Connections between van der Corput's Difference Theorem and the Hierarchy of Mixing Properties.

Sohail Farhangi

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Some Existing Variants of van der Corput's Difference Theorem

vdC1: Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a bounded sequence of vectors. If

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle = 0,$$

for every $h \in \mathbb{N}$, then

$$\lim_{N\to\infty} ||\frac{1}{N}\sum_{n=1}^{N} x_n|| = 0.$$

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vdC2: Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a bounded sequence of vectors. If

$$\overline{\lim_{h\to\infty}}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{N}\langle x_{n+h}, x_n\rangle\right| = 0,$$

then

$$\lim_{N\to\infty}||\frac{1}{N}\sum_{n=1}^{N}x_n|| = 0.$$

Some Existing Variants of van der Corput's Difference Theorem

vdC3: Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a bounded sequence of vectors. If

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Generalizing Existing Variants of van der Corput's Difference Theorem

MvdC1: Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ a bounded sequence of vectors. If

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{\infty}\langle x_{n+h}, x_n\rangle = 0,$$

for every $h \in \mathbb{N}$, then $(x_n)_{n=1}^\infty$ is a **nearly orthogonal sequence**.

Intuitively, $(x_n)_{n=1}^\infty \subseteq \mathcal{H}$ is a nearly orthogonal sequence if there exists another (to be specified) Hilbert space $\mathscr{H}$ whose elements are sequences of elements from $\mathcal{H}$, and $\{(x_{n+h})_{n=1}^\infty\}_{h=1}^\infty$ is an orthogonal set of vectors in $\mathscr{H}$.

Generalizing Existing Variants of van der Corput's Difference Theorem

MvdC2: Let $\mathcal{H}$ be a Hilbert space and $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ a bounded sequence of vectors. If

$$\overline{\lim_{h\to\infty}}\left|\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^{\infty}\langle x_{n+h}, x_n\rangle\right| = 0,$$

then $(x_n)_{n=1}^{\infty}$ is a **nearly strongly mixing sequence**.

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[1] As defined by Berend and Bergelson

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[2] As defined by Berend and Bergelson

A New Variant of van der Corput's Difference Theorem

MvdC4: Let $\mathcal{H}$ be a Hilbert space and let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequence. If

$$\lim_{K\to\infty} \sup_{N,H\geq K} \left| \frac{1}{NH} \sum_{\substack{1\leq h\leq H \\ 1\leq n\leq N}} \langle x_{n+h}, x_n\rangle \right| = 0,$$

then

$$\lim_{N\to\infty} || \sum_{n=1}^{N} x_n|| = 0.$$

A Closer look at MvdC4

Example 1: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be arbitrary and let $L : [0,1) \to [0,\frac{1}{2})$ denote the map $x \mapsto \frac{1}{2}x$. The sequence $(y_n)_{n=1}^\infty$ given by $y_{2n} = L(2n\alpha)$ and $y_{2n-1} = L((2n-1)\alpha) + \frac{1}{2}$ for all $n \in \mathbb{N}$ satisfies

(1) $(y_n)_{n=1}^\infty$ is (well) uniformly distributed.

(2) $(y_n)_{n=1}^\infty$ is not totally uniformly distributed (and therefore not 'weakly mixing').

(3) $(y_{n+h} - y_n)_{(n,h)\in\mathbb{N}^2}$ is uniformly distributed.

(4) For each $h \in \mathbb{N}$ the sequence $(y_{n+h} - y_n)_{n=1}^\infty$ takes on exactly 2 distinct values.

A Closer look at MvdC4

Example 2: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be arbitrary, and consider the sequence $(z_n)_{n=1}^{\infty} \subseteq \mathbb{T}$ given by $z_n = m\alpha$ for $\binom{m}{2} < n \leq \binom{m+1}{2}$.

$(z_n)_{n=1}^{\infty} = \alpha, 2\alpha, 2\alpha, 3\alpha, 3\alpha, 3\alpha, 4\alpha, \cdots$

$(z_n)_{n=1}^{\infty}$ is a uniformly distributed sequence, but $(z_{n+h} - z_n)_{(n,h)\in\mathbb{N}^2}$ is not uniformly distributed.

