1) tells us that $\{U_p(a_n)_{n=1}^\infty\}_{p\in\mathbb{P}}$ is an orthogonal set of vectors that is bounded in norm. We now proceed as Katai did in [5] and use the Turán-Kubilius inequality to deduce that for any $\epsilon > 0$ there is a finite set of primes $P(\epsilon)$ with $\min(P(\epsilon)) > \frac{1}{\epsilon}$ and $\sum_{p\in P(\epsilon)}\frac{1}{p} > \frac{1}{\epsilon}$ for which

$$\langle (a_n)_{n=1}^\infty, (f(n))_{n=1}^\infty\rangle_{\mathscr{H}} \overset{\sqrt{\epsilon}}{\approx} \left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{1}{p}\langle U_p(a_n)_{n=1}^\infty, U_p(f(n))_{n=1}^\infty\rangle_{\mathscr{H}} \tag{4}$$

$$\overset{M\epsilon}{\approx} \left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{f(p)}{p}\langle U_p(a_n)_{n=1}^\infty, (f(n))_{n=1}^\infty\rangle_{\mathscr{H}}. \tag{5}$$

Since $\{U_p(a_n)_{n=1}^\infty\}_{p\in\mathbb{P}}$ is a bounded set of orthogonal vectors, we see that the sequence $(\langle U_p(a_n)_{n=1}^\infty, (f(n))_{n=1}^\infty\rangle_{\mathscr{H}})_{p\in\mathbb{P}}$ is square summable. Since f is bounded, we conclude that the quantity in Equation (5) can be made arbitrarily close to 0, which proves Equation (2).

We see now that the assumption of multiplicativity on f is only used to show that $U_p(f(n))_{n=1}^\infty \approx f(p)(f(n))_{n=1}^\infty$ for large enough primes p. Similarly, the assumption of Equation (1) is used to show that each term of the average in Equation (5) is going to 0, so that we can conclude that the average is itself going to 0. These observations suggest that we can generalize Katai's orthogonality criterion by either relaxing the sense in which $U_p(f(n))_{n=1}^\infty \approx (f(n))_{n=1}^\infty$, or by relaxing the assumptions on $(a_n)_{n=1}^\infty$ so that the average in Equation (5) is still small even if not every constituent term of the average is small. This leads us to the following generalizations of Katai's criterion that are related to the ergodic hierarchy of mixing.

Theorem 1.2 (Pseudo Ergodic Katai)**.** Let $(a_n)_{n=1}^\infty \subseteq \mathbb{C}$ be a bounded sequence such that for any sequence of primes $(q_k)_{k=1}^\infty$ satisfying $\sum_{k=1}^\infty \frac{1}{q_k} = \infty$ we have

$$\lim_{K\to\infty}\limsup_{N\to\infty}\left|\left(\sum_{k=1}^K\frac{1}{q_k}\right)^{-2}\sum_{1\le k_1,k_2\le K}\frac{1}{N}\sum_{n=1}^N\frac{a_{q_{k_1}n}}{q_{k_1}}\frac{\overline{a_{q_{k_2}n}}}{q_{k_2}}\right| = 0. \tag{6}$$

If $f:\mathbb{N}\to\mathbb{C}$ is a bounded function such that for all $\epsilon>0$ there exists a set of primes $\mathbb{P}(\epsilon) = \{p_k\}_{k=1}^\infty$ and a function $g_\epsilon:\mathbb{N}\to\mathbb{C}$ uniformly bounded by M (independent of ϵ) satisfying

(i) $\sum_{k=1}^\infty \frac{1}{p_k} = \infty$, and

(ii) $\lim_{K\to\infty}\limsup_{N\to\infty}\left(\sum_{k=1}^K\frac{1}{p_k}\right)^{-1}\sum_{k=1}^K\left|\frac{1}{p_kN}\sum_{n=1}^N(f(p_kn)-g_\epsilon(n))\right| < \epsilon$

then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N a_nf(n) = 0. \tag{7}$$

We see that Equation (6) is implied by Equation (1), and it can be shown with the aid of Lemma 3.19 of [2] that every multiplicative function f satisfies Inequality (ii) with $g_\epsilon(n) = uf(n)$ where u is some limit point of $(f(p))_{p \text{ prime}}$, so Theorem 1.2 is a generalization of Theorem 1.1. Furthermore, it is shown in [2] that for a wide class of level sets of multiplicative functions, the indicator function $f = \mathbb{1}_E$ satisfies Inequality (ii), which allows us to recover the following generalization of Katai's criterion.

