Ramsey Theory, Rado's c_p colorings, and diophantine equations

Seminarium Zakładu Arytmetycznej Geometrii Algebraicznej
Uniwersytet Im. Adama Mickiewicza w Poznaniu

Sohail Farhangi
Slides available on sohailfarhangi.com

March 17, 2023

Overview

1. A review of Ramsey Theory on rings
2. Rado's c_p colorings
   - The c_p colorings
   - Applications of the c_p colorings
3. Some special diophantine equations
   - The Pythagorean Equation
   - The Markov Equation
   - The end goal

Partition regularity

Definition

Let R be a ring, $S \subseteq R$, $n, m \in \mathbb{N}$, and $p_1, \cdots, p_m \in R[x_1, \cdots, x_n]$ be polynomials. The system of equations

$$\begin{aligned} p_1(x_1, \cdots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \cdots, x_n) &= 0 \end{aligned} \tag{1}$$

is **partition regular (p.r.) over S** if for any partition $S = \bigcup_{i=1}^{r} C_i$, there is some $1 \le i_0 \le r$ for which C_{i_0} contais a solution to the system of equations in (1). We remove 0 since most equations that we consider will be homogeneous and we want to omit trivial solutions.

Positive results 1/2

The following systems of equations are partition regular over $\mathbb{N}$.

1) $x + y = z$, Schur 1916 [19]

2) van der Waerden 1927 [22] (arithmetic progressions or A.P.s)

$$\begin{aligned} x_1 - x_2 &= x_2 - x_3 \\ &\vdots \\ x_{n-2} - x_{n-1} &= x_{n-1} - x_n \end{aligned}$$

3) Brauer 1928 [5] (A.P.s and their common difference)

$$\begin{aligned} x_1 - x_2 &= x_0 \\ &\vdots \\ x_{n-1} - x_n &= x_0 \end{aligned}$$

Positive results 2/2

4) Rado 1933 [17] classified which finite systems of linear equations are p.r.
5) $x - y = p(z)$ with $p(z) \in z\mathbb{Z}[z]$, Bergelson 1996 [3] (page 53)
6) Bergelson, Moreira, and Johnson 2017 [4], for $p_i(x) \in x\mathbb{Z}[x]$

$$\begin{aligned} x_1 - x_2 &= p_1(x_0) \\ &\vdots \\ x_{n-1} - x_n &= p_{n-1}(x_0) \end{aligned}$$

7) $x^2 - y^2 = z$, Moreira 2017 [15]
8) $z = x^y$, Sahasrabudhe 2018 [18]
9) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2$, Chow, Lindqvist, Prendiville 2021 [7]
(Some results here use the circle method, see also Prendiville [16])

Negative results (all using Rado's c_p colorings)

The following systems of equations are not partition regular over $\mathbb{N}$.

1) $2x + 3y = z$, Rado 1933 [17]

2) Rado 1933 [17]

$$\begin{aligned} 2x + y &= z \\ 3w + y &= z \end{aligned}$$

3) $x + y = z^2$ (ignoring $2 + 2 = 2^2$), Csikvári, Gyarmati, and Sárközy 2012 [8] (see also [13])

4) $x - 2y = z^2$, Di Nasso and Luperi Baglini 2018 [9]

5) $x^2 - 2y^2 = z$, Di Nasso and Luperi Baglini 2018 [9]

6) $x + y = w^3 z^2$, F. and Magner 2022 [11]

7) $2x + 3y = wz^2$, F. and Magner 2022 [11]

8) F. and Magner 2022 [11]

$$\begin{aligned} x_1 + 17y_1 &= w_1 z_1^{100} \\ 9x_2 + 18y_2 &= w_2 z_2^2 \end{aligned}$$

Open problems

The partition regularity of the following systems of equations over $\mathbb{N}$ is not known.

