A generalization of van der Corput's difference theorem and applications to recurrence

Sohail Farhangi (sohailfarhangi.com)
the Ohio State University

A generalization of van der Corput's difference theorem

Definition 1: Let $\mathcal{H}$ be a Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ a unitary operator.

(i) f is a **strongly mixing element** of $(\mathcal{H}, U)$ if for any $g \in \mathcal{H}$ we have $\lim_{n\to\infty}\langle U^n f, g\rangle = 0$.

(ii) f is a **rigid element** of $(\mathcal{H}, U)$ if there exists a sequence $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ for which $\lim_{n\to\infty} ||U^{k_n}f - f|| = 0$.

Definition 2: Let $\mathcal{H}$ be a Hilbert space and $(f_n)_{n=1}^\infty$ a bounded sequence of vectors in $\mathcal{H}$. $(f_n)_{n=1}^\infty$ is a **nearly strongly mixing sequence** if for all weakly permissible triples of the form $((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$, we have

$$\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, g_n\rangle = 0. \tag{1}$$

I.e., $(f_n)_{n=1}^\infty$ is a strongly mixing (rigid) element of $(\mathscr{H}, S)$ where $\mathscr{H}$ is a Hilbert space of sequences of vectors from $\mathcal{H}$ and S is a unitary operator induced by the left shift.

Theorem 3(cf. Corollary 2.2.15 and 2.2.17 in [1]): Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^\infty$ be a bounded sequence of vectors in $\mathcal{H}$. If

$$\lim_{h\to\infty}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle f_{n+h}, f_n\rangle\right| = 0, \text{ or} \tag{2}$$

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}\langle f_{n+h}, f_n\rangle\right| = 0, \tag{3}$$

then $(f_n)_{n=1}^\infty$ is a nearly strongly mixing sequence.

Definition 4: A bounded sequence of vectors $(c_n)_{n=1}^\infty$ in a Hilbert space $\mathcal{H}$ is a **rigid sequence** if for all permissible triples $((c_n)_{n=1}^\infty, (c_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ there exists $(k_m)_{m=1}^\infty \subseteq \mathbb{N}$ for which

$$\lim_{m\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||c_{n+k_m} - c_n||^2 = 0. \tag{4}$$

See end of Definition 2.

Theorem 5: Let $\mathcal{H}$ be a Hilbert space, $(x_n)_{n=1}^\infty$ be a bounded sequence of vectors that is nearly strongly mixing, $(y_n)_{n=1}^\infty$ a bounded sequence of vectors that is rigid, and $(c_n)_{n=1}^\infty$ a bounded sequence of complex numbers that is rigid.

(i) $\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}c_n x_n|| = 0.$

(ii) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_n, y_n\rangle = 0.$

(iii) If $\mathcal{H} = L^2(X,\mu)$, then $\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}y_n x_n|| = 0.$

Remark 6: Analogues of Theorem 2 and Corollary 3 for other levels of the ergodic hierarchy of mixing as well as variations in the context of uniform distribution are discussed in Chapter 2 of [1]. Furthermore, Theorem 2 and Corollary 3 have analogues for any Følner sequences (not just $([1,N])_{N=1}^\infty$).

Recurrence with noncommuting transformations

Theorem 7(Corollary 1.7 in [2]): Let $a : \mathbb{R}_+ \to \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{n\to\infty}\frac{a(n)}{t^{d+\epsilon}} = \lim_{n\to\infty}\frac{t^{d+1}}{a(n)} = \infty. \;(E.g.\; a(n) = n^\beta, \beta \in \mathbb{R}_{\geq 1}\setminus\mathbb{N}) \tag{5}$$

Furthermore, let $(X, \mathscr{B}, \mu)$ be a probability space and $T, S : X \to X$ be measure preserving transformations (not necessarily commuting). Suppose that the system $(X, \mathscr{B}, \mu, T)$ has zero entropy. Then

(i) For every $f, g \in L^\infty(X,\mu)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\cdot S^{\lfloor a(n)\rfloor}g = \mathbb{E}[f|\mathcal{I}_T]\cdot\mathbb{E}[g|\mathcal{I}_S], \tag{6}$$

where the limit is taken in $L^2(X,\mu)$.

(ii) For every $A \in \mathscr{B}$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A\cap S^{-\lfloor a(n)\rfloor}A) \geq \mu(A)^3. \tag{7}$$

Question 8(cf. Problem 2 in [2]): Let $(X, \mathscr{B}, \mu, \{S, T\})$ be as in Theorem 7 and $f, g \in L^\infty(X,\mu)$.

(i) Is it true that the averages

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\cdot S^{p(n)}g \tag{8}$$

converge in $L^2(X,\mu)$ when $p(n) = n$ or $p(n) = n^2$?

(ii) Is it true that for every $A \in \mathscr{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that

$$\mu(A\cap T^{-n}A\cap S^{-p(n)}A) > 0 \tag{9}$$

when $p(n) = n$ or $p(n) = n^2$?

A positive answer to Question 8 is forthcoming in [3].