Corollary 1.3. Let $(a_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ be a bounded sequence for which

$$\lim_{K\to\infty} \limsup_{N\to\infty} \left| \left(\sum_{k=1}^{K} \frac{1}{q_k}\right)^{-2} \sum_{1\le k_1,k_2\le K} \frac{1}{N} \sum_{n=1}^{N} \frac{a_{q_{k_1} n}}{q_{k_1}} \frac{\overline{a_{q_{k_2} n}}}{q_{k_2}} \right| = 0. \tag{8}$$

If $m : \mathbb{N} \to \mathbb{C}$ is a bounded multiplicative function, and $R = \{(r,\theta) \mid r \in I_1 \ \& \ \theta \in I_2\}$ where I_1 and I_2 are bounded intervals in $[0,\infty)$ that may be open, closed, or half open, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n \mathbb{1}_{m^{-1}(R)}(n) = 0. \tag{9}$$

At this point, it is worth mentioning that the condition on f in Theorem 1.2 more closely resembles a rigidity condition (the dual of strong mixing) than an invariance condition (the dual of ergodicity) since the sequence of primes $\mathbb{P}(\epsilon)$ is almost arbitrary. To better understand this remark, let us consider the following theorem that does not even imply Theorem 1.1.

Theorem 1.4 (Ergodic Katai)**.** Let $(q_k)_{k=1}^{\infty}$ be the increasing enumeration of the primes. Let $(a_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ be a bounded sequence for which

$$\lim_{K\to\infty} \limsup_{N\to\infty} \left| \left(\sum_{k=1}^{K} \frac{1}{q_k}\right)^{-2} \sum_{1\le k_1,k_2\le K} \frac{1}{N} \sum_{n=1}^{N} \frac{a_{q_{k_1} n}}{q_{k_1}} \frac{\overline{a_{q_{k_2} n}}}{q_{k_2}} \right| = 0. \tag{10}$$

If $f : \mathbb{N} \to \mathbb{C}$ is a bounded function such that there exists a bounded function $g : \mathbb{N} \to \mathbb{C}$ satisfying

$$\lim_{K\to\infty} \limsup_{N\to\infty} \left(\sum_{k=1}^{K} \frac{1}{q_k}\right)^{-1} \sum_{k=1}^{K} \left| \frac{1}{q_k N} \sum_{n=1}^{N} \left(f(q_k n) - g(n)\right) \right| = 0, \tag{11}$$

then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n f(n) = 0. \tag{12}$$

In Section 3 we state three more generalizations of Theorem 1.2 (and hence of Theorem 1.1) corresponding to weak mixing, strong mixing, and countable Lebesgue spectrum as Theorems 3.4, 3.5, and 3.6 respectively. We defer them to Section 3 since they are more abstract and require terminology from Section 2. The main reason we encounter these difficulty is that the family of operators $\{U_p\}_{p \text{ prime}}$ is not a semigroup, so we cannot use the ideas introduced in

`https://sohailfarhangi.files.wordpress.com/2022/11/talkfortorunnovember2022.pdf`

and we cannot state an analogue of Theorem 1.1 for mild mixing.

2 A Hilbert space of sequences

The contents of this section are a slight modification of the contents of Chapters 2.2 and 2.3 of [4] that we need to construct $\mathscr{H}$. The only difference is that we will be constructing $\mathscr{H}$ from a countable collection of sequences $\{(x_{n,m})_{n=1}^\infty\}_{m=1}^\infty$ instead of just a pair of sequences, because for a given pair of sequences $(a_n)_{n=1}^\infty$ and $(f(n))_{n=1}^\infty$, we are required to also consider the countable collection of sequences $\{(a_{pn})_{n=1}^\infty\}_{p \text{ prime}}$.