1) $x^2 + y^2 = z^2$ (**VERY** popular, [10]), and $w^2 + x^2 + y^2 = z^2$
2) $a(x^2 - y^2) = bz^2 + dw$ (important, cf. [16])
3) $x^3 + y^3 + z^3 = w^3$ (cf. [7])
4) $x^3 + y^3 = 1 + z^3$
5) $x^4 + y^4 + z^4 = w^4$ (cf. [7])
6) (**VERY** popular, cf. [15])

$$\begin{aligned} w &= xy \\ z &= x + y \end{aligned}$$

7) $2x - 8y = wz^3$ (cf. [11])
8) (cf. [11])

$$\begin{aligned} 16x_1 + 17y_1 &= w_1 z_1^8 \\ 33x_2 - 17y_2 &= w_2 z_2^8 \end{aligned}$$

The c_p-coloring

For $p \in \mathbb{N}$ a prime define a partition $c_p : \mathbb{Q} \setminus \{0\} \to [1, p-1]$ by the first nonzero digit of the p-adic expansion, i.e.,

$$c_p(\frac{r}{s}) \equiv (p^{-v_p(r)} r)(p^{-v_p(s)} s)^{-1} \pmod{p}. \qquad (2)$$

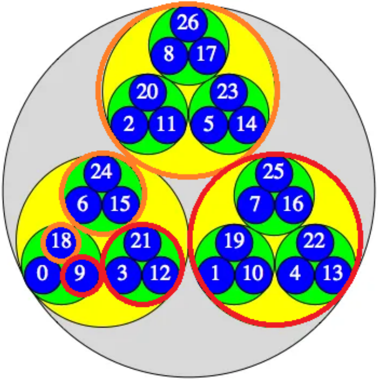


Figure: $c_3^{-1}(\{1\})$ is red and $c_3^{-1}(\{2\})$ is orange. See also [2] and [6].

Using the c_p colorings 1/3

The equation $2x + 3y = z$ is not partition regular over $\mathbb{N}$, as it contains no solutions in any cell of c_7. To see this, let us assume for the sake of contradiction that for some $i \in [1, 6]$ and $x, y, z \in c_7^{-1}(\{i\})$ we have $2x + 3y = z$. By considering the fact that $c_7(2x + 3y) = c_7(z)$, we see that

$$i \equiv c_7(z) \equiv \begin{cases} 2i \equiv c_7(2x) & \text{if } v_7(x) < v_7(y) \\ 3i \equiv c_7(3y) & \text{if } v_7(x) > v_7(y) \\ 5i \equiv c_7(2x+3y) & \text{if } v_7(x) = v_7(y), \end{cases} \pmod 7$$

but none of these congruences can hold modulo 7, which yields the desired contradiction.

Using the c_p colorings 2/3

The following system of equations is not partition regular over $\mathbb{N}$.

$$\begin{aligned} 2x + y &= z \\ 3w + y &= z \end{aligned} \tag{3}$$

We again assume for the sake of contradiction that there is some $i \in [1, 6]$ and $w, x, y, z \in C_i$ satisfying the above system. The considerations of the previous slide show us that we must have $i = 1$ and $v_7(x), v_7(w) > v_7(y) = v_7(z)$. WLOG, $v_7(x) \geq v_7(w)$, so a contradiction is obtained by considering the digit $z_{v_7(w)}$ in position $v_7(w)$ of the base 7 expansion of z. In particular, we have that

$$z_{v_7(w)} \equiv y_{v_7(w)} + 3i \notin y_{v_7(w)} + \{0, 2i\} \pmod 7. \tag{4}$$

Using the c_p colorings 3/3

The equation $2x + 3y = wz^2$ is not partition regular over $\mathbb{N}$. Let us assume for the sake of contradiction that there was some $i \in [1, 42]$ and $w, x, y, z \in c_{43}^{-1}(\{i\})$ satisfying the given equation. Since we have $c_{43}(2x + 3y) = c_{43}(wz^2)$, we see that

$$i^3 \equiv c_{43}(wz^2) \equiv \begin{cases} 2i \equiv c_{43}(2x) & \text{if } v_{43}(x) < v_{43}(y) \\ 3i \equiv c_{43}(3y) & \text{if } v_{43}(x) > v_{43}(y) \\ 5i \equiv c_{43}(2x + 3y) & \text{if } v_{43}(x) = v_{43}(y), \end{cases} \pmod{43}$$

but none of the above congruences are solvable since $2, 3,$ and 5 are not squares modulo 43, which yields the desired contradiction.

