Ramsey Theory, Rado's c_p colorings, and the Pythagorean Equation

Seminarium Zakładu Algebry i Teorii Liczb
Uniwersytet Im. Adama Mickiewicza w Poznaniu

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January 10, 2023

Overview

1. A review of Ramsey Theory on rings
2. Rado's c_p colorings
3. An approach to the partition regularity of the Pythagorean Equation

Partition regularity

Definition

Let R be a ring, $n, m \in \mathbb{N}$, and $p_1, \cdots, p_m \in R[x_1, \cdots, x_n]$ be polynomials. The system of equations

$$\begin{aligned} p_1(x_1, \cdots, x_n) &= 0 \\ &\vdots \\ p_m(x_1, \cdots, x_n) &= 0 \end{aligned} \tag{1}$$

is **partition regular (p.r.)** if for any partition $R \setminus \{0\} = \bigcup_{i=1}^{r} C_i$, there is some $1 \leq i_0 \leq r$ for which C_{i_0} contais a solution to the system of equations in (1). We remove 0 since most equations that we consider will be homogeneous and we want to omit trivial solutions.

Positive results 1/2

The following systems of equations are partition regular over $\mathbb{N}$.

1) $x + y = z$, Schur 1916 [17]

2) van der Waerden 1927 [20] (arithmetic progressions or A.P.s)

$$\begin{aligned} x_1 - x_2 &= x_2 - x_3 \\ &\vdots \\ x_{n-2} - x_{n-1} &= x_{n-1} - x_n \end{aligned}$$

3) Brauer 1928 [4] (A.P.s and their common difference)

$$\begin{aligned} x_1 - x_2 &= x_0 \\ &\vdots \\ x_{n-1} - x_n &= x_0 \end{aligned}$$

Positive results 2/2

4) Rado 1933 [15] classified which finite systems of linear equations are p.r.
5) $x - y = p(z)$ with $p(z) \in z\mathbb{Z}[z]$, Bergelson 1996 [2] (page 53)
6) Bergelson, Moreira, and Johnson 2017 [3], for $p_i(x) \in x\mathbb{Z}[x]$

$$\begin{aligned} x_1 - x_2 &= p_1(x_0) \\ &\vdots \\ x_{n-1} - x_n &= p_{n-1}(x_0) \end{aligned}$$

7) $x^2 - y^2 = z$, Moreira 2017 [13]
8) $z = x^y$, Sahasrabudhe 2018 [16]
9) $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2$, Chow, Lindqvist, Prendiville [6], 2021.
(Some results here use the circle method, see also Prendiville [14])

Negative results (all using Rado's c_p colorings)

The following systems of equations are not partition regular over $\mathbb{N}$.

1) $2x + 3y = z$, Rado 1933 [15]

2) Rado 1933 [15]

$$\begin{aligned} 2x + y &= z \\ 3w + y &= z \end{aligned}$$

3) $x + y = z^2$ (ignoring $2 + 2 = 2^2$), Csikvári, Gyarmati, and Sárközy 2012 [7]

4) $x - 2y = z^2$, Di Nasso and Luperi Baglini 2018 [8]

5) $x^2 - 2y^2 = z$, Di Nasso and Luperi Baglini 2018 [8]

6) $x + y = w^3 z^2$, F. and Magner 2022 [10]

7) $2x + 3y = wz^2$, F. and Magner 2022 [10]

8) F. and Magner 2022 [10]

$$\begin{aligned} x_1 + 17y_1 &= w_1 z_1^{100} \\ 9x_2 + 18y_2 &= w_2 z_2^2 \end{aligned}$$

Open problems

The partition regularity of the following systems of equations over $\mathbb{N}$ is not known.

1) $x^2 + y^2 = z^2$ (**VERY** popular, [9]), and $w^2 + x^2 + y^2 = z^2$
2) $a(x^2 - y^2) = bz^2 + dw$ (important, cf. [14])
3) $x^3 + y^3 + z^3 = w^3$ (cf. [6])
4) $x^3 + y^3 = 1 + z^3$
5) $x^4 + y^4 + z^4 = w^4$ (cf. [6])
6) (**VERY** popular, cf. [13])

$$\begin{aligned} w &= xy \\ z &= x + y \end{aligned}$$

7) $2x - 8y = wz^3$ (cf. [10])
8) (cf. [10])

$$\begin{aligned} 16x_1 + 17y_1 &= w_1 z_1^8 \\ 33x_2 - 17y_2 &= w_2 z_2^8 \end{aligned}$$

The c_p-coloring

Let $p \in \mathbb{N}$ be a prime, and define a coloring (partition) $c_p : \mathbb{Q} \setminus \{0\} \to [1, p-1]$ by

$$c_p\left(\frac{r}{s}\right) \equiv p^{-v_p\left(\frac{r}{s}\right)} rs^{-1} \pmod{p}. \tag{2}$$

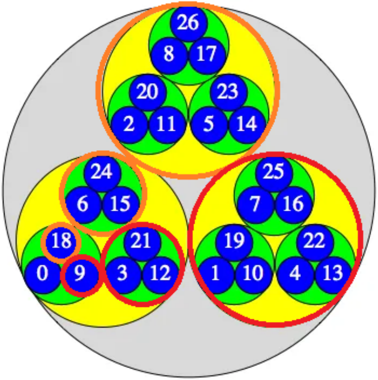

Figure: $c_3^{-1}(\{1\})$ is red and $c_3^{-1}(\{2\})$ is orange. See also [1] and [5].