Let $\mathcal{H}$ be a Hilbert space. In this section we will discuss how to construct a Hilbert space $\mathscr{H}$ out of sequences of vectors coming from $\mathcal{H}$. We will then use $\mathscr{H}$ to prove our generalizations of Katai's orthogonality criterion for sequences of vectors coming from $\mathcal{H}$.

Let $||\cdot||$ and $\langle\cdot,\cdot\rangle$ denote the norm and inner product on $\mathcal{H}$ and let $||\cdot||_{\mathscr{H}}$ and $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ denote the norm and inner product on $\mathscr{H}$. We denote the collection of square averageable sequences by

$$SA(\mathcal{H}) := \{(f_n)_{n=1}^\infty \subseteq \mathcal{H} \mid \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N ||f_n||^2 < \infty\}. \tag{13}$$

Let $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty \in SA(\mathcal{H})$ and observe that

$$\limsup_{N\to\infty}\frac{1}{N}\left|\sum_{n=1}^N\langle f_n, g_n\rangle\right| \le \limsup_{N\to\infty}\frac{1}{N}\sum ||f_n||\cdot||g_n|| \tag{14}$$
$$\le \left(\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N||f_n||^2\right)^{\frac{1}{2}}\left(\limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N||g_n||^2\right)^{\frac{1}{2}} < \infty.$$

It follows that we may use diagonalization to construct an increasing sequence of positive integers $(N_q)_{q=1}^\infty$ for which

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle x_{n,m_1}, x_{n,m_2}\rangle \tag{15}$$

exists for all $m_1, m_2 \in \mathbb{N}$. We now construct a new Hilbert space $\mathscr{H} = \mathscr{H}(\{(x_{n,m})_{n=1}^\infty\}_{m=1}^\infty, (N_q)_{q=1}^\infty)$ from $\{(x_{n,m})_{n=1}^\infty\}_{m=1}^\infty$ and $(N_q)_{q=1}^\infty$ as follows. For all $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty \in \{(x_{n,m})_{n=1}^\infty\}_{m=1}^\infty$, we define

$$\langle(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty\rangle_{\mathscr{H}} = \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_n, g_n\rangle. \tag{16}$$

We see that $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ is a sesquilinear form on $\mathscr{H}' = \text{Span}_{\mathbb{C}}(\{(x_{n,m})_{n=1}^\infty\}_{m=1}^\infty)$ with scalar multiplication and addition occuring pointwise. Letting

$$\mathscr{H}'' = \{(e_n)_{n=1}^\infty \in SA(\mathcal{H}) \mid \forall\ \epsilon > 0\ \exists\ (h_n(\epsilon))_{n=1}^\infty \in \mathscr{H}' \text{ s.t.} \tag{17}$$
$$\limsup_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||e_n - h_n(\epsilon)||^2 < \epsilon\}, \text{ and}$$

$$S = \{(x_n)_{n=1}^{\infty} \in \mathscr{H}'' \mid \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||x_n||^2 = 0\}, \tag{18}$$

we see that $\mathscr{H}''/S$ is a pre-Hilbert space. We will soon see that $\mathscr{H}''$ is sequentially closed under the topology induced by $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ (cf. Theorem 2.1), so we define $\mathscr{H}(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty}) = \mathscr{H}''/S$. We call $\mathscr{H}(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$ the Hilbert space induced by $(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$, and we may write $\mathscr{H}$ in place of $\mathscr{H}(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$ if $(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is understood from the context.

For $\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty} \subseteq SA(\mathcal{H})$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ we say that $(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a **permissible pair** if $\mathscr{H}(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is well defined. Given $(f_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ for which $(f_n)_{n=1}^{\infty} \in \mathscr{H}''$, we may view $(f_n)_{n=1}^{\infty}$ as an element of $\mathscr{H}$ by identifying $(f_n)_{n=1}^{\infty}$ with its equivalence class in $\mathscr{H}''/S$. We will now show that $\mathscr{H}$ is a Hilbert space by verifying that it is complete.

