The Ergodic Hierarchy of Mixing, van der Corput's difference theorem, and Katai's orthogonality criterion

Ergodic Theory and Dynamical Systems Seminar
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Overview

1. Review of mixing and van der Corput's difference theorem within ergodic theory
2. Mixing van der Corput difference theorems and applications
   - Statements of new vdC difference theorems
   - Applications to Noncommuting ergodic averages
3. Katai's orthogonality criterion
   - An update
   - Variations of Katai's criterion

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The Classical van der Corput Difference Theorem

Definition

A sequence $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is **uniformly distributed** if for any open interval $(a,b) \subseteq [0,1]$ we have

$$\lim_{N\to\infty} \frac{1}{N}\Big|\{1 \leq n \leq N \mid x_n \in (a,b)\}\Big| = b - a. \tag{1}$$

Theorem (van der Corput, [13])

If $(x_n)_{n=1}^{\infty} \subseteq [0,1]$ is such that $(x_{n+h} - x_n \pmod 1)_{n=1}^{\infty}$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is itself uniformly distributed.

Corollary

If $\alpha \in \mathbb{R}$ is irrational, then $(n^2\alpha \pmod 1)_{n=1}^{\infty}$ is uniformly distributed.

Hilbertian van der Corput Difference Theorems 1/3

Theorem (HvdCDT1, [1])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle = 0, \tag{2}$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} \left|\left| \frac{1}{N} \sum_{n=1}^{N} x_n \right|\right| = 0. \tag{3}$$

Hilbertian van der Corput Difference Theorems 2/3

Theorem (HvdCDT2, [1])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \ \textit{then} \tag{4}$$

$$\lim_{N\to\infty} || \frac{1}{N} \sum_{n=1}^{N} x_n || = 0. \tag{5}$$

Hilbertian van der Corput Difference Theorems 3/3

Theorem (HvdCDT3, [1])

If $\mathcal{H}$ is a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \text{ then} \tag{6}$$

$$\lim_{N\to\infty} || \frac{1}{N} \sum_{n=1}^{N} x_n || = 0. \tag{7}$$

Question

Why would we ever use HvdCDT1 or HvdCDT2 when they are both corollaries of HvdCDT3? Why are there at least 3 Hilbertian vdCDTs and only 1 vdCDT in the theory of uniform distribution?

Applications of HvdCDTs 1/2

Theorem (Poincaré)

For any measure preserving system (m.p.s.) $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0. \tag{8}$$

Does not need vdCDT.

Theorem (Furstenberg-Sárközy)

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0. \tag{9}$$

Furstenberg's proof implicitly uses a form of vdCDT.

Applications of HvdCDTs 2/2

Theorem (Furstenberg, [10])

For any m.p.s. $(X, \mathscr{B}, \mu, T)$, *any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\ell \in \mathbb{N}$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A) > 0. \tag{10}$$

Furstenberg's proof uses an equivalent form of HvdCT3. Other proofs directly use HvdCT3.

Theorem (Bergelson and Leibman, [3])

For any m.p.s. $(X, \mathscr{B}, \mu, \{T_i\}_{i=1}^{\ell})$ *with the* T_i*s commuting, any* $A \in \mathscr{B}$ *with* $\mu(A) > 0$, *and any* $\{p_i(x)\}_{i=1}^{\ell} \subseteq x\mathbb{N}[x]$, *there exists* $n \in \mathbb{N}$ *for which*

$$\mu(A \cap T_1^{-p_1(n)}A \cap T_2^{-p_2(n)}A \cap \cdots \cap T_\ell^{-p_\ell(n)}A) > 0. \tag{11}$$

Uses an equivalent form of HvdCT3.

The Ergodic Hierarchy of Mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If for every $f, g \in L^2_0(X, \mu)$

1. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, g \rangle = 0$, then $\mathcal{X}$ is ergodic.
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, g \rangle| = 0$, then $\mathcal{X}$ is weakly mixing,
3. $\text{IP}^* - \lim_{n\to\infty} \langle U_T^n f, g \rangle = 0$, then $\mathcal{X}$ is mildly mixing,
4. $\lim_{n\to\infty} \langle U_T^n f, g \rangle = 0$, then $\mathcal{X}$ is strongly mixing,
5. and if $L^2_0(X, \mu)$ has an orthogonal basis of the form $\{U_T^n f_m\}_{n,m\in\mathbb{Z}}$, then $\mathcal{X}$ has Lebesgue spectrum.

These definitions also apply to individual elements $f \in L^2_0(X, \mu)$.

