Connections between van der Corput's Difference Theorem and the Ergodic Hierarchy of Mixing Properties.

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The Classical van der Corput Difference Theorem

Definition: A sequence $(x_n)_{n=1}^\infty \subseteq [0,1]$ is uniformly distributed if for any open interval $(a,b) \subseteq [0,1]$ we have

$$\lim_{N\to\infty}\frac{1}{N}\Big|\{1 \le n \le N \mid x_n \in (a,b)\}\Big| = b-a.$$

Theorem(van der Corput)**:** If $(x_n)_{n=1}^\infty \subseteq [0,1]$ is such that $(x_{n+h}-x_n \pmod 1)_{n=1}^\infty$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^\infty$ is itself uniformly distributed.

Hilbertian van der Corput Difference Theorem

vdC1: If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0,$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} x_n|| = 0.$$

Theorem(Poincaré)**:** For any measure preserving system $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A) > 0.$$

Theorem(Furstenberg-Sárközy)**:** For any measure preserving system $(X, \mathscr{B}, \mu, T)$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n^2}A) > 0.$$

Theorem(Furstenberg)**:** For any measure preserving system $(X, \mathscr{B}, \mu, T)$, any $\ell \in \mathbb{N}$, and any $A \in \mathscr{B}$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-\ell n}A) > 0.$$

A Hilbertian van der Corput Difference Theorem Variant

vdC2: If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0,$$

then

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0.$$

Another Hilbertian van der Corput Difference Theorem Variant

vdC3: If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h},x_n\rangle\right| = 0,$$

then

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0.$$

Question: Why is this the variant of van der Corput's Difference Theorem that is used in the proof of Furstenberg's multiple recurrence Theorem?

The Ergodic Hierarchy of Mixing

Definition: Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a measure preserving system. If for every $A, B \in \mathscr{B}$ we have

- $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}B) = \mu(A)\mu(B)$, then $\mathcal{X}$ is ergodic.
- $\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{\lim_{N\to\infty}}\left|\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}B) - \mu(A)\mu(B)\right| = 0$, then $\mathcal{X}$ is weakly mixing.
- $\lim_{n\to\infty}\mu(A\cap T^{-n}B) = \mu(A)\mu(B)$, then $\mathcal{X}$ is strongly mixing.

If there exists a σ-algebra $\mathscr{A}$ such that $\{T^{-n}A \mid A \in \mathscr{A}, n \geq 0\}$ generates $\mathscr{B}$, and for every $A, B \in \mathscr{A}$ and $n \geq 1$ we have $\mu(A\cap T^{-n}B) = \mu(A)\mu(B)$, then $\mathcal{X}$ is Bernoulli.

Symmetry and Mixing

Theorem: Let $\mathcal{X} = (X, \mathscr{B}, \mu, T)$ be a measure preserving system. If for every $A \in \mathscr{B}$ we have

- $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A) = \mu(A)^2$, then $\mathcal{X}$ is ergodic.
- $\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{\lim_{N\to\infty}}|\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-n}A)-\mu(A)^2| = 0$, then $\mathcal{X}$ is weakly mixing.
- $\lim_{n\to\infty}\mu(A\cap T^{-n}A) = \mu(A)^2$, then $\mathcal{X}$ is strongly mixing.

Mixing van der Corput Theorems

Theorem(A. Tserunyan)**:** Let $\mathscr{P}$ be a nice filter. If $(e_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a nice bounded sequence, then

$$\mathscr{P}-\lim_{h\to\infty}\mathscr{P}-\lim_{n\to\infty}\langle e_n, e_{n+h}\rangle = 0 \Rightarrow \mathscr{P}-\lim_{n\to\infty}\langle f, e_n\rangle = 0\ \forall f \in \mathcal{H}.$$

Remark: To see the resemblance with our previous van der Corput Theorems, we first conisder a special case in which $e_n = U^n e_1$, where $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator. In this case, we see that

$$\mathscr{P}-\lim_{h\to\infty}\mathscr{P}-\lim_{n\to\infty}\langle e_n, e_{n+h}\rangle = \mathscr{P}-\lim_{h\to\infty}\mathscr{P}-\lim_{n\to\infty}\langle U^n e_1, U^{n+h}e_1\rangle$$

$$= \mathscr{P}-\lim_{h\to\infty}\mathscr{P}-\lim_{n\to\infty}\langle e_1, U^h e_1\rangle = \mathscr{P}-\lim_{h\to\infty}\langle e_1, U^h e_1\rangle$$

Hilbertian (Cesàro) van der Corput Difference Theorems Revisited

Theorem: Let $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ be a bounded sequences which satisfies any of $(i), (ii)$, and (iii).