Theorem 2.1. Let $\mathcal{H}$ be a Hilbert space and $\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty} \subseteq SA(\mathcal{H})$. Let $(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$ be a permissible pair and $\mathscr{H} = \mathscr{H}(\{(x_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty}, (N_q)_{q=1}^{\infty})$. If $\{(\xi_{n,m})_{n=1}^{\infty}\}_{m=1}^{\infty} \subseteq \mathscr{H}''$ is a Cauchy sequence with respect to the metric induced by $||\cdot||_{\mathscr{H}}$, then there exists $(\xi_n)_{n=1}^{\infty} \in \mathscr{H}''$ for which

$$\lim_{m\to\infty}\left(\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||\xi_{n,m} - \xi_n||^2\right) = 0. \tag{19}$$

In particular, $\mathscr{H}$ is a Hilbert space.

Proof. We proceed by modifying the proof of the main result in section §2 of chapter II of [3]. Let $(\epsilon_m)_{m=1}^{\infty}$ be a sequence of real numbers tending to 0 for which

$$\lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} ||\xi_{n,m} - \xi_{n,k}||^2 < \epsilon_m \tag{20}$$

whenever $k \geq m$. By induction, let $T_0 = N_0 = 0$ and let $(T_m)_{m=1}^{\infty} \subseteq \mathbb{N}$ be such that conditions (i)-(iii) below hold.

(i) For every $m \geq 1$, every $k \geq m$, and every $T \geq T_k$

$$\frac{1}{N_T} \sum_{n=1}^{N_T} ||\xi_{n,k} - \xi_{n,m}||^2 < \epsilon_m. \tag{21}$$

(ii) For every $m \geq 1$ and every $k \geq m$

$$\frac{1}{N_{T_k} - N_{T_{k-1}}} \sum_{n=N_{T_{k-1}}+1}^{N_{T_k}} ||\xi_{n,k} - \xi_{n,m}||^2 < \epsilon_m. \tag{22}$$

(iii) For every $m \geq 1$

$$\frac{1}{N_{T_m}}\sum_{j=1}^{m-1}\sum_{n=N_{T_{j-1}}+1}^{N_{T_j}}||\xi_{n,j}-\xi_{n,m}||^2 < \epsilon_m. \tag{23}$$

Now let us define $(\xi_n)_{n=1}^{\infty}$ by $\xi_n = \xi_{n,m}$ where m is such that $N_{T_{m-1}} < n \leq N_{T_m}$. To conclude the proof, we note that for $m \geq 1$, $k > m$, and $T_{k-1} < T \leq T_k$ we have

$$\begin{aligned}
&\sum_{n=1}^{N_T}||\xi_{n,m}-\xi_n||^2 \qquad (24)\\
=&\sum_{j=1}^{m-1}\sum_{n=N_{T_{j-1}}+1}^{N_{T_j}}||\xi_{n,j}-\xi_{n,m}||^2+\sum_{j=m}^{k-1}\sum_{n=N_{T_{j-1}}+1}^{N_{T_j}}||\xi_{n,m}-\xi_n||^2+\sum_{n=N_{T_{k-1}}+1}^{N_T}||\xi_{n,m}-\xi_n||^2\\
\leq&N_{T_m}\epsilon_m+\sum_{j=m}^{k-1}(N_{T_j}-N_{T_{j-1}})\epsilon_m+\sum_{n=1}^{N_T}||\xi_{n,k}-\xi_{n,m}||^2\\
\leq&N_{T_{k-1}}\epsilon_m+N_T\epsilon_m \leq 2N_T\epsilon_m.
\end{aligned}$$

□

3 Proofs

We begin with a lemma that is a well known consequence of the Turán-Kubilius inequality.

