Connections between van der Corput's Difference Theorem and the Hierarchy of Mixing Properties Part 2.

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Standing Assumption

For the rest of this presentation, we will assume that any sequence $(x_n)_{n=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ satisfies

$$\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}||x_n||^2 < \infty. \tag{1}$$

Nearly Weakly Mixing sequences

Definition 1: $(f_n)_{n=1}^\infty$ is a **nearly weakly mixing sequence** if for any permissible triple of the form $((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$, we have

$$\lim_{H\to\infty}\frac{1}{H}\sum_{h=1}^{H}\left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, g_n\rangle\right| = 0. \tag{2}$$

Example: If $(X, \mathscr{B}, \mu, T)$ is a weakly mixing m.p.s., then for any $f \in L^2(X, \mu)$, $(U_T^n f)_{n=1}^\infty$ is a nearly weakly mixing sequence.

Weakly Mixing van der Corput

MvdC3: Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^\infty \subseteq \mathcal{H}$ be a bounded sequence. If

$$\overline{\lim_{H\to\infty}} \sum_{h=1}^{H} \frac{1}{H} \left| \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \langle f_{n+h}, f_n \rangle \right| = 0, \tag{3}$$

then $(f_n)_{n=1}^\infty$ is an nearly weakly mixing sequence.

Nearly Strongly Mixing Sequences

Definition 2: $(f_n)_{n=1}^\infty$ is a **nearly strongly mixing sequence** if for any permissible triple of the form $((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$, we have

$$\lim_{h\to\infty}\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, g_n\rangle = 0. \tag{4}$$

Example: If $(X, \mathscr{B}, \mu, T)$ is a strongly mixing m.p.s., then for any $f \in L^2(X,\mu)$, $(U_T^n f)_{n=1}^\infty$ is a nearly strongly mixing sequence.

Strongly Mixing van der Corput

MvdC2: Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequence. If

$$\overline{\lim_{H\to\infty}}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\langle f_{n+h}, f_n\rangle\right| = 0, \tag{5}$$

then $(f_n)_{n=1}^{\infty}$ is an nearly strongly mixing sequence.

Nearly Orthogonal Sequences

Definition 2: $(f_n)_{n=1}^\infty$ is a **nearly orthogonal sequence** if for any permissible triple of the form $((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$, we have

$$\sum_{h=0}^\infty |\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, g_n\rangle|^2 \le \lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}||g_n||^2. \tag{6}$$

'Orthogonally Mixing' van der Corput

MvdC1: Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequence. If

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n=1}^{N} \langle f_{n+h}, f_n \rangle = 0, \tag{7}$$

for every $h \in \mathbb{N}$, then $(f_n)_{n=1}^{\infty}$ is an nearly orthogonal sequence.

A Sad Fact of Life

Sad Fact: For any sequence of complex numbers $(x_n)_{n=1}^{\infty}$ satisfying

$$\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}|x_n| < \infty, \tag{8}$$

there exists a sequence $(y_n)_{n=1}^{\infty} \subseteq \mathbb{S}^1$ for which

$$\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}x_{n+h}\overline{y_n} = \overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}|x_n|, \tag{9}$$

for every $h \geq 0$.

Characterization of Nearly Weakly Mixing

Theorem 1: $(f_n)_{n=1}^\infty$ is a nearly weakly mixing sequence if and only if for any permissible triple of the form $((f_n)_{n=1}^\infty, (f_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ we have

$$\lim_{H\to\infty}\sum_{h=1}^{H}\frac{1}{H}\left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, f_n\rangle\right| = 0. \tag{10}$$

Characterization of Nearly Strongly Mixing

Theorem 2: $(f_n)_{n=1}^\infty$ is a nearly strongly mixing sequence if and only if for any permissible triple of the form $((f_n)_{n=1}^\infty, (f_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ we have

$$\lim_{h\to\infty}\left|\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, f_n\rangle\right| = 0. \tag{11}$$

Characterization of Nearly Orthogonal

Theorem 3: $(f_n)_{n=1}^\infty$ is a nearly orthogonal sequence if and only if for any permissible triple of the form $((f_n)_{n=1}^\infty, (f_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ we have

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, f_n\rangle = 0, \tag{12}$$

for every $h \in \mathbb{N}$.

A Brief Review of Mixing

Mixing Notion	Ergodicity	Weak Mixing	Mild Mixing	Strong Mixing
Complementary Notion	Invariance	Compactness	Rigidity	???

Mixing Notion	K-Mixing	Bernoulli
Complementary Notion	Zero Entropy	???