The Symmetric Ergodic Hierarchy of Mixing

Theorem

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ *be a m.p.s. If for every* $f \in L^2_0(X, \mu)$

1. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle U_T^n f, f \rangle = 0$, *then* $\mathcal{X}$ *is ergodic,*
2. $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\langle U_T^n f, f \rangle| = 0$, *then* $\mathcal{X}$ *is weakly mixing,*
3. $IP^* - \lim_{n\to\infty} \langle U_T^n f, f \rangle = 0$, *then* $\mathcal{X}$ *is mildly mixing,*
4. $\lim_{n\to\infty} \langle U_T^n f, f \rangle = 0$, *then* $\mathcal{X}$ *is strongly mixing.*

This theorem also applies to individual elements $f \in L^2_0(X, \mu)$.

Anti-Mixing

Definition

Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a m.p.s. If $f \in L^2_0(X, \mu)$ satisfies

1. $U_T f = f$, then f is invariant.
2. $p - \lim_{n\to\infty} U_T^n f = f$ in the weak (and hence norm) topology for some minimal idempotent ultrafilter p, then f is compact (cf. Theorem 2.25 in [4]), i.e., $f \in L^2(X, K, \mu)$ where (X, K, μ, T) is the Kronecker factor of $(X, \mathscr{B}, \mu, T)$.
3. $\lim_{k\to\infty} ||U_T^{n_k} f - f||_2 = 0$, for some $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$, then f is rigid.

Theorem

For $f, g \in L^2_0(X, \mu)$, we have $\langle f, g\rangle = 0$ if

1. *f is invariant and g is ergodic.*
2. *f is compact and g is weakly mixing.*
3. *f is rigid and g is mildly mixing.*

The dual of strong mixing?

Theorem (Blum and Hanson, [5])

The m.p.s. $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ *is strongly mixing if and only if for any* $f \in L^2_0(X, \mu)$ *and any increasing* $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ *we have*

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} U_T^{n_k} f = 0. \tag{12}$$

Definition

An element $g \in L^2(X, \mu)$ is weakly rigid if there exists $h \in L^2(X, \mu)$ and $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ for which

$$g = \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} U_T^{n_k} h. \tag{13}$$

If f is strongly mixing and g is weakly rigid, then $\langle f, g \rangle = 0$.

The dual of countable Lebesgue spectrum?

Definition

An element $g \in L^2(X,\mu)$ is very weakly rigid if there exists $(h_k)_{k=1}^{\infty} \subseteq\in L^2(X,\mu)$, $(c_{n,k})_{n,k\in\mathbb{Z}} \subseteq \mathbb{C}$ and $(n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ for which

$$g = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} c_{n_k,K} U_T^{n_k} h_K, \text{ and} \tag{14}$$

$$\left(\sum_{k=1}^{K} |c_{n_k,K}|^2\right) ||h_K||^2 = o(K^2). \tag{15}$$

If f is well approximated by the "Lebesgue spectrum subspace" and g is very weakly rigid, then $\langle f, g\rangle = 0$.
In [2] it is shown that a 'typical' m.p.s. is weakly mixing and rigid, hence the class of very weakly rigid system is also quite large, and **might** include many strongly mixing systems.

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A strong mixing van der Corput difference theorem

Theorem (MvdCT)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{16}$$

then $(x_n)_{n=1}^{\infty}$ is a nearly strongly mixing sequence. This means that for any other bounded sequence $(y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ we morally (but not literally) have that

$$\lim_{h\to\infty} \left| \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, y_n \rangle \right| = 0. \tag{17}$$

A Lebesgue spectrum vdCdt

Theorem (MvdCT)

If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying for all $h \in \mathbb{N}$

$$\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle x_{n+h}, x_n \rangle \right| = 0, \tag{18}$$

then $(x_n)_{n=1}^{\infty}$ is a nearly orthogonal sequence. This means that for any other in some other Hilbert space $\mathscr{H}$ whose vectors are sequences of vectors from $\mathcal{H}$, the set $\{(x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}$ consists of orthogonal vectors.

See Chapter 2 of [6] for variations of MvdCT related to other levels of mixing, as well as uniform distribution. See also [12].