Using the c_p colorings 1/3

The equation $2x + 3y = z$ is not partition regular over $\mathbb{N}$, as it contains no solutions in any cell of c_7. To see this, let us assume for the sake of contradiction that for some $i \in [1, 6]$ and $x, y, z \in c_7^{-1}(\{i\})$ we have $2x + 3y = z$. By considering the fact that $c_7(2x + 3y) = c_7(z)$, we see that

$$i \equiv c_7(z) \equiv \begin{cases} 2i \equiv c_7(2x) & \text{if } v_7(x) < v_7(y) \\ 3i \equiv c_7(3y) & \text{if } v_7(x) > v_7(y) \\ 5i \equiv c_7(2x + 3y) & \text{if } v_7(x) = v_7(y), \end{cases} \pmod 7$$

but none of these congruences can hold modulo 7, which yields the desired contradiction.

Using the c_p colorings 2/3

The following system of equations is not partition regular over $\mathbb{N}$.

$$\begin{aligned} 2x + y &= z \\ 3w + y &= z \end{aligned} \tag{3}$$

We again assume for the sake of contradiction that there is some $i \in [1,6]$ and $w, x, y, z \in C_i$ satisfying the above system. The considerations of the previous slide show us that we must have $i = 1$ and $v_7(x), v_7(w) > v_7(y) = v_7(z)$. WLOG, $v_7(x) \geq v_7(w)$, so a contradiction is obtained by considering the digit $z_{v_7(w)}$ in position $v_7(w)$ of the base 7 expansion of z. In particular, we have that

$$z_{v_7(w)} \equiv y_{v_7(w)} + 3i \notin y_{v_7(w)} + \{0, 2i\} \pmod 7. \tag{4}$$

Using the c_p colorings 3/3

The equation $2x + 3y = wz^2$ is not partition regular over $\mathbb{N}$. Let us assume for the sake of contradiction that there was some $i \in [1, 42]$ and $w, x, y, z \in c_{43}^{-1}(\{i\})$ satisfying the given equation. Since we have $c_{43}(2x + 3y) = c_{43}(wz^2)$, we see that

$$i^3 \equiv c_{43}(wz^2) \equiv \begin{cases} 2i \equiv c_{43}(2x) & \text{if } v_{43}(x) < v_{43}(y) \\ 3i \equiv c_{43}(3y) & \text{if } v_{43}(x) > v_{43}(y) \\ 5i \equiv c_{43}(2x + 3y) & \text{if } v_{43}(x) = v_{43}(y), \end{cases} \pmod{43}$$

but none of the above congruences are solvable since $2, 3,$ and 5 are not squares modulo 43, which yields the desired contradiction.

Ramsey Theory of the Pythagorean Equation

Graham and Erdős [9] asked whether or not the equation $x^2 + y^2 = z^2$ is partition regular over $\mathbb{N}$, and Erdős offered \$250 for a solution. Heule, Kullman, and Marek [12] showed using (a sophisticated) computer search that for any partition of the form $\mathbb{N} = C_1 \cup C_2$, one of the C_i contains a pythagorean triple, but the problem remains open for partitions of size 3 or more. In fact, it is still open as to whether or not for any partition $\mathbb{N} = \bigcup_{i=1}^r C_i$, there exists $1 \leq i \leq r$ and $x, y \in C_i$ for which $x^2 + y^2 = \lambda^2$, where λ need not come from C_i. The analagous problem for $x^2 + \lambda^2 = z^2$ is also open, but Frantzikinakis and Host [11] showed that many equations such as $9x^2 + 16y^2 = \lambda^2$ are partition regular. Sun [18], [19] (as a corollary of work related to Sarnak's möbius disjointness conjecture) showed that $ax^2 + by^2 = \lambda^2$ is partition regular over the ring of integers of $\mathbb{Q}(\sqrt{a}, \sqrt{b}, \sqrt{a+b})$, so $x^2 + y^2 = \lambda^2$ is p.r. over $\mathbb{Z}[\sqrt{2}]$ and $x^2 + \lambda^2 = z^2$ is p.r. over $\mathbb{Z}[i]$.

Pythagorean Triples

We recall that Pythagorean Triples are parameterized by $(x, y, z) = (2kmn, k(m^2 - n^2), k(m^2 + n^2))$. What can be said about the range of y and z when $x = N$ is fixed?

Conjecture 1: For each $L \in \mathbb{N}$, there exists $N, w \in \mathbb{N}$ for which

$$w, 2w, \cdots, Lw \subseteq \left\{km^2 - k\left(\frac{N}{2mk}\right)^2 \mid m, k \in \mathbb{N}\right\}. \tag{5}$$

Conjecture 2: For each $L \in \mathbb{N}$, there exists $N, k_1, k_2 \in \mathbb{N}$ for which

$$\{(iw_1, jw_2) \mid 1 \leq i, j \leq L\} \subseteq \left\{\left(km^2 - k\left(\frac{N}{2mk}\right)^2, km^2 + \left(\frac{N}{2mk}\right)^2\right) \mid m, k \in \mathbb{N}\right\}. \tag{6}$$

Some implications

If Conjecture 1 is true, then the equation $x^2 + y^2 = \lambda^2$ is partition regular over $\mathbb{N}$. If Conjecture 2 is true, then the Pythagorean Equation $x^2 + y^2 = z^2$ is partition regular over $\mathbb{N}$. In fact, we only need weaker versions of Conjecture 1 or Conjecture 2 to hold to get these implications, but the weaker versions are too technical to state here since they concern minimal idempotent ultrafilters.

Question: Is there a pleasant parameterization of the quadraples (w, x, y, z) for which $w^2 + x^2 + y^2 = z^2$? If so, a variant of Conjecture 2 can be formulated to help and show that the equation $w^2 + x^2 + y^2 = z^2$ is partition regular over $\mathbb{N}$.

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