

A generalization of van der Corput's difference theorem and applications to recurrence

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A generalization of van der Corput's difference theorem

Definition 1: Let $\mathcal{H}$ be a Hilbert space and $U : \mathcal{H} \rightarrow \mathcal{H}$ a unitary operator.

- (i) f is a **strongly mixing element** of $(\mathcal{H}, U)$ if for any $g \in \mathcal{H}$ we have $\lim_{n \rightarrow \infty} \langle U^n f, g \rangle = 0$.
- (ii) f is a **rigid element** of $(\mathcal{H}, U)$ if there exists a sequence $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ for which $\lim_{n \rightarrow \infty} \|U^{k_n} f - f\| = 0$.

Definition 2: Let $\mathcal{H}$ be a Hilbert space and $(f_n)_{n=1}^\infty$ a bounded sequence of vectors in $\mathcal{H}$. $(f_n)_{n=1}^\infty$ is a **nearly strongly mixing sequence** if for all **weakly permissible triples** of the form $((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$, we have

$$\lim_{h \rightarrow \infty} \lim_{q \rightarrow \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle f_{n+h}, g_n \rangle = 0. \quad (1)$$

i.e., $(f_n)_{n=1}^\infty$ is a strongly mixing (**rigid**) element of $(\mathcal{H}, S)$ where $\mathcal{H}$ is a Hilbert space of sequences of vectors from $\mathcal{H}$ and S is a unitary operator induced by the left shift.

Theorem 3(cf. Corollary 2.2.15 and 2.2.17 in [1]): Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^\infty$ be a bounded sequence of vectors in $\mathcal{H}$. If

$$\lim_{h \rightarrow \infty} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle f_{n+h}, f_n \rangle \right| = 0, \text{ or} \quad (2)$$

$$\lim_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=1}^N \langle f_{n+h}, f_n \rangle \right| = 0, \quad (3)$$

then $(f_n)_{n=1}^\infty$ is a nearly strongly mixing sequence.

Definition 4: A bounded sequence of vectors $(c_n)_{n=1}^\infty$ in a Hilbert space $\mathcal{H}$ is a **rigid sequence** if for all **permissible triples** $((c_n)_{n=1}^\infty, (c_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ there exists $(k_m)_{m=1}^\infty \subseteq \mathbb{N}$ for which

$$\lim_{m \rightarrow \infty} \lim_{q \rightarrow \infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \|c_{n+k_m} - c_n\|^2 = 0. \quad (4)$$

See end of Definition 2.

Theorem 5: Let $\mathcal{H}$ be a Hilbert space, $(x_n)_{n=1}^\infty$ be a bounded sequence of vectors that is nearly strongly mixing, $(y_n)_{n=1}^\infty$ a bounded sequence of vectors that is rigid, and $(c_n)_{n=1}^\infty$ a bounded sequence of complex numbers that is rigid.

- (i) $\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N c_n x_n \right\| = 0$.
- (ii) $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \langle x_n, y_n \rangle = 0$.
- (iii) If $\mathcal{H} = L^2(X, \mu)$, then $\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=1}^N y_n x_n \right\| = 0$.

Remark 6: Analogues of Theorem 2 and Corollary 3 for other levels of the ergodic hierarchy of mixing as well as variations in the context of uniform distribution are discussed in Chapter 2 of [1]. Furthermore, Theorem 2 and Corollary 3 have analogues for any Følner sequences (not just $([1, N])_{N=1}^\infty$).

Recurrence with noncommuting transformations

Theorem 7(Corollary 1.7 in [2]): Let $a : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a Hardy field function for which there exist some $\epsilon > 0$ and $d \in \mathbb{Z}_+$ satisfying

$$\lim_{n \rightarrow \infty} \frac{a(n)}{t^{d+\epsilon}} = \lim_{n \rightarrow \infty} \frac{t^{d+1}}{a(n)} = \infty. \text{ (E.g. } a(n) = n^\beta, \beta \in \mathbb{R}_{\geq 1} \setminus \mathbb{N} \text{)} \quad (5)$$

Furthermore, let $(X, \mathcal{B}, \mu)$ be a probability space and $T, S : X \rightarrow X$ be measure preserving transformations (not necessarily commuting). Suppose that the system $(X, \mathcal{B}, \mu, T)$ has zero entropy. Then

- (i) For every $f, g \in L^\infty(X, \mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{\lfloor a(n) \rfloor} g = \mathbb{E}[f | \mathcal{I}_T] \cdot \mathbb{E}[g | \mathcal{I}_S], \quad (6)$$

where the limit is taken in $L^2(X, \mu)$.

- (ii) For every $A \in \mathcal{B}$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n} A \cap S^{-\lfloor a(n) \rfloor} A) \geq \mu(A)^3. \quad (7)$$

Question 8(cf. Problem 2 in [2]): Let $(X, \mathcal{B}, \mu, \{S, T\})$ be as in Theorem 7 and $f, g \in L^\infty(X, \mu)$.

- (i) Is it true that the averages

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{p(n)} g \quad (8)$$

converge in $L^2(X, \mu)$ when $p(n) = n$ or $p(n) = n^2$?