Noncommutative ergodic theorems 1/2

Theorem ((Corollary 1.7 in [7]))

Let $a : \mathbb{R}_+ \to \mathbb{R}$ *be a Hardy field function for which there exist some* $\epsilon > 0$ *and* $d \in \mathbb{Z}_+$ *satisfying*

$$\lim_{n\to\infty} \frac{a(n)}{t^{d+\epsilon}} = \lim_{n\to\infty} \frac{t^{d+1}}{a(n)} = \infty. \tag{19}$$

Furthermore, let $(X, \mathscr{B}, \mu)$ *be a probability space and* $T, S : X \to X$ *be measure preserving transformations. Suppose that the system* $(X, \mathscr{B}, \mu, T)$ *has zero entropy. Then*

(i) *For every* $f, g \in L^\infty(X, \mu)$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f|\mathcal{I}_T] \cdot \mathbb{E}[g|\mathcal{I}_S], \tag{20}$$

where the limit is taken in $L^2(X, \mu)$.

Noncommutative ergodic theorems 2/2

Theorem (Continued)

(ii) *For every $A \in \mathscr{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap S^{-\lfloor a(n) \rfloor}A) \geq \mu(A)^3. \qquad (21)$$

In [8] a similar theorem is proven for $a(n) = p(n)$ with $p(x) \in \mathbb{Z}[x]$ of degree at least 2. It is also mentioned in [7] that the zero entropy assumption on T cannot, in general, be weakened.

Application 1/4

Theorem (F., 2022)

Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving transformations. Suppose that the m.p.s. $(X, \mathscr{B}, \mu, T)$ is very weakly rigid, and that the m.p.s. $(X, \mathscr{B}, \mu, S)$ is totally ergodic. Let $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha \pmod 1)_{n=1}^{\infty}$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$.

(i) *For any $f, g \in L^\infty(X, \mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{k_n} g = \mathbb{E}[f|\mathcal{I}_T]\mathbb{E}[g|\mathcal{I}_S], \tag{22}$$

with convergence taking place in $L^2(X, \mu)$.

Application 1/4 continued

Theorem (Continued)

(ii) *If* $A \in \mathscr{B}$ *then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap S^{-k_n}A) \geq \mu(A)^3. \qquad (23)$$

(iii) *If* $((k_{n+h} - k_n)\alpha \pmod 1)_{n=1}^{\infty}$ *is uniformly distributed in its orbit closure for all* $\alpha \in \mathbb{R}$ *then (i) and (ii) hold even when* $(X, \mathscr{B}, \mu, S)$ *is not ergodic.*

Sets of K but not $K+1$ recurrence?

Theorem ((Theorem 1.4 and Corollary 4.4 of [9]))

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$.

(i) *If $(X, \mathscr{B}, \mu)$ is a probability space and $T_1, T_2, \cdots, T_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which*

$$\mu(A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_{k-1}^{-n}A) > 0. \tag{24}$$

(ii) *There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R_k$ we have*

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A) = 0. \tag{25}$$

Application 2/4

Theorem (F., 2022)

Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R_k = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $T_1, T_2, \cdots, T_{k-1} : X \to X$ commuting measure preserving transformations. Let $S : X \to X$ be an measure preserving transformation for which $(X, \mathscr{B}, \mu, S)$ is very weakly rigid, and $\{S, T_1, T_2, \cdots, T_{k-1}\}$ generate a nilpotent group. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R_k$ for which

$$\mu(A \cap S^{-n}A \cap T_1^{-n}A \cap T_2^{-n}A \cap \cdots \cap T_{k-1}^{-n}A) > 0. \tag{26}$$

It is worth noting that the skew system $(\mathbb{T}^2, \mathscr{B}^2, m^2, T)$ with $T(x, y) = (x + \alpha, y + x)$ is the system used in item (ii) of the last slide when $k = 2$. Consequently, the current theorem does not hold for a general S with zero entropy.

Application 3/4

Theorem (F., 2022)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations. Suppose that $(X, \mathscr{B}, \mu, T)$ is very weakly rigid and $(X, \mathscr{B}, \mu, S)$ is totally ergodic. Let $p_1, \cdots, p_K \in \mathbb{Q}[x]$ be a collection of integer polynomials such that $\{p_1(n+h) - p_1(n), p_2(n+h) - p_1(n) \cdots, p_K(n+h) - p_1(n), p_2(n) - p_1(n), \cdots, p_K(n) - p_1(n)\}$ is independent for all $h \in \mathbb{N}$. For any $f_0, f_1, \cdots, f_K \in L^\infty(X, \mu)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_0 \prod_{i=1}^{K} S^{p_i(n)} f_i = \mathbb{E}[f_0 | \mathcal{I}_T] \prod_{i=1}^{K} \int_X f_i \, d\mu, \tag{27}$$

with convergence taking place in $L^2(X, \mu)$.

