What are.... some nice conjectures in Graph Theory?

What is Seminar 2021

Sohail Farhangi

June 24, 2021

Overview

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Cayley Graphs

Definition

Given a group G and a set $S \subseteq G$, the Cayley Graph $\mathcal{T} = \mathcal{T}(G, S) = (V, E)$ is defined by $V = G$, and $(g_1, g_2) \in E$ iff $g_2 g_1^{-1} \in S$.

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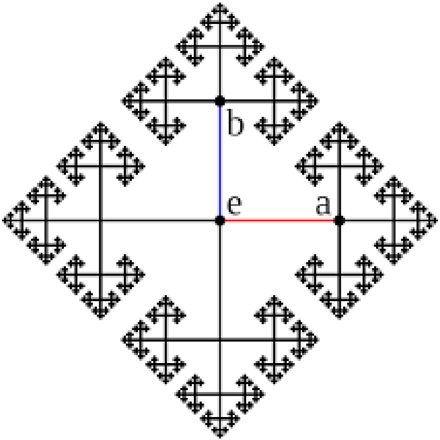
In this talk we will usually assume that G is finite, $S = S^{-1}$, and S generates G. This will result in a finite connected undirected graph.

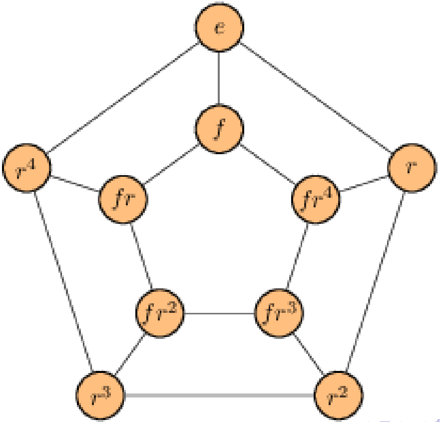
A Cayley Graph of F_2

Letting F_2 denote the free group generated by a and b, we see that $\mathcal{T}(F_2, \{a, b, a^{-1}, b^{-1}\})$ is given by



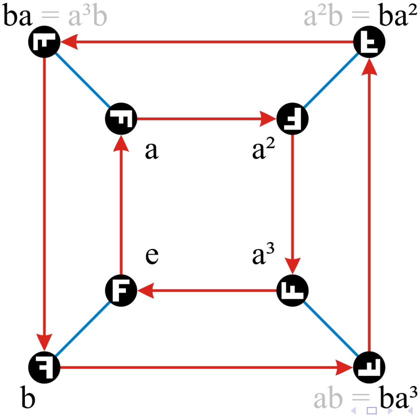
A Cayley Graph of D_5

Letting $D_5 = \langle f, r \rangle$ denote the group of symmetries of a regular pentagon with f for reflection and r for rotation, we see that $\mathcal{T}(D_5, \{f, r, r^{-1}\})$ is given by



A Cayley Digraph of D_4

Letting $D_4 = \langle a, b \rangle$ denote the group of symmetries of a square with b for reflection and a for rotation, we see that $\mathcal{T}(D_5, \{a, b\})$ is given by



The Lovász Conjecture (version 1)

Conjecture

The Cayley graph of a finite group has a Hamiltonian cycle.

Known Results (also cf. [4])

cf. [11] Every finite group G has a generating set S of size at most $\log_2(|G|)$ for which the Cayley graph $\mathcal{T}(G, S)$ is Hamiltonian.

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  cf. [7] Every cyclic group whose order is not a prime power has a Cayley digraph that does not have a Hamiltonian cycle.

Known Results (also cf. [4])

cf. [5] Let p and q be primes. A Cayley graph on a group of order $pq, 4q(q > 3), p^2q(2 < p < q), 2p^2, 2pq, 8p$, or $4p^2$ is Hamiltonian.

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cf. [6] If $1 \leq k \leq 31$ and $k \neq 24$, then for any prime p, any group G of order kp satisfies the Lovász conjecture.

The Lovász Conjecture (version 2)

Definition

A graph $G = (V, E)$ is **vertex-transitive** if for any $v_1, v_2 \in V$, there exists a graph automorphism $\tau : G \to G$ for which $\tau(v_1) = v_2$.