(i) $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\langle x_{n+h}, x_n\rangle = 0$ for every $h \in \mathbb{N}$.

(ii) $\lim_{h\to\infty}\overline{\lim_{N\to\infty}}|\frac{1}{N}\sum_{n=1}^N\langle x_{n+h}, x_n\rangle| = 0.$

(iii) $\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^H\overline{\lim_{N\to\infty}}|\frac{1}{N}\sum_{n=1}^N\langle x_{n+h}, x_n\rangle| = 0.$

Then

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^N x_n|| = 0.$$

Bernoulli-Mixing van der Corput's Difference Theorem

MvdC1: If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{\infty}\langle x_{n+h}, x_n\rangle = 0,$$

for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ is a **nearly orthogonal sequence**.

Remark: One way to understand this result is to consider a new Hilbert space $\mathcal{H}'$, whose elements are sequences $(x_n)_{n=1}^{\infty}$ of vectors coming from $\mathcal{H}$. Intuitively, we may let

$$\langle (x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}\rangle_{\mathcal{H}'} = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_n, y_n\rangle$$

be the inner product on $\mathcal{H}'$.

The hypothesis that

$$0 = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{\infty}\langle x_{n+h}, x_n\rangle = \langle U^h(x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}\rangle_{\mathcal{H}'},$$

$$(cf.\mu(A\cap T^{-n}B) = \mu(A)\mu(B)\ \forall\ A, B \in \mathscr{A}, n \geq 1)$$

for every $h \in \mathbb{N}$ verifies that $\{U^h(x_n)_{n=1}^{\infty}\}_{h=0}^{\infty}$ is an orthonormal set in $\mathcal{H}'$, where U denotes the left shift operator. It follows that

$$\sum_{h=0}^{\infty}|\langle U^h(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}\rangle_{\mathcal{H}'}|^2 \leq ||(y_n)_{n=1}^{\infty}||_{\mathcal{H}'}^2\ \forall\ (y_n)_{n=1}^{\infty} \in \mathcal{H}'$$

Corollary: For any totally ergodic measure preserving system $(X, \mathscr{B}, \mu, T)$, any rigid μ-preserving transformation $S : X \to X$, and any $A, B \in \mathscr{B}$, we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(S^{-n}A\cap T^{-n^2}B) = \mu(A)\mu(B).$$

Strong Mixing van der Corput's Difference Theorem

MvdC2: If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{h\to\infty}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{\infty}\langle x_{n+h}, x_n\rangle\right| = 0,$$

then $(x_n)_{n=1}^\infty$ is a **nearly strongly mixing sequence**.

Remark: Let $\mathcal{H}'$, $\langle\cdot,\cdot,\rangle_{\mathcal{H}'}$, and U be as before. The given hypothesis implies

$$0 = \lim_{h\to\infty}\langle U^h(x_n)_{n=1}^\infty, (x_n)_{n=1}^\infty\rangle_{\mathcal{H}'},$$

$$(\text{cf. } \lim_{h\to\infty}\mu(A\cap T^{-n}A) = \mu(A)^2\ \forall\ A\in\mathscr{B})$$

verifies that $\{U^h(x_n)_{n=1}^\infty\}_{h=0}^\infty$ is a strongly mixing sequence in $\mathcal{H}'$. It follows that

$$\lim_{h\to\infty}\langle U^h(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty\rangle_{\mathcal{H}'} = 0\ \forall\ (y_n)_{n=1}^\infty \in \mathcal{H}'.$$

Theorem: Let $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ be a nearly strongly mixing sequence, $(r_n)_{n=1}^\infty \subseteq \mathcal{H}$ a rigid sequence, and $(c_n)_{n=1}^\infty \subseteq \mathbb{C}$ a rigid sequence. We have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\langle x_n, r_n\rangle = 0$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N c_n x_n = 0,$$

with convergence taking place in the weak topology.

Weak Mixing van der Corput's Difference Theorem

MvdC3: If $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H\rightarrow\infty}\frac{1}{H}\sum_{h=1}^H\left|\overline{\lim_{N\rightarrow\infty}}\frac{1}{N}\sum_{n=1}^\infty\langle x_{n+h},x_n\rangle\right| = 0,$$

then $(x_n)_{n=1}^\infty$ is a **nearly weakly mixing sequence**.