Ramsey Theory of the Pythagorean Equation

Graham and Erdős [10] asked whether or not the equation $x^2 + y^2 = z^2$ is partition regular over $\mathbb{N}$, and Erdős offered \$250 for a solution. Heule, Kullman, and Marek [14] showed using (a sophisticated) computer search that for any partition of the form $\mathbb{N} = C_1 \cup C_2$, one of the C_i contains a pythagorean triple, but the problem remains open for partitions of size 3 or more. In fact, it is still open as to whether or not for any partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exists $1 \leq i \leq r$ and $x, y \in C_i$ for which $x^2 + y^2 = \lambda^2$, where λ need not come from C_i. The analagous problem for $x^2 + \lambda^2 = z^2$ is also open, but Frantzikinakis and Host [12] showed that many equations such as $9x^2 + 16y^2 = \lambda^2$ are partition regular. Sun [20], [21] (as a corollary of work related to Sarnak's möbius disjointness conjecture) showed that $ax^2 + by^2 = \lambda^2$ is partition regular over the ring of integers of $\mathbb{Q}(\sqrt{a}, \sqrt{b}, \sqrt{a+b})$, so $x^2 + y^2 = \lambda^2$ is p.r. over $\mathbb{Z}[\sqrt{2}]$ and $x^2 + \lambda^2 = z^2$ is p.r. over $\mathbb{Z}[i]$.

Pythagorean Triples

We recall that Pythagorean Triples are parameterized by $(x, y, z) = (2kmn, k(m^2 - n^2), k(m^2 + n^2))$. Using this structure we can show that the partition regularity of the Pythagorean equation is resistant to many extensions of Rado's c_p colorings. Let us first consider an alternative point of view on partitions of $\mathbb{N}$. We see that some of the most basic partitions are $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, where $C_i = \{x \in \mathbb{N} \mid x \equiv i \pmod{r}\}$. We observe that these partitions can also be defined by declaring $x, y \in \mathbb{N}$ to be in the same cell of the partition if and only if $x \equiv y \pmod{r}$. In particular, the partition can be created as equivalence classes under the equivalence relation induced by reduction modulo r.

Resilience of the Pythagorean Equation!

Exercise: Fix $\ell \in \mathbb{N}$, let $p_1, \cdots, p_\ell \in \mathbb{N}$ be distinct primes, and let $\mathbb{N} = \bigcup_{i=1}^{r} C_{i,j}$ be a partition for each $1 \leq j \leq \ell$. Create a partition $\mathbb{N} = \bigcup_{i=1}^{r'} D_i$ as follows. For $x, y \in \mathbb{N}$, but x and y in the same cell of the partition if and only if for each $1 \leq j \leq \ell$ we have

1. The first ℓ digits of the base p_j expansion of x and y starting at their respective first nonzero digits agree.
2. The first ℓ nonzero digits of the base p_j expansion of x and y agree.
3. The last ℓ digits of the base p_j expansion of x and y.
4. The last ℓ nonzero digits of the base p_j expansion of x and y.
5. $v_{p_j}(x), v_{p_j}(y) \in C_{i,j}$ for some $1 \leq i \leq r$.
6. $\lfloor \log_{p_j}(x) \rfloor, \lfloor \log_{p_j}(y) \rfloor \in C_{i,j}$ for some $1 \leq i \leq r$.

Show that there is a $1 \leq i \leq r'$ and $x, y, z \in D_i$ satisfying $x^2 + y^2 = z^2$.

Resilience of the Pythagorean Equation?

Question: Suppose that we refine the partition $\mathbb{N} = \bigcup_{i=1}^{r'} D_i$ further by placing the following additional restrictions on x and y. Let $\mathbb{N} = \bigcup_{i=1}^{w} E_i$ be a partition (maybe $w = r'$ and $E_i = D_i$?).

