Connections between van der Corput's Difference Theorem and the Hierarchy of Mixing Properties Part 3.

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The Classical van der Corput Difference Theorem

Theorem: If $(x_n)_{n=1}^\infty \subseteq \mathbb{T}^d$ is a sequence for which $(x_{n+h}-x_n)_{n=1}^\infty$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^\infty$ is itself a uniformly distributed sequence.

Example: For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ the sequence $(n^2\alpha \pmod 1)_{n=1}^\infty$ is uniformly distributed in $[0,1]$.

Some New Notions

Definition: Let us recall that $\mathbb{C}$ is a Hilbert space when equipped with the inner product $\langle c_1, c_2\rangle = c_1\overline{c_2}$. By abuse of notation, let $C_0(\mathbb{T})$ denote the set of continuous complex valued functions f on $\mathbb{T}$ with $\int_{\mathbb{T}} f dm = 0$. Let $(x_n)_{n=1}^\infty \subseteq \mathbb{T}^d$ be a sequence.

(1) $(x_n)_{n=1}^\infty$ is a **wm-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^\infty$ is a nearly weakly mixing sequence.

(1.5) $(x_n)_{n=1}^\infty$ is a **mm-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^\infty$ is a nearly mildly mixing sequence.

(2) $(x_n)_{n=1}^\infty$ is a **sm-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^\infty$ is a nearly strongly mixing sequence.

(3) $(x_n)_{n=1}^\infty$ is a **o-sequence** if for every $f \in C_0(\mathbb{T})$, $(f(x_n))_{n=1}^\infty$ is a nearly orthogonal sequence.

Notions Complementary to Mixing

Let $A := (n_k)_{k=1}^\infty \subseteq \mathbb{N}$ have positive lower natural density.

(1) A is **invariant** if $d(A \cap (A-1)) = 0$.

(2) A is **compact** if $(\mathbb{1}_A(n))_{n=1}^\infty$ is a compact sequence of complex numbers.

(3) A is **rigid** if $(\mathbb{1}_A(n))_{n=1}^\infty$ is a rigid sequence of complex numbers.

(4) A has **zero-entropy** if $(\mathbb{1}_A(n))_{n=1}^\infty$ is a zero-entropy sequence of complex numbers.

A Consequence of the Pointwise Ergodic Theorem

Definition: $(x_n)_{n=1}^{\infty} \subseteq [0,1]^d$ is **totally uniformly distributed** if for any $a, b \in \mathbb{N}$ the sequence $(x_{an+b})_{n=1}^{\infty}$ is totally uniformly distributed.

Fact: If $\mathcal{X} := ([0,1]^d, \mathscr{B}, m, T)$ is an ergodic m.p.s. then for Lebesgue a.e. $x \in [0,1]^d$, the sequence $(T^n x)_{n=1}^{\infty}$ is uniformly distributed. If $\mathcal{X}$ is totally ergodic, then for Lebesgue a.e. $x \in [0,1]^d$, the sequence $(T^n x)_{n=1}^{\infty}$ is totally uniformly distributed.

Remark: The points $x \in [0,1]$ for which the fact holds are precisely that x that are generic for T.

The Consequence of Higher Order Pointwise Ergodic Theorems

Theorem: Let $\mathcal{X} := ([0,1]^d, \mathscr{B}, m, T)$ be an ergodic m.p.s. and let $x \in [0,1]^d$ be a generic point for T.

(1) If $\mathcal{X}$ is weakly mixing, then $(T^n x)_{n=1}^\infty$ is a wm-sequence.

(1.5) If $\mathcal{X}$ is mildly mixing, then $(T^n x)_{n=1}^\infty$ is a mm-sequence.

(2) If $\mathcal{X}$ is strongly mixing, then $(T^n x)_{n=1}^\infty$ is a sm-sequence.

(3) $(T^n x)_{n=1}^\infty$ is **not** an o-sequence.

Discrepancy

Given a sequence $(x_n)_{n=1}^N \subseteq [0,1]^d$, the **discrepancy** of $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is denoted by $D_N((x_n)_{n=1}^N)$ and given by

$$D_N((x_n)_{n=1}^N) = \sup_{B \in \mathcal{R}} \left| \frac{1}{N} |\{1 \le n \le N \mid x_n \in B\}| - m^d(B) \right|, \tag{1}$$

where $\mathcal{R}$ denotes the collection of all rectangular prisms contained in $[0,1]^d$. For an infinite sequence $(x_n)_{n=1}^\infty \subseteq [0,1]^d$, we let

$$\overline{D}((x_n)_{n=1}^\infty) = \overline{\lim_{N\to\infty}} D_N((x_n)_{n=1}^N), \text{ and we let} \tag{2}$$

$$D((x_n)_{n=1}^\infty, (N_q)_{q=1}^\infty) = \lim_{q\to\infty} D_{N_q}((x_n)_{n=1}^{N_q}), \tag{3}$$

provided that the limit exists.

Ergodic van der Corput

Theorem: $\{x_{(n,m)}\}_{(n,m)\in\mathbb{N}^2} \subseteq \mathbb{T}$ is uniformly distributed if and only if for every $k \in \mathbb{N}$, we have

$$\lim_{K\to\infty} \sup_{N,M\geq K} \left|\frac{1}{NM} \sum_{\substack{1\leq n\leq N \\ 1\leq m\leq M}} e^{2\pi i k x_{n,m}}\right| = 0. \tag{4}$$

Theorem: If $(x_n)_{n=1}^{\infty} \subseteq \mathbb{T}$ is such that $(x_{n+h} - x_n)_{(n,h)\in\mathbb{N}^2}$ is uniformly distributed, then $(x_n)_{n=1}^{\infty}$ is also uniformly distributed.