Remark: Let $\mathcal{H}'$, $\langle\cdot,\cdot,\rangle_{\mathcal{H}'}$, and U be as before. The given hypothesis implies

$$0 = \lim_{H\rightarrow\infty}\frac{1}{H}\sum_{h=1}^H|\langle U^h(x_n)_{n=1}^\infty,(x_n)_{n=1}^\infty\rangle_{\mathcal{H}'}|,$$

$$\text{(cf. }\lim_{H\rightarrow\infty}\frac{1}{H}\sum_{h=1}^H|\mu(A\cap T^{-n}A)-\mu(A)^2| = 0\ \forall\ A\in\mathscr{B})$$

verifies that $\{U^h(x_n)_{n=1}^\infty\}_{h=0}^\infty$ is a weakly mixing sequence in $\mathcal{H}'$. It follows that

$$\lim_{H\rightarrow\infty}\frac{1}{H}\sum_{h=1}^H|\langle U^h(x_n)_{n=1}^\infty,(y_n)_{n=1}^\infty\rangle_{\mathcal{H}'}| = 0\ \forall\ (y_n)_{n=1}^\infty\in\mathcal{H}'.$$

Theorem: Let $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ be a nearly weakly mixing sequence, $(r_n)_{n=1}^\infty \subseteq \mathcal{H}$ a compact sequence, and $(c_n)_{n=1}^\infty \subseteq \mathbb{C}$ a compact sequence. We have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_n, r_n\rangle = 0$$

and

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}c_nx_n|| = 0.$$

Corollary: For any measure preserving system $(X, \mathscr{B}, \mu, T)$, any $\ell \in \mathbb{N}$, and any compact μ-preserving transformation $S : X \to X$, there exists $n \in \mathbb{N}$ for which

$$\mu(S^{-n}A \cap T^{-n}A \cap T^{-2n}A \cap \cdot \cap T^{-\ell n}A) > 0$$

Ergodic van der Corput's Difference Theorem

MvdC4: If $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a bounded sequence satisfying

$$\lim_{H\to\infty} \overline{\lim_{N\to\infty}} \left| \frac{1}{NH} \sum_{\substack{1\le h\le H \\ 1\le n\le N}} \langle x_{n+h}, x_n \rangle \right| = 0,$$

then $(x_n)_{n=1}^{\infty}$ is a **completely ergodic sequence**.

Remark: Let $\mathcal{H}'$, $\langle\cdot,\cdot,\rangle_{\mathcal{H}'}$, and U be as before. The given hypothesis implies

$$0 = \lim_{h\to\infty} \langle U^h(x_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty} \rangle_{\mathcal{H}'},$$

$$(\text{cf. } \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A) = \mu(A)^2 \ \forall\ A \in \mathscr{B})$$

verifies that $\{U^h(x_n)_{n=1}^{\infty}\}_{h=0}^{\infty}$ is a ergodic sequence in $\mathcal{H}'$. It follows that

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\langle U^h(x_n)_{n=1}^{\infty},(y_n)_{n=1}^{\infty}\rangle_{\mathcal{H}'}=0\ \forall\ (y_n)_{n=1}^{\infty}\in\mathcal{H}'.$$

Theorem: Let $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ be a completely ergodic sequence, $(r_n)_{n=1}^\infty \subseteq \mathcal{H}$ a invariant sequence, and $(c_n)_{n=1}^\infty \subseteq \mathbb{C}$ a invariant sequence. We have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle x_n, r_n\rangle = 0$$

and

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}c_n x_n|| = 0.$$

The live talk had ended on this slide.

Mixing and Uniform Distribution

Definition: Let us recall that $\mathbb{C}$ is a Hilbert space when equipped with the inner product $\langle c_1, c_2\rangle = c_1\overline{c_2}$. By abuse of notation, let $C_0(\mathbb{T})$ denote the set of continuous complex valued functions f on $\mathbb{T}$ with $\int_{\mathbb{T}} f dm = 0$. Let $(x_n)_{n=1}^{\infty} \subseteq \mathbb{T}^d$ be a sequence.

(1) $(x_n)_{n=1}^{\infty}$ is a **e-sequence** if for every $f \in \mathbb{T}$, $f(x_n))_{n=1}^{\infty}$ is a completely ergodic sequence.

(2) $(x_n)_{n=1}^{\infty}$ is a **wm-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^{\infty}$ is a nearly weakly mixing sequence.

(3) $(x_n)_{n=1}^{\infty}$ is a **mm-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^{\infty}$ is a nearly mildly mixing sequence.

(4) $(x_n)_{n=1}^{\infty}$ is a **sm-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^{\infty}$ is a nearly strongly mixing sequence.

(5) $(x_n)_{n=1}^{\infty}$ is a **o-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^{\infty}$ is a nearly orthogonal sequence.

Notions Complementary to Mixing

Let $A := (n_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ have positive lower natural density.