Compact Sequences

Definition: $(c_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a **compact sequence** if for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ for which

$$\sup_{m\in\mathbb{N}} \min_{1\leq k\leq K} \overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=1}^{N} ||c_{n+m} - c_{n+k}||^2 < \epsilon. \tag{13}$$

Example: If $(X, \mathscr{B}, \mu, T)$ is a discrete spectrum m.p.s., then for any $f \in L^2(X,\mu)$, $(U_T^n f)_{n=1}^\infty$ is a compact sequence.

Rigid Sequences

Definition: $(r_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a **rigid sequence** if there exists $(k_m)_{m=1}^\infty \subseteq \mathbb{N}$ for which

$$\lim_{m\to\infty} \overline{\lim_{N\to\infty}} \frac{1}{N}\sum_{n=1}^{N} ||r_{n+k_m} - r_n||^2 < \epsilon. \tag{14}$$

Example: If $(X, \mathscr{B}, \mu, T)$ is a m.p.s., and $f \in L^2(X, \mu)$ is rigid with respect to U_T, then $(U_T^n f)_{n=1}^\infty$ is a rigid sequence.

Zero Entropy Sequences

Definition: $(z_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ is a **zero entropy sequence** if there exists a dynamical system (X, T) with zero topological entropy, a $x \in X$, and a $f : X \to \mathcal{H}$ that is continuous with respect to the weak topology of $\mathcal{H}$ for which $z_n = f(T^n x)$.

The Orthogonality between Compactness and Weak Mixing

Theorem 4: If $(w_n)_{n=1}^\infty \subseteq \mathcal{H}$ is nearly weakly mixing and $(c_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a compact sequence, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle w_n, c_n\rangle = 0. \tag{15}$$

Conjecture 1: If $(w_n)_{n=1}^\infty \subseteq \mathcal{H}$ is nearly weakly mixing and $(c_n')_{n=1}^\infty$ is a compact sequence of complex numbers then

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} c_n' w_n|| = 0. \tag{16}$$

The Orthogonality between Rigidity and Mild Mixing

Theorem 5: If $(s_n)_{n=1}^\infty \subseteq \mathcal{H}$ is nearly strongly (mildly) mixing and $(r_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a rigid sequence, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle s_n, r_n\rangle = 0. \tag{17}$$

Conjecture 2: If $(s_n)_{n=1}^\infty \subseteq \mathcal{H}$ is nearly strongly (mildly) mixing and $(r'_n)_{n=1}^\infty$ is a rigid sequence of complex numbers, then

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}r'_n s_n|| = 0. \tag{18}$$

At Long Last.... An Application!!!!

Let $(X, \mathscr{B}, \mu)$ be a probability space.
Exercise 1: Show that if $T : X \to X$ is a totally ergodic m.p.t., and $p : \mathbb{Z} \to \mathbb{Z}$ is a polynomial of degree at least 2 then $(\mathbb{1}_{T^n A} - \mu(T^n A))_{n=1}^{\infty}$ is a nearly orthogonal sequence in $L^2(X, \mu)$.

Exercise 2: Show that if $T : X \to X$ is a totally ergodic m.p.t. and $S : X \to X$ is a discrete spectrum m.p.t., then for any polynomial $p : \mathbb{Z} \to \mathbb{Z}$ of degree at least 2 and any $A, B \in \mathscr{B}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(T^{p(n)}(A) \cap S^n(B)) = \mu(A)\mu(B). \tag{19}$$

Remark: Can discrete spectrum be relaxed to rigidity?

The Orthogonality between Zero Entropy and K-Mixing

Conjecture 3: If $(o_n)_{n=1}^\infty \subseteq \mathcal{H}$ is nearly orthgonal (K-mixing) and $(z_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a 'zero entropy' sequence, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle o_n, z_n\rangle = 0. \tag{20}$$

Conjecture 4: If $(o_n)_{n=1}^\infty \subseteq \mathcal{H}$ is nearly orthgonal (K-mixing) and $(z'_n)_{n=1}^\infty$ is a zero entropy sequence of complex numbers, then

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{\infty} z'_n o_n|| = 0. \tag{21}$$

Mildly Mixing van der Corput

MvdC6: Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequence. If

$$\text{IP}^* - \lim_{h\to\infty}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\langle f_{n+h}, f_n\rangle\right| = 0, \tag{22}$$

then $(f_n)_{n=1}^{\infty}$ is an nearly mildly mixing sequence. In particular, if $((f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is a permissible triple, then

$$\text{IP}^* - \lim_{h\to\infty}\lim_{q\to\infty}|\frac{1}{N_q}\sum_{n=1}^{N_q}\langle f_{n+h}, g_n\rangle| = 0. \tag{23}$$

K-Mixing van der Corput

Exercise 3: Formulate and prove a van der Corput difference Theorem corresponding to K-mixing.

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