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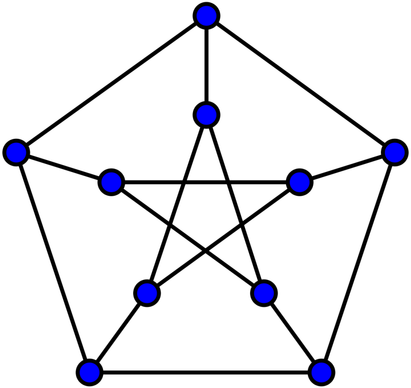
Conjecture

Every finite vertex-transitive graph contains a Hamiltonian path.

The Petersen Graph

We see that the Petersen graph is vertex transitive and contains a Hamiltonian path, but does not contain a Hamiltonian cycle.



The Lovász Conjecture (version 3)

Conjecture

Every finite connected vertex-transitive graph has a Hamiltonian cycle except for the 5 known counterexamples.

The 5 Counterexamples

(1) K_2, the complete graph on 2 vertices.

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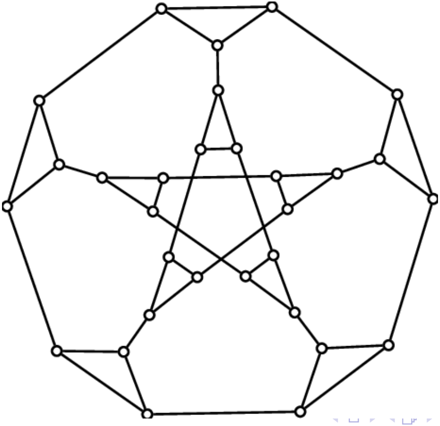
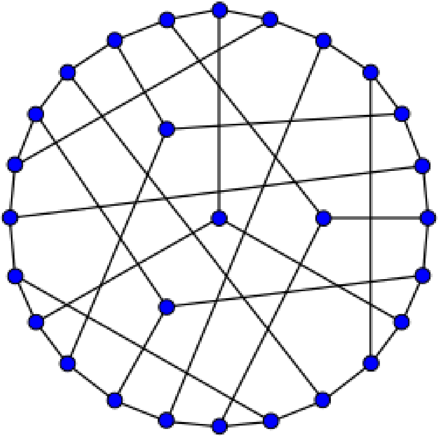


The 5 Counterexamples

(4) The Coxeter graph.



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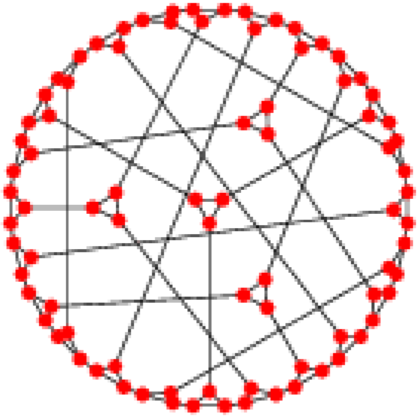

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edge-weighting vertex colourings

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An **edge-weighting vertex colouring** of a graph $G = (V, E)$ is an edge-weighting assignment $f : E \to \mathbb{R}$ such that the accumulated weights at the vertices yields a proper vertex-colouring,

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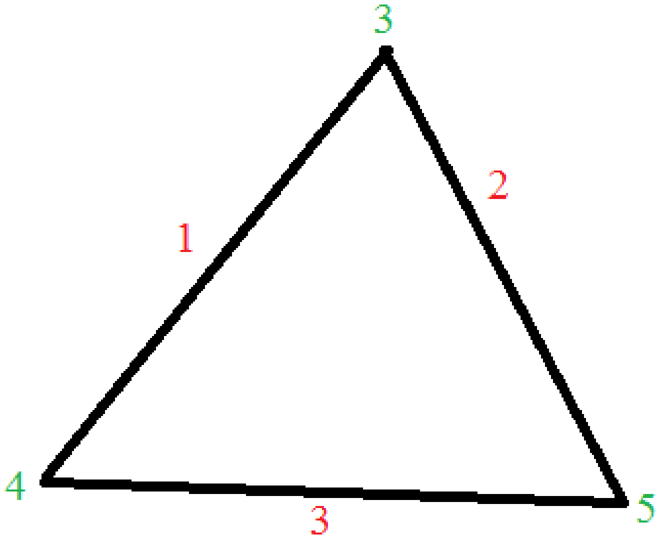
$$a(v) = \sum_{(v,v')\in E} f(v, v'). \qquad (1)$$