Application 4/4

Theorem (F., 2022)

Let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S_1, S_2 : X \to X$ be measure preserving transformations. Suppose that $(X, \mathscr{B}, \mu, T)$ is very weakly rigid, S_1 and S_2 commute, and $(X, \mathscr{B}, \mu, S_2)$ is weakly mixing. Let $p : \mathbb{R} \to \mathbb{R}$ either be a polynomial of degree at least 2, or of the form $p(n) = n^\alpha \log(n)^\beta$ with $\alpha > 1$ and $\beta \leq 0$. For any $f_0, f_1, f_2 \in L^\infty(X, \mu)$ satisfying $\int_X f_2 d\mu = 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_0 \cdot S_1^n f_1 \cdot S_2^{\lfloor p(n) \rfloor} f_2 = 0, \tag{28}$$

with convergence taking place in $L^2(X, \mu)$.

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An update

The following slides contain statements which I believed to be true when I gave this talk, but for which I had not yet written down a formal proof. Upon trying to write down a formal proof, I obtained different statements which are discussed in the file in the link below.

https://sohailfarhangi.files.wordpress.com/2023/03/van$_der_corput_and_ka$
$//sohailfarhangi.files.wordpress.com/$2023/03/
van_der_corput_and_katai.pdf

Consequently, the variations of Katai's criterion presented in the following slides are now stated as conjectures instead of theorems.

Katai's orthogonality criterion

Theorem (Katai, [11])

Let $(c_n)_{n=1}^{\infty}$ *be a bounded sequence of complex numbers satisfying*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_{pn}\overline{c_{qn}}=0, \tag{29}$$

for all distinct primes p *and* q*. If* $f:\mathbb{N}\to\mathbb{C}$ *is a bounded multiplicative function, then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_n\overline{f(n)}=0. \tag{30}$$

Ergodicity and Katai's criterion

Conjecture (F., 2022)

Let $(p_k)_{k=1}^{\infty}$ be the increasing enumeration of the primes. Let $(c_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers such that

$$\lim_{K\to\infty}\limsup_{N\to\infty}\left|\frac{1}{K^2}\sum_{k_1,k_2=1}^{K}\frac{1}{N}\sum_{n=1}^{N}c_{p_{k_1}n}\overline{c_{p_{k_2}n}}\right| = 0. \tag{31}$$

If $f:\mathbb{N}\to\mathbb{C}$ is a bounded multiplicative function, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}c_n\overline{f(n)} = 0. \tag{32}$$

Strong mixing and Katai's criterion

Conjecture (F., 2022)

Let $(a_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ be increasing. Let $(c_n)_{n=1}^{\infty}$ be a bounded sequence of complex numbers such that for all $W \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} c_{a_W n} \overline{c_{a_k n}} \right| = 0. \tag{33}$$

If $f : \mathbb{N} \to \mathbb{C}$ satisfies

$$\lim_{K\to\infty} \limsup_{N\to\infty} \left| \frac{1}{K} \sum_{k=1}^{K} \frac{1}{N} \sum_{n=1}^{N} \left(d_k g\left(\frac{n}{a_k}\right) - f(n) \right) \right| = 0, \tag{34}$$

for some bounded $(d_k)_{k=1}^{\infty} \subseteq \mathbb{C}$ and $g : \mathbb{Q} \to \mathbb{C}$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \overline{f(n)} = 0. \tag{35}$$

Conjecture (F., 2022)

Let $(a_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ *be increasing. Let* $(c_n)_{n=1}^{\infty}$ *be a bounded sequence of complex numbers such that for all* $k_2 \neq k_2 \in \mathbb{N}$ *we have*

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} c_{a_{k_1}n}\overline{c_{a_{k_2}n}}\right| = 0. \tag{36}$$

If $f : \mathbb{N} \to \mathbb{C}$ *satisfies*

$$\lim_{K\to\infty}\limsup_{N\to\infty}\left|\frac{1}{K}\sum_{k=1}^{K}\frac{1}{N}\sum_{n=1}^{N}\left(d_{n_k,K}g_K\left(\frac{n}{a_k}\right)-f(n)\right)\right| = 0, \tag{37}$$

for some "very weakly rigid pair" $(d_k)_{k=1}^{\infty} \subseteq \mathbb{C}$, $g : \mathbb{Q} \to \mathbb{C}$, *then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} c_n\overline{f(n)} = 0. \tag{38}$$

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