Lemma 3.1. Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ bounded sequences. Let P be a finite collection of primes with $S_p := \sum_{p\in P}\frac{1}{p}$. We have

$$\lim_{N\to\infty}\left|\left|\left(\sum_{n=1}^{N}\langle x_n, y_n\rangle\right)-\left(\frac{1}{S_p}\sum_{p\in P}\sum_{n=1}^{\frac{N}{p}}\langle x_{pn}, y_{pn}\rangle\right)\right|\right| \leq \frac{1}{\sqrt{S_p}}. \tag{25}$$

Theorem 3.2 (Pseudo Ergodic Katai)**.** Let $(a_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ be a bounded sequence such that for any sequence of primes $(q_k)_{k=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty}\frac{1}{q_k} = \infty$ we have

$$\lim_{K\to\infty}\limsup_{N\to\infty}\left|\left(\sum_{k=1}^{K}\frac{1}{q_k}\right)^{-2}\sum_{1\leq k_1,k_2\leq K}\frac{1}{N}\sum_{n=1}^{N}\frac{a_{q_{k_1}n}}{q_{k_1}}\frac{\overline{a_{q_{k_2}n}}}{q_{k_2}}\right| = 0. \tag{26}$$

If $f:\mathbb{N}\to\mathbb{C}$ is a bounded function such that for all $\epsilon > 0$ there exists a set of primes $\mathbb{P}(\epsilon) = \{p_k\}_{k=1}^{\infty}$ and a function $g_\epsilon:\mathbb{N}\to\mathbb{C}$ uniformly bounded by M (independent of ϵ) satisfying

(i) $\sum_{k=1}^{\infty}\frac{1}{p_k} = \infty$, and

(ii) $\lim_{K\to\infty}\limsup_{N\to\infty}\left(\sum_{k=1}^{K}\frac{1}{p_k}\right)^{-1}\sum_{k=1}^{K}\left|\frac{1}{p_kN}\sum_{n=1}^{N}(f(p_kn)-g_\epsilon(n))\right|<\epsilon$

then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}a_nf(n)=0. \tag{27}$$

Proof. Let $(N_k)_{k=1}^{\infty}$ be any sequence for which

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{n=1}^{N_k}a_nf(n) \tag{28}$$

exists, and let $(M_k)_{k=1}^{\infty}$ be a subsequence of $(N_k)_{k=1}^{\infty}$ for which $\mathscr{H}=\mathscr{H}(\{(a_n)_{n=1}^{\infty},(f(n))_{n=1}^{\infty}\}\cup\{(a_{pn})_{n=1}^{\infty}\}_{p\text{ prime}},(N_k)_{k=1}^{\infty})$ is well defined. We see that

$$\lim_{K\to\infty}\left|\left|\left(\sum_{k=1}^{K}\frac{1}{p_k}\right)^{-1}\sum_{k=1}^{K}\frac{1}{p_k}(a_{p_kn})_{n=1}^{\infty}\right|\right|_{\mathscr{H}}^{2} \tag{29}$$

$$=\lim_{K\to\infty}\limsup_{N\to\infty}\left|\left(\sum_{k=1}^{K}\frac{1}{p_k}\right)^{-2}\sum_{1\le k_1,k_2\le K}\langle\frac{1}{p_{k_1}}(a_{p_{k_1}n})_{n=1}^{\infty},\frac{1}{p_{k_2}}(a_{p_{k_2}n})_{n=1}^{\infty}\rangle_{\mathscr{H}}\right|=0. \tag{30}$$

Now let $\epsilon>0$ be arbitrary, and let $P(\epsilon)\subseteq\mathbb{P}(\epsilon)$ be a finite set of primes for which $\sum_{p\in P(\epsilon)}\frac{1}{p}>\frac{1}{\epsilon^2}$. By Lemma 3.1 we see that

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{n=1}^{N_k}a_nf(n)=\lim_{k\to\infty}\frac{1}{M_k}\sum_{n=1}^{M_k}a_nf(n)=\langle(a_n)_{n=1}^{\infty},(f(n))_{n=1}^{\infty}\rangle_{\mathscr{H}} \tag{31}$$

$$\overset{\epsilon}{\approx}\left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{1}{p}\langle U_p(a_n)_{n=1}^{\infty},U_p(f(n))_{n=1}^{\infty}\rangle_{\mathscr{H}}. \tag{32}$$