7. $\Omega(x), \Omega(y) \in E_i$ for some i, where Ω counts the number of prime divisors with repetition.
8. $\omega(x), \omega(y) \in E_i$ for some i, where Ω counts the number of prime divisors without repetition.
9. $\text{sqf}(x), \text{sqf}(y) \in E_i$ for some i, where sqf denotes the square free part.
10. $\phi(x), \phi(y) \in E_i$ for some i where ϕ is Euler's totient function.
11. $\tau(x), \tau(y) \in E_i$ for some i where $\tau(x)$ counts the number of positive divisors of x.

Does there exist a $1 \leq i \leq r'$ and $x, y, z \in D_i'$ satisfying $x^2 + y^2 = z^2$?

The Markov Equation 1/3

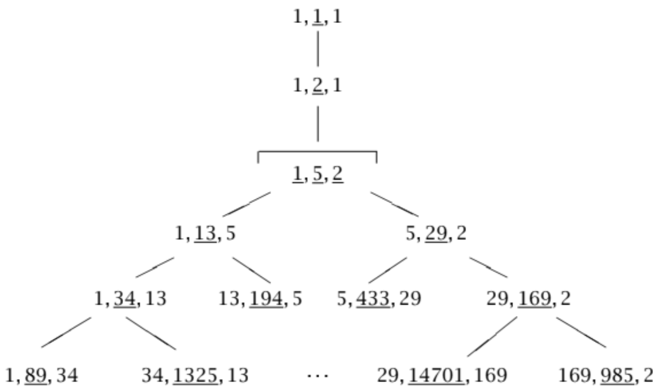
The Markov Equation is

$$x^2 + y^2 + z^2 = 3xyz. \tag{5}$$

The Markov Equation is connected to the study of continued fractions, hyperbolic geometry, quadratic forms, combinatorics, and many other parts of math as discussed in [1]. The partition regularity of the Markov equation cannot be disproven using the methods of slides 9-11. Can it be disproven using the methods of slides 14-15? We will show that it can be disproven if we assume the uniqueness conjecture, which is the other main topic of [1]. We remark this if the methods of slides 14-15 were insufficient, but the uniqueness conjecture was true, then the Markov Equation would be a crucial example in Ramsey theory by objectively showing that partition regularity of polynomial equations is vastly more complex than that of linear equations.

The Markov Equation 2/3

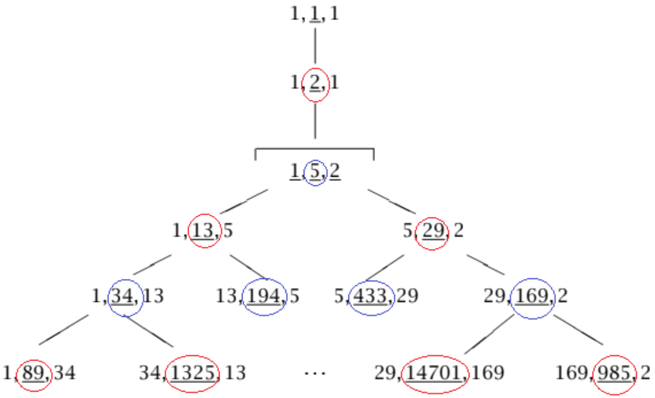
As discussed in [1, pg. 45-47], solutions of the Markov Equation naturally have the structure of a binary tree as shown below.



Markov tree T_M

The uniqueness conjecture is that the underlined terms of the binary tree appear exactly once.

If the uniqueness conjecture is true, then the partition of $\mathbb{N}$ into 2 sets as indicated in the following picture will avoid all nontrivial solutions to the Markov equation in each cell.



Markov tree T_M

The end goal

Question: Can we find examples of diophantine equations other than the Markov equation (or $x^n + y^n = z^n$ with $n \geq 3$) whose partition regularity cannot be determined using the methods of slides 9-11 or 14, but can be determined through knowledge about the structure of their solution sets? Such examples may help advance the understanding of partition regularity of polynomial equations. Is $(x^2 - 1)(y^2 - 1)(z^2 - 1) = (k^2 - 1)$ such an example?