'Theorem': If $(x_n)_{n=1}^{\infty} \subseteq \mathbb{T}$ is such that $(x_{n+h} - x_n)_{(n,h)\in\mathbb{N}^2}$ is uniformly distributed, then $(x_{n_k})_{k=1}^{\infty}$ is uniformly distributed for any invariant sequence $(n_k)_{k=1}^{\infty}$.

Weakly Mixing van der Corput

Theorem: Let $(x_n)_{n=1}^\infty \subseteq [0,1]$ be a sequence for which

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\overline{D}((x_{n+h}-x_n)_{n=1}^\infty)=0. \tag{5}$$

Then $(x_n)_{n=1}^\infty$ is a wm-sequence.

Theorem: $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is a wm-sequence if and only if $(x_{n_k})_{k=1}^\infty$ is uniformly distributed whenever $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ is compact.

Mildly Mixing van der Corput

Theorem: Let $(x_n)_{n=1}^\infty \subseteq [0,1]$ be a sequence for which

$$\text{IP}^* - \lim_{h\to\infty} \overline{D}((x_{n+h}-x_n)_{n=1}^\infty) = 0. \tag{6}$$

Then $(x_n)_{n=1}^\infty$ is a mm-sequence.

'Theorem': $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is a mm-sequence if and only if $(x_{n_k})_{k=1}^\infty$ is uniformly distributed whenever $(n_k)_{k=1}^\infty \subseteq \mathbb{N}$ is rigid.

Strongly Mixing van der Corput

Theorem: Let $(x_n)_{n=1}^\infty \subseteq [0,1]$ be a sequence for which

$$\lim_{h\to\infty} \overline{D}((x_{n+h}-x_n)_{n=1}^\infty) = 0. \tag{7}$$

Then $(x_n)_{n=1}^\infty$ is a sm-sequence.

Nearly Orthogonal van der Corput and A Counter Example

Theorem: $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is an o-sequence if and only if for each $h \in \mathbb{N}$ $(x_n, x_{n+h})_{n=1}^\infty \subseteq [0,1]^{2d}$ is uniformly distributed.

Example: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be artbirary and consider the sequence $(x_n)_{n=1}^\infty$ defined by $x_n = n^2\alpha \pmod 1$ if n is odd and $x_n = 2(n-1)^2\alpha \pmod 1$ if n is even.

(1) $(x_n)_{n=1}^\infty$ is **not** an o-sequence.

(2) For each $h \in \mathbb{N}$ the sequence $(x_{n+h} - x_n)_{n=1}^\infty$ is uniformly distributed.

A Conjecture

Conjecture: If $(x_n)_{n=1}^\infty \subseteq [0,1]^d$ is such that $(x_{n+h}-x_n)_{n=1}^\infty$ is uniformly distributed for every $h \in \mathbb{N}$, then $(x_{n_k})_{k=1}^\infty$ is uniformly distribtued for any zero-entropy sequence $(n_k)_{k=1}^\infty$.

If and only If Weakly Mixing van der Corput

Theorem: For $(x_n)_{n=1}^\infty \subseteq [0,1]^{d_1}$ the following are equivalent:

(1) $(x_n)_{n=1}^\infty$ is a wm-sequence.

(2) For any uniformly distributed $(y_n)_{n=1}^\infty \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^\infty\}_{h=1}^\infty, (N_q)_{q=1}^\infty)$ is a permissible pair, we have

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} D((x_n, y_{n+h})_{n=1}^\infty, (N_q)_{q=1}^\infty) = 0. \tag{8}$$

(3) For any $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ for which $(\{(x_n, x_{n+h})_{n=1}^\infty\}_{h=1}^\infty, (N_q)_{q=1}^\infty)$ is a permissible pair, we have

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} D((x_n, x_{n+h})_{n=1}^\infty, (N_q)_{q=1}^\infty) = 0. \tag{9}$$

(4) For any $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ that makes $(\{(x_{n+h} - x_n)_{n=1}^\infty\}_{h=1}^\infty, (N_q)_{q=1}^\infty))$ a permissible pair, we have

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} D((x_{n+h} - x_n)_{n=1}^\infty, (N_q)_{q=1}^\infty) = 0. \tag{10}$$

If and only If Mildly Mixing van der Corput

'Theorem': For $(x_n)_{n=1}^{\infty} \subseteq [0,1]^{d_1}$ the following are equivalent:

(1) $(x_n)_{n=1}^{\infty}$ is a mm-sequence.

(2) For any uniformly distributed $(y_n)_{n=1}^{\infty} \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\mathrm{IP}^* - \lim_{h\to\infty} D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{11}$$

(3) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\mathrm{IP}^* - \lim_{h\to\infty} D((x_n, x_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{12}$$

(4) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ that makes $((\{(x_{n+h} - x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty}))$ a permissible pair, we have

$$\mathrm{IP}^* - \lim_{h\to\infty} D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{13}$$

If and only If Strongly Mixing van der Corput

Theorem: For $(x_n)_{n=1}^{\infty} \subseteq [0,1]^{d_1}$ the following are equivalent:

(1) $(x_n)_{n=1}^{\infty}$ is a sm-sequence.

(2) For any uniformly distributed $(y_n)_{n=1}^{\infty} \subseteq [0,1]^{d_2}$ and $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, y_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\lim_{h\to\infty} D((x_n, y_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{14}$$

(3) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ for which $(\{(x_n, x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible pair, we have

$$\lim_{h\to\infty} D((x_n, x_{n+h})_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{15}$$

(4) For any $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ that makes $((\{(x_{n+h} - x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}, (N_q)_{q=1}^{\infty}))$ a permissible pair, we have

$$\lim_{h\to\infty} D((x_{n+h} - x_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty}) = 0. \tag{16}$$

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