By increasing the size of $P(\epsilon)$ if necessary, we may further assume that

$$\left|\left|\left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{1}{p}U_p(a_n)_{n=1}^{\infty}\right|\right|<\epsilon,\text{ and} \tag{33}$$

$$\limsup_{N\to\infty}\left|\left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{1}{pN}\sum_{n=1}^{N}(f(pn)-g_\epsilon(n))\right|<\epsilon. \tag{34}$$

We now see that

$$\left|\left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{1}{p}\langle U_p(a_n)_{n=1}^\infty, U_p(f(n))_{n=1}^\infty\rangle_{\mathscr{H}}\right| \tag{35}$$

$$\overset{\epsilon}{\approx}\left|\left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{1}{p}\langle U_p(a_n)_{n=1}^\infty, (g_\epsilon(n))_{n=1}^\infty\rangle_{\mathscr{H}}\right| \tag{36}$$

$$=\left|\left\langle\left(\sum_{p\in P(\epsilon)}\frac{1}{p}\right)^{-1}\sum_{p\in P(\epsilon)}\frac{1}{p}U_p(a_n)_{n=1}^\infty, (g_\epsilon(n))_{n=1}^\infty\right\rangle_{\mathscr{H}}\right| < \epsilon||(g_\epsilon(n))_{n=1}^\infty||_{\mathscr{H}} \leq M\epsilon. \tag{37}$$

Since the sequence $(N_k)_{k=1}^\infty$ and ϵ were both arbitrary, the desired result follows. □

Proof of Corollary 1.3. Since any bounded function g can be uniformly approximated by functions of the form $\sum_{k=1}^K c_k\mathbb{1}_{g^{-1}(R_k)}(x)$, it suffices to show that $f = \mathbb{1}_{m^{-1}(R)}(x)$ satisfies equation (ii) for some set of prime numbers $P = P(R, m)$. If $\overline{d}(m^{-1}(R)) = 0$, then it is clear that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N a_n\mathbb{1}_{m^{-1}(R)}(n) = 0, \tag{38}$$

so let us assume that $\overline{d}(m^{-1}(R)) > 0$. Let $\epsilon > 0$ be arbitrary. Propositions 3.5 and 3.8 of [2] provide us with sets $E_1, E_2 \subseteq \mathbb{N}$ and a set of prime numbers $P = \{p_k\}_{k=1}^\infty$ which satisfy

(i) $\overline{d}(E_2 \setminus E_1) < \epsilon$.

(ii) $\sum_{k=1}^\infty \frac{1}{p_k} = \infty$.

(iii) For all $p \in P$ and $n \in \mathbb{N}$ with $\gcd(n, p) = 1$, we have $\mathbb{1}_{E_1}(n) \leq \mathbb{1}_{m^{-1}(R)}(np) \leq \mathbb{1}_{E_2}$.

We now see that it suffices to take $g_\epsilon = \mathbb{1}_{E_2}$ in Theorem 3.2, so we see that Equation (38) holds in this case as well. □

Before proving our next theorem, we require a lemma that is a variation of Theorem 3.1 of [1].

Lemma 3.3. Let $\mathcal{H}$ be a Hilbert space, $(f_n)_{n=1}^\infty \subseteq \mathcal{H}$ a bounded sequence, and $(q_k)_{k=1}^\infty$ an increasing sequence of primes for which $\sum_{k=1}^\infty \frac{1}{p_k} = \infty$. We have that $(i) \to (ii) \to (iii)$.

(i) $\lim_{K\to\infty}\left(\sum_{k=1}^K\frac{1}{q_k}\right)^{-2}\sum_{m,n=1}^K\left|\left\langle\frac{f_m}{q_m}, \frac{f_n}{q_n}\right\rangle\right| = 0.$

(ii) If $\underline{d}((n_k)_{k=1}^\infty) > 0$, then

$$\lim_{K\to\infty}\left|\left|\left(\sum_{k=1}^K\frac{1}{q_{n_k}}\right)^{-1}\sum_{k=1}^K\frac{f_{n_k}}{q_{n_k}}\right|\right| = 0. \tag{39}$$

(iii) For any $g \in \mathcal{H}$ we have

$$\lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_k}\right)^{-1}\sum_{k=1}^{K}\left|\left\langle\frac{f_k}{q_k},g\right\rangle\right| = 0. \tag{40}$$