An Example

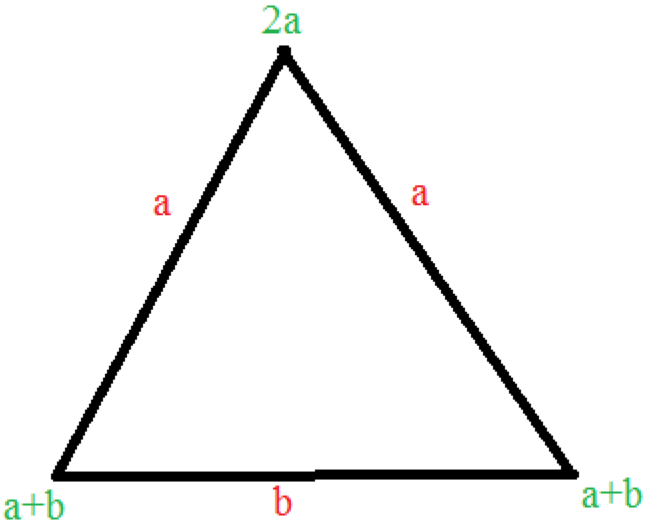


The 1-2-3 Conjecture

Conjecture (M. Karoński, T. Łuczak, A. Thomason (2004, cf. [3]))

For any finite connected graph $G = (V, E)$ that is not K_2, there exists an edge-weighting assignment $f : E \to \{1, 2, 3\}$ which results in an edge-weighting vertex colouring.

The 1-2 Conjecture is False



Known Results

- The 1-2-3 Conjecture was proven for 3-colorable graphs by M. Karoński, T. Łuczak, and A. Thomason in 2004. (cf. [10])

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- The 1-2-3-4-5-6 Conjecture was proven by M. Kalkowski, M. Karoński, and F. Pfender in 2009. (5 pages, cf. [8])

Known Results

- The 1-2-3-4-5 Conjecture was proven by M. Kalkowski, M. Karoński, and F. Pfender in 2010. (3 pages, cf. [9])

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- The 1-2-3-4-5 Conjecture was proven by M. Kalkowski, M. Karoński, and F. Pfender in 2010. (3 pages, cf. [9])
- The 1-2-3-4 Conjecture for regular graphs and the 1-2-3 Conjecture for d-regular graphs with $d \geq 10^8$ was proven by J. Przybyło in 2021. (cf. [12])

The Oriented 1-2-3 "Conjecture"

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An **edge-weighting vertex colouring** of an oriented graph $G = (V, E)$ is an edge-weighting assignment $f : E \to \mathbb{R}$ such that the accumulated weights at the vertices yields a proper vertex-colouring,

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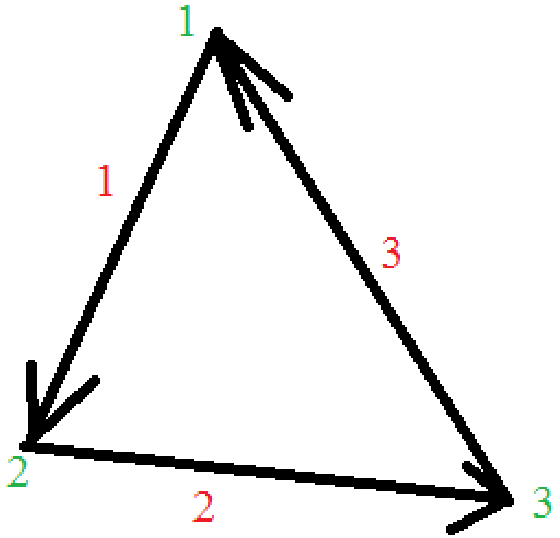
An Example



The Oriented 1-2-3 Theorem

Theorem (O. Baudon, J. Bensmail, E. Sopena (2015, cf. [3]))

Every oriented graph $G = (V, E)$ admits an edge-weighting vertex colouring with an edge-weighting assignment $f : E \to \{1, 2, 3\}$.

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