(1) A is **invariant** if $d(A \cap (A-1)) = 0$.

(2) A is **compact** if $(\mathbb{1}_A(n))_{n=1}^{\infty}$ is a compact sequence of complex numbers.

(3) A is **rigid** if $(\mathbb{1}_A(n))_{n=1}^{\infty}$ is a rigid sequence of complex numbers.

(4) A has **zero-entropy** if $(\mathbb{1}_A(n))_{n=1}^{\infty}$ is a zero-entropy sequence of complex numbers.

A Consequence of the Pointwise Ergodic Theorem

Definition: $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is **totally uniformly distributed** if for any $a, b \in \mathbb{N}$ the sequence $(x_{an+b})_{n=1}^\infty$ is totally uniformly distributed.

Fact: If $\mathcal{X} := ([0,1]^d, \mathscr{B}, m, T)$ is an ergodic m.p.s. then for Lebesgue a.e. $x \in [0,1]^d$, the sequence $(T^n x)_{n=1}^\infty$ is uniformly distributed. If $\mathcal{X}$ is totally ergodic, then for Lebesgue a.e. $x \in [0,1]^d$, the sequence $(T^n x)_{n=1}^\infty$ is totally uniformly distributed.

Remark: The points $x \in [0,1]$ for which the fact holds are precisely that x that are generic for T.

The Consequence of Higher Order Pointwise Ergodic Theorems

Theorem: Let $\mathcal{X} := ([0,1]^d, \mathscr{B}, m, T)$ be an ergodic m.p.s. and let $x \in [0,1]^d$ be a generic point for T.

(1) If $\mathcal{X}$ is weakly mixing, then $(T^n x)_{n=1}^{\infty}$ is a wm-sequence.

(1.5) If $\mathcal{X}$ is mildly mixing, then $(T^n x)_{n=1}^{\infty}$ is a mm-sequence.

(2) If $\mathcal{X}$ is strongly mixing, then $(T^n x)_{n=1}^{\infty}$ is a sm-sequence.

(3) $(T^n x)_{n=1}^{\infty}$ is **not** an o-sequence.

Discrepancy

Given a sequence $(x_n)_{n=1}^N \subseteq [0,1]^d$, the **discrepancy** of $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is denoted by $D_N((x_n)_{n=1}^N)$ and given by

$$D_N((x_n)_{n=1}^N) = \sup_{B\in\mathcal{R}} \left|\frac{1}{N}|\{1 \le n \le N \mid x_n \in B\}| - m^d(B)\right|, \tag{1}$$

where $\mathcal{R}$ denotes the collection of all rectangular prisms contained in $[0,1]^d$. For an infinite sequence $(x_n)_{n=1}^\infty \subseteq [0,1]^d$, we let

$$\overline{D}((x_n)_{n=1}^\infty) = \overline{\lim_{N\to\infty}} D_N((x_n)_{n=1}^N), \text{ and we let} \tag{2}$$

$$D((x_n)_{n=1}^\infty, (N_q)_{q=1}^\infty) = \lim_{q\to\infty} D_{N_q}((x_n)_{n=1}^{N_q}), \tag{3}$$

provided that the limit exists.

Ergodic van der Corput

Theorem: $\{x_{(n,m)}\}_{(n,m)\in\mathbb{N}^2} \subseteq \mathbb{T}$ is uniformly distributed if and only if for every $k \in \mathbb{N}$, we have

$$\lim_{K\to\infty} \sup_{N,M\geq K} \Big|\frac{1}{NM} \sum_{\substack{1\leq n\leq N \\ 1\leq m\leq M}} e^{2\pi i k x_{n,m}}\Big| = 0. \tag{4}$$

Theorem: If $(x_n)_{n=1}^{\infty} \subseteq \mathbb{T}$ is such that $(x_{n+h} - x_n)_{(n,h)\in\mathbb{N}^2}$ is uniformly distributed, then $(x_n)_{n=1}^{\infty}$ is also uniformly distributed.

'Theorem': If $(x_n)_{n=1}^{\infty} \subseteq \mathbb{T}$ is such that $(x_{n+h} - x_n)_{(n,h)\in\mathbb{N}^2}$ is uniformly distributed, then $(x_{n_k})_{k=1}^{\infty}$ is uniformly distributed for any invariant sequence $(n_k)_{k=1}^{\infty}$.

Weakly Mixing van der Corput

Theorem: Let $(x_n)_{n=1}^\infty \subseteq [0,1]$ be a sequence for which

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{D}((x_{n+h}-x_n)_{n=1}^\infty)=0. \tag{5}$$

Then $(x_n)_{n=1}^\infty$ is a wm-sequence.