Theorem 9(cf. Theorem 2.5.5 in [1]): Let $(X, \mathscr{B}, \mu)$ be a probability space and let $T, S : X \to X$ be measure preserving transformations. Suppose that the m.p.s. $(X, \mathscr{B}, \mu, T)$ is rigid, and that the m.p.s. $(X, \mathscr{B}, \mu, S)$ is totally ergodic. Let $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha \pmod 1)_{n=1}^\infty$ is uniformly distributed for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $h \in \mathbb{N}$.

(i) For any $f, g \in L^\infty(X,\mu)$ we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^n f\cdot S^{k_n}g = \mathbb{E}[f|\mathcal{I}_T]\cdot\mathbb{E}[g|\mathcal{I}_S], \tag{10}$$

with convergence taking place in $L^2(X,\mu)$.

(ii) If $A \in \mathscr{B}$ then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A\cap S^{-k_n}A) = \mu(A)\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A) \geq \mu(A)^3. \tag{11}$$

(iii) If $((k_{n+h} - k_n)\alpha)_{n=1}^\infty$ is uniformly distributed in its orbit closure for all $\alpha \in \mathbb{R}$ (e.g. $k_n = \lfloor n\log^2(n)\rfloor$) then (i) holds even when $(X, \mathscr{B}, \mu, S)$ is not ergodic.

Szemerédi's theorem and multiple recurrence

Theorem 10(Furstenberg's correspondence principle): If $A \subseteq \mathbb{N}$ is such that $\overline{d}(A) > 0$, then there exists a measure preserving system $(X, \mathscr{B}, \mu, T)$ and $E \in \mathscr{B}$ with $\mu(E) > 0$ such that for all $n, \ell \in \mathbb{N}$ we have

$$\overline{d}(A\cap(A-n)\cap(A-2n)\cap\cdots\cap(A-\ell n)) \geq \mu(E\cap T^{-n}E\cap T^{-2n}E\cap\cdots\cap T^{-\ell n}E)$$

Theorem 11(Furstenberg, 1977): For any measure preserving system $(X, \mathscr{B}, \mu, T)$ and any $E \in \mathscr{B}$ with $\mu(E) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(E\cap T^{-n}E\cap T^{-2n}E\cap\cdots\cap T^{-\ell n}E) > 0. \tag{12}$$

Sets of K but not $K+1$ Recurrence

Theorem 12(Theorem 1.4 and Corollary 4.4 of [4]): Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$.

(i) If $(X, \mathscr{B}, \mu)$ is a probability space and $T_1, T_2, \cdots, T_{k-1} : X \to X$ are commuting measure preserving transformations, then for any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A\cap T_1^{-n}A\cap T_2^{-n}A\cap\cdots\cap T_{k-1}^{-n}A) > 0. \tag{13}$$

(ii) There exists a m.p.s. $(X, \mathscr{B}, \mu, T)$ and a set $A \in \mathscr{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R$ we have

$$\mu(A\cap T^{-n}A\cap T^{-2n}A\cap\cdots\cap T^{-kn}A) > 0. \tag{14}$$

Theorem 13(Theorem 2.5.8 in [1]): Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k\alpha \in [\frac{1}{4}, \frac{3}{4}]\}$. Let $(X, \mathscr{B}, \mu)$ be a probability space and $S, T_1, T_2, \cdots, T_{k-1} : X \to X$ *commuting* invertible measure preserving transformations for which $(X, \mathscr{B}, \mu, S)$ is rigid[a]. For any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A\cap S^{-n}A\cap T_1^{-n}A\cap T_2^{-n}A\cap\cdots\cap T_{k-1}^{-n}A) > 0. \tag{15}$$

It is worth noting that the example of a m.p.s. $(X, \mathscr{B}, \mu, T)$ given in [4] satisfying Theorem 10(ii) has zero entropy. Consequently, the analogue of Theorem 11 in which S has zero entropy is not true.

References

[1] S. Farhangi. *Topics in ergodic theory and ramsey theory*. PhD dissertation, the Ohio State University, 2022.

[2] N. Frantzikinakis. Furstenberg systems of hardy field sequences and applications. *J. D'Analyse Math.*, 2022 (To Appear).

[3] N. Frantzikinakis and B. Host. Multiple recurrence and convergence without commutativity. *arXiv:2111.01518v2* ***[math.DS]***, 2022.

[4] N. Frantzikinakis, E. Lesigne, and M. Wierdl. Sets of k-recurrence but not $(k+1)$-recurrence. *Ann. Inst. Fourier (Grenoble)*, 56(4):839–849, 2006.

[a] $(X, \mathscr{B}, \mu, S)$ is rigid if there exists a sequence $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ for which $(S^{k_n})_{n=1}^\infty$ converges to the identity operator. If $(X, \mathscr{B}, \mu, S)$ is rigid, then each $f \in L^2(X,\mu)$ is a rigid element of $(L^2(X,\mu), U_S)$. Every group rotation is rigid, and every rigid system has zero (measurable) entropy. "Most" weakly mixing systems $(X, \mathscr{B}, \mu, S)$ are also rigid.