Proof. Let us first show that $(i) \to (ii)$. There exists a positive integer C such that $n_k \le Ck$ for all $k \ge 1$. One consequence of this is that

$$C\sum_{k=1}^{\infty}\frac{1}{q_{n_k}} \ge \sum_{k=1}^{\infty}\frac{1}{q_k} = \infty. \tag{41}$$

We now see that

$$\lim_{K\to\infty}\left|\left|\left(\sum_{k=1}^{K}\frac{1}{q_{n_k}}\right)^{-1}\sum_{k=1}^{K}\frac{f_{n_k}}{q_{n_k}}\right|\right| = \lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_{n_k}}\right)^{-2}\sum_{i,j=1}^{K}\left\langle\frac{f_{n_i}}{q_{n_i}},\frac{f_{n_j}}{q_{n_j}}\right\rangle \tag{42}$$

$$\le \lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_{n_k}}\right)^{-2}\sum_{i,j=1}^{CK}\left|\left\langle\frac{f_i}{q_i},\frac{f_j}{q_j}\right\rangle\right| \le \lim_{K\to\infty}C^2\left(\sum_{k=1}^{CK}\frac{1}{q_k}\right)^{-2}\sum_{i,j=1}^{CK}\left|\left\langle\frac{f_i}{q_i},\frac{f_j}{q_j}\right\rangle\right| = 0. \tag{43}$$

Now let us show that $(ii) \to (iii)$. Let us assume for the sake of contradiction that

$$\limsup_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_k}\right)^{-1}\sum_{k=1}^{K}\left|\left\langle\frac{f_k}{q_k},g\right\rangle\right| = \epsilon > 0. \tag{44}$$

It follows that there is some sequence $(n'_k)_{k=1}^{\infty}$ for which $|\langle\frac{f_{n'_k}}{q_{n'_k}},g\rangle| > \frac{\epsilon}{2}$ for all k and $\overline{d}((n'_k)_{k=1}^{\infty}) > \frac{\epsilon}{2-\epsilon}$. Consequently, there exists a subsequence $(n''_k)_{k=1}^{\infty}$ for which $\alpha := \overline{d}((n''_k)_{k=1}^{\infty}) > 0$, and $\langle\frac{f_{n''_k}}{q_{n''_k}},g\rangle \in B_{\frac{\epsilon}{10}}(p)$ for some $p \in \mathbb{C}$. Let $C > 1000\alpha^{-1}$ be an integer and let $(n_k)_{k=1}^{\infty}$ be the increasing enumeration of $(n''_k)_{k=1}^{\infty} \cup C\mathbb{N}$. Since $\underline{d}((n_k)_{k=1}^{\infty}) > \frac{1}{C}$, we see that

$$0 = \lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_{n_k}}\right)^{-1}\sum_{k=1}^{K}\left\langle\frac{f_{n_k}}{q_{n_k}},g\right\rangle \tag{45}$$

$$\ge \lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_{n_k}}\right)^{-1}\left(\left(\sum_{n''_k}\left\langle\frac{f_{n_k}}{q_{n_k}},g\right\rangle\right)-\left(\sum_{C\mathbb{N}}\left\langle\frac{f_{n_k}}{q_{n_k}},g\right\rangle\right)\right) \ge \frac{\alpha}{\alpha+C^{-1}}\cdot\frac{2\epsilon}{5}, \tag{46}$$

which yields the desired contradiction. □

Theorem 3.4 (Weak Mixing Katai)**.** Let $(q_k)_{k=1}^{\infty}$ be an increasing sequence of primes for which $\sum_{k=1}^{\infty}\frac{1}{q_k} = \infty$. Let $(a_n)_{n=1}^{\infty} \subseteq \mathbb{C}$ be a bounded sequence for which

$$\lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_k}\right)^{-2}\sum_{1\le k_1,k_2\le K}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\frac{a_{q_{k_1}n}}{q_{k_1}}\frac{\overline{a_{q_{k_2}n}}}{q_{k_2}}\right|=0. \tag{47}$$