Theorem: $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is a wm-sequence if and only if $(x_{n_k})_{k=1}^\infty$ is uniformly distributed whenever $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ is compact.

Mildly Mixing van der Corput

Theorem: Let $(x_n)_{n=1}^\infty \subseteq [0,1]$ be a sequence for which

$$\text{IP}^* - \lim_{h\to\infty} \overline{D}((x_{n+h}-x_n)_{n=1}^\infty) = 0. \tag{6}$$

Then $(x_n)_{n=1}^\infty$ is a mm-sequence.

'Theorem': $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is a mm-sequence if and only if $(x_{n_k})_{k=1}^\infty$ is uniformly distributed whenever $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ is rigid.

Strongly Mixing van der Corput

Theorem: Let $(x_n)_{n=1}^\infty \subseteq [0,1]$ be a sequence for which

$$\lim_{h\to\infty} \overline{D}((x_{n+h}-x_n)_{n=1}^\infty) = 0. \tag{7}$$

Then $(x_n)_{n=1}^\infty$ is a sm-sequence.

Nearly Orthogonal van der Corput and A Counter Example

Theorem: $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is an o-sequence if and only if for each $h \in \mathbb{N}$ $(x_n, x_{n+h})_{n=1}^\infty \subseteq [0,1]^{2d}$ is uniformly distributed.

Example: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be artbirary and consider the sequence $(x_n)_{n=1}^\infty$ defined by $x_n = n^2\alpha \pmod 1$ if n is odd and $x_n = 2(n-1)^2\alpha \pmod 1$ if n is even.

(1) $(x_n)_{n=1}^\infty$ is **not** an o-sequence.

(2) For each $h \in \mathbb{N}$ the sequence $(x_{n+h} - x_n)_{n=1}^\infty$ is uniformly distributed.

A Conjecture

Conjecture: If $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is such that $(x_{n+h}-x_n)_{n=1}^\infty$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_{n_k})_{k=1}^\infty$ is uniformly distribtued for any zero-entropy sequence $(n_k)_{k=1}^\infty$.

If and only If Weakly Mixing van der Corput

Theorem: For $(x_n)_{n=1}^\infty \subseteq [0,1]^{d_1}$ the following are equivalent:

(1) $(x_n)_{n=1}^\infty$ is a wm-sequence.

(2) For any uniformly distributed $(y_n)_{n=1}^\infty \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^\infty\}_{h=1}^\infty, (N_q)_{q=1}^\infty)$ is a permissible pair, we have

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} D((x_n, y_{n+h})_{n=1}^\infty, (N_q)_{q=1}^\infty) = 0. \tag{8}$$

(3) For any $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ for which $(\{(x_n, x_{n+h})_{n=1}^\infty\}_{h=1}^\infty, (N_q)_{q=1}^\infty)$ is a permissible pair, we have

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} D((x_n, x_{n+h})_{n=1}^\infty, (N_q)_{q=1}^\infty) = 0. \tag{9}$$

(4) For any $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ that makes $(\{(x_{n+h} - x_n)_{n=1}^\infty\}_{h=1}^\infty, (N_q)_{q=1}^\infty))$ a permissible pair, we have

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} D((x_{n+h} - x_n)_{n=1}^\infty, (N_q)_{q=1}^\infty) = 0. \tag{10}$$

If and only If Mildly Mixing van der Corput

'Theorem': For $(x_n)_{n=1}^{\infty} \subseteq [0,1]^{d_1}$ the following are equivalent:

(1) $(x_n)_{n=1}^{\infty}$ is a mm-sequence.

(2) For any uniformly distributed $(y_n)_{n=1}^{\infty} \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\text{IP}^* - \lim_{h\to\infty} D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{11}$$

(3) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\text{IP}^* - \lim_{h\to\infty} D((x_n, x_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{12}$$

(4) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ that makes $((\{(x_{n+h} - x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty}))$ a permissible pair, we have

$$\text{IP}^* - \lim_{h\to\infty} D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{13}$$

If and only If Strongly Mixing van der Corput

Theorem: For $(x_n)_{n=1}^{\infty} \subseteq [0,1]^{d_1}$ the following are equivalent:

(1) $(x_n)_{n=1}^{\infty}$ is a sm-sequence.

(2) For any uniformly distributed $(y_n)_{n=1}^{\infty} \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\lim_{h\to\infty} D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{14}$$

(3) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\lim_{h\to\infty} D((x_n, x_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{15}$$

(4) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ that makes $((\{(x_{n+h} - x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty}))$ a permissible pair, we have

$$\lim_{h\to\infty} D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{16}$$