- (ii) Is it true that for every $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $n \in \mathbb{N}$ such that

$$\mu(A \cap T^{-n} A \cap S^{-p(n)} A) > 0 \quad (9)$$

when $p(n) = n$ or $p(n) = n^2$?

A positive answer to Question 8 is forthcoming in [3].

Theorem 9(cf. Theorem 2.5.5 in [1]): Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T, S : X \rightarrow X$ be measure preserving transformations. Suppose that the m.p.s. $(X, \mathcal{B}, \mu, T)$ is **rigid**, and that the m.p.s. $(X, \mathcal{B}, \mu, S)$ is totally ergodic. Let $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ be a sequence for which $((k_{n+h} - k_n)\alpha \pmod{1})_{n=1}^\infty$ is uniformly distributed for all $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $h \in \mathbb{N}$.

- (i) For any $f, g \in L^\infty(X, \mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^n f \cdot S^{k_n} g = \mathbb{E}[f | \mathcal{I}_T] \cdot \mathbb{E}[g | \mathcal{I}_S], \quad (10)$$

with convergence taking place in $L^2(X, \mu)$.

- (ii) If $A \in \mathcal{B}$ then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n} A \cap S^{-k_n} A) = \mu(A) \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(A \cap T^{-n} A) \geq \mu(A)^3. \quad (11)$$

- (iii) If $((k_{n+h} - k_n)\alpha)_{n=1}^\infty$ is uniformly distributed in its orbit closure for all $\alpha \in \mathbb{R}$ (e.g. $k_n = \lfloor n \log^2(n) \rfloor$) then (i) holds even when $(X, \mathcal{B}, \mu, S)$ is not ergodic.

Szemerédi's theorem and multiple recurrence

Theorem 10(Furstenberg's correspondence principle): If $A \subseteq \mathbb{N}$ is such that $\bar{d}(A) > 0$, then there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$ and $E \in \mathcal{B}$ with $\mu(E) > 0$ such that for all $n, \ell \in \mathbb{N}$ we have

$$\bar{d}(A \cap (A - n) \cap (A - 2n) \cap \dots \cap (A - \ell n)) \geq \mu(E \cap T^{-n} E \cap T^{-2n} E \cap \dots \cap T^{-\ell n} E)$$

Theorem 11(Furstenberg, 1977): For any measure preserving system $(X, \mathcal{B}, \mu, T)$ and any $E \in \mathcal{B}$ with $\mu(E) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(E \cap T^{-n} E \cap T^{-2n} E \cap \dots \cap T^{-\ell n} E) > 0. \quad (12)$$

Sets of K but not $K + 1$ Recurrence

Theorem 12(Theorem 1.4 and Corollary 4.4 of [4]): Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k \alpha \in [\frac{1}{4}, \frac{3}{4}]\}$.

- (i) If $(X, \mathcal{B}, \mu)$ is a probability space and $T_1, T_2, \dots, T_{k-1} : X \rightarrow X$ are commuting measure preserving transformations, then for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A \cap T_1^{-n} A \cap T_2^{-n} A \cap \dots \cap T_{k-1}^{-n} A) > 0. \quad (13)$$

- (ii) There exists a m.p.s. $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ satisfying $\mu(A) > 0$ such that for all $n \in R$ we have

$$\mu(A \cap T^{-n} A \cap T^{-2n} A \cap \dots \cap T^{-kn} A) > 0. \quad (14)$$

Theorem 13(Theorem 2.5.8 in [1]): Let $k \geq 2$ be an integer and $\alpha \in \mathbb{R}$ be irrational. Let $R = \{n \in \mathbb{N} \mid n^k \alpha \in [\frac{1}{4}, \frac{3}{4}]\}$. Let $(X, \mathcal{B}, \mu)$ be a probability space and $S, T_1, T_2, \dots, T_{k-1} : X \rightarrow X$ commuting invertible measure preserving transformations for which $(X, \mathcal{B}, \mu, S)$ is **rigid**^a. For any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $n \in R$ for which

$$\mu(A \cap S^{-n} A \cap T_1^{-n} A \cap T_2^{-n} A \cap \dots \cap T_{k-1}^{-n} A) > 0. \quad (15)$$

It is worth noting that the example of a m.p.s. $(X, \mathcal{B}, \mu, T)$ given in [4] satisfying Theorem 10(ii) has zero entropy. Consequently, the analogue of Theorem 11 in which S has zero entropy is not true.

References

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^a $(X, \mathcal{B}, \mu, S)$ is rigid if there exists a sequence $(k_n)_{n=1}^\infty \subseteq \mathbb{N}$ for which $(S^{k_n})_{n=1}^\infty$ converges to the identity operator. If $(X, \mathcal{B}, \mu, S)$ is rigid, then each $f \in L^2(X, \mu)$ is a rigid element of $(L^2(X, \mu), \mathcal{U}_S)$. Every group rotation is rigid, and every rigid system has zero (measurable) entropy. "Most" weakly mixing systems $(X, \mathcal{B}, \mu, S)$ are also rigid.