References I

[1] M. Aigner.
Markov's theorem and 100 years of the uniqueness conjecture.

Springer, Cham, 2013.
A mathematical journey from irrational numbers to perfect matchings.

[2] J. M. Barrett, M. Lupini, and J. Moreira.
On Rado conditions for nonlinear Diophantine equations.
European J. Combin., 94:103277, 20, 2021.

References II

[3] V. Bergelson.
Ergodic Ramsey theory—an update.
In *Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994)*, volume 228 of *London Math. Soc. Lecture Note Ser.*, pages 1–61. Cambridge Univ. Press, Cambridge, 1996.

[4] V. Bergelson, J. H. Johnson, Jr., and J. Moreira.
New polynomial and multidimensional extensions of classical partition results.
J. Combin. Theory Ser. A, 147:119–154, 2017.

[5] R. Brauer.
Untersuchungen über die arithmetischen Eigenschaften von Gruppen linearer Substitutionen.
Math. Z., 28(1):677–696, 1928.

References III

[6] J. Byszewski and E. Krawczyk.
Rado's theorem for rings and modules.
J. Combin. Theory Ser. A, 180:105402, 28, 2021.

[7] S. Chow, S. Lindqvist, and S. Prendiville.
Rado's criterion over squares and higher powers.
J. Eur. Math. Soc. (JEMS), 23(6):1925–1997, 2021.

[8] P. Csikvári, K. Gyarmati, and A. Sárközy.
Density and Ramsey type results on algebraic equations with restricted solution sets.
Combinatorica, 32(4):425–449, 2012.

[9] M. Di Nasso and L. Luperi Baglini.
Ramsey properties of nonlinear Diophantine equations.
Adv. Math., 324:84–117, 2018.

References IV

[10] P. Erdős and R. L. Graham.
Old and new problems and results in combinatorial number theory, volume 28 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*.
Université de Genève, L'Enseignement Mathématique, Geneva, 1980.

[11] S. Farhangi and R. Magner.
On the partition regularity of $ax + by = cw^m z^n$.
Integers, 23:Paper No. A18, 52, 2023.

[12] N. Frantzikinakis and B. Host.
Higher order Fourier analysis of multiplicative functions and applications.
J. Amer. Math. Soc., 30(1):67–157, 2017.

References V

[13] B. J. Green and S. Lindqvist.
Monochromatic solutions to $x + y = z^2$.
Canad. J. Math., 71(3):579–605, 2019.

[14] M. J. H. Heule, O. Kullmann, and V. W. Marek.
Solving and verifying the Boolean Pythagorean triples problem via cube-and-conquer.
In *Theory and applications of satisfiability testing—SAT 2016*, volume 9710 of *Lecture Notes in Comput. Sci.*, pages 228–245. Springer, [Cham], 2016.

[15] J. Moreira.
Monochromatic sums and products in $\mathbb{N}$.
Ann. of Math. (2), 185(3):1069–1090, 2017.

References VI

[16] S. Prendiville.
Counting monochromatic solutions to diagonal diophantine equations.
Discrete Anal., pages Paper No. 14, 47, 2021.

[17] R. Rado.
Studien zur Kombinatorik.
Math. Z., 36(1):424–470, 1933.

[18] J. Sahasrabudhe.
Exponential patterns in arithmetic Ramsey theory.
Acta Arith., 182(1):13–42, 2018.

[19] I. Schur.
Uber die kongruenz $x^m + y^m = z^m \pmod{p}$.
Jahresber. Dtsch. Math, 25:114–117, 1916.

References VII

[20] W. Sun.
A structure theorem for multiplicative functions over the Gaussian integers and applications.
J. Anal. Math., 134(1):55–105, 2018.

[21] W. Sun.
Sarnak's conjecture for nilsequences on arbitrary number fields and applications.
Arxiv, pages 1–65, 2019.

[22] B. van der Waerden.
Beweis einer baudetschen vermutung.
Nieuw Arch. Wiskd, 15:212–216, 1927.