For any bounded sequence $(b_n)_{n=1}^{\infty}$ and any $(N_w)_{w=1}^{\infty}\subseteq\mathbb{N}$, we have

$$\lim_{K\to\infty}\left(\sum_{k=1}^{K}\frac{1}{q_k}\right)^{-1}\sum_{k=1}^{K}\frac{1}{q_k}\left|\lim_{w\to\infty}\frac{1}{N_w}\sum_{n=1}^{N_w}a_{q_kn}b_n\right|=0, \tag{48}$$

provided that all of the above limits exist. In other words, whenever $\mathscr{H}=\mathscr{H}(\{(a_n)_{n=1}^{\infty},(b_n)_{n=1}^{\infty}\}\cup\{(a_{pn})_{n=1}^{\infty}\}_{p\text{ prime}},(N_w)_{w=1}^{\infty})$, the collection $\{(a_{q_kn})_{n=1}^{\infty}\}_{k=1}^{\infty}$ is a "weakly mixing sequence".

Proof. This is a direct consequence of Lemma 3.3. □

We observe that if $\mathcal{H}$ is an infinite dimensional Hilbert space, and $\xi\in\mathcal{H}$, then for the i.i.d. process $(X_n)_{n=1}^{\infty}$ taking values in $\mathbb{S}^1$ uniformly, the sequence of vectors $X_n\xi$ will be ergodic in a sense similar to that of Theorem 3.2, but will not satisfy Lemma 3.3.

Theorem 3.5 (Strong Mixing Katai)**.** Let $(a_n)_{n=1}^{\infty}\subseteq\mathbb{C}$ be a bounded sequence satisfying

$$\lim_{K\to\infty}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}a_{q_Kn}\overline{a_{q_kn}}\right|=0, \tag{49}$$

for all $k\in\mathbb{N}$. For any bounded sequence $(b_n)_{n=1}^{\infty}$ and any $(N_w)_{w=1}^{\infty}\subseteq\mathbb{N}$, we have

$$\lim_{k\to\infty}\lim_{w\to\infty}\frac{1}{N_w}\sum_{n=1}^{N_w}a_{p_kn}b_n=0, \tag{50}$$

provided that all of the above limits exist. In other words, whenever $\mathscr{H}=\mathscr{H}(\{(a_n)_{n=1}^{\infty},(b_n)_{n=1}^{\infty}\}\cup\{(a_{pn})_{n=1}^{\infty}\}_{p\text{ prime}},(N_w)_{w=1}^{\infty})$, the collection $\{(a_{pn})_{n=1}^{\infty}\}_{p\text{ prime}}$ is a strongly mixing sequence.[1]

Proof. This is a consequence of Lemma 1 of [6]. □

Theorem 3.6 (Lebesgue Spectrum Katai)**.** Let $(a_n)_{n=1}^{\infty}\subseteq\mathbb{C}$ be a sequence uniformly bounded by 1 for which

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}a_{pn}\overline{a_{qn}}\right|=0, \tag{51}$$

for all distinct primes p and q. For any bounded sequence $(b_n)_{n=1}^{\infty}$ and any $(N_w)_{w=1}^{\infty}\subseteq\mathbb{N}$, we have

$$\sum_{k=1}^{\infty}\left|\lim_{w\to\infty}\frac{1}{N_w}\sum_{n=1}^{N_w}a_{p_kn}b_n\right|^2\le\lim_{w\to\infty}\frac{1}{N_w}\sum_{n=1}^{N_w}|b_n|^2, \tag{52}$$

provided that all of the above limits exist. In other words, whenever $\mathscr{H}=\mathscr{H}(\{(a_n)_{n=1}^{\infty},(b_n)_{n=1}^{\infty}\}\cup\{(a_{pn})_{n=1}^{\infty}\}_{p\text{ prime}},(N_w)_{w=1}^{\infty})$, the collection $\{(a_{pn})_{n=1}^{\infty}\}_{p\text{ prime}}$ consistents of orthogonal vectors.

[1] It has not yet been proven that this sequence is strongly mixing in the sense discussed in [1].

Proof. This is immediate from the construction of $\mathscr{H}$. $\square$

4 Applications

References

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