Another New Variant of van der Corput's Difference Theorem

MvdC5: Let $\mathcal{H}$ be a Hilbert space and let $(f_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequence. We have that

$$\lim_{K\to\infty} \sup_{N,H\geq K,N_0,H_0\geq 1} \left|\frac{1}{NH} \sum_{\substack{H_0\leq h\leq H_0+H-1 \\ N_0\leq n\leq N_0+N-1}} \langle f_{n+h}, f_n\rangle\right| = 0,$$

if and only if

$$\lim_{K\to\infty} \sup_{N\geq K,N_0\geq 1} ||\frac{1}{N} \sum_{n=N_0}^{N_0+N-1} f_n|| = 0.$$

The Correct Point of View

Let $\mathcal{H}$ be a Hilbert space, let $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty \subseteq \mathcal{H}$ be sequences of vectors satisfying

$$\overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^N ||f_n||^2 < \infty \text{ and } \overline{\lim_{N\to\infty}}\frac{1}{N}\sum_{n=1}^N ||g_n||^2 < \infty. \tag{1}$$

Let $(N_q)_{q=1}^\infty$ be an increasing sequence of positive integers for which

$$\lim_{q\to\infty}\frac{1}{N_q}\sum_{n=1}^{N_q}\langle x_{n+h}, y_n\rangle_{\mathcal{H}}$$

exists whenever $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in \{(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty\}$ and $h \in \mathbb{N}$.

The Correct Point of View

We can now construct a new Hilbert space $\mathscr{H}((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ from $(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty$ and $(N_q)_{q=1}^\infty$ as follows. For any $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in \{(f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty\}$ and $h \in \mathbb{N}$, we define

$$\langle (x_{n+h})_{n=1}^\infty, (y_n)_{n=1}^\infty \rangle_{\mathscr{H}} = \lim_{q\to\infty} \frac{1}{N_q} \sum_{n=1}^{N_q} \langle x_{n+h}, y_n \rangle_{\mathcal{H}}.$$

We see that $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ is a sesquilinear form on $\mathscr{H}' = \text{Span}_{\mathbb{C}}(\{(f_{n+h})_{n=1}^\infty\}_{h=1}^\infty \cup \{(g_{n+h})_{n=1}^\infty\}_{h=1}^\infty)$. Letting $S = \{x \in \mathscr{H}' \mid \langle x, x \rangle_{\mathscr{H}} = 0\}$, we see that $\mathscr{H}'/S$ is a pre-Hilbert space, so we define $\mathscr{H}((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ to be the completion of $\mathscr{H}'/S$. We call $\mathscr{H}((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ the Hilbert space induced by $((f_n)_{n=1}^\infty, (g_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$. We note that the shift operator U given by $U((x_n)_{n=1}^\infty) = (x_{n+1})_{n=1}^\infty$ induces (by abuse of notation) a unitary operator $U : \mathscr{H} \to \mathscr{H}$.

van der Corput and Mixing

MvdC1: Let $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subseteq \mathcal{H}$ be a bounded sequence. Let $(N_q)_{q=1}^{\infty} \subseteq \mathbb{N}$ be such that $\mathscr{H} := \mathcal{H}((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}, (N_q)_{q=1}^{\infty})$ is well defined. If

$$\langle (x_{n+h})_{n=1}^{\infty}, (x_n)_{n=1}^{\infty} \rangle_{\mathscr{H}} = 0, \text{ for every } h \in \mathbb{N},$$

then $\{(x_{n+h})_{n=1}^{\infty}\}_{h=1}^{\infty} = \{U^h (x_n)_{n=1}^{\infty}\}_{h=1}^{\infty}$ is an orthogonal set in $\mathscr{H}$. In particular, for any over $\vec{v} \in \mathscr{H}$, we have

$$\sum_{h=1}^{\infty} |\langle (x_{n+h})_{n=1}^{\infty}, \vec{v} \rangle_{\mathscr{H}}|^2 \leq ||\vec{v}||_{\mathscr{H}}^2 = \sum_{h=1}^{\infty} |\langle U^h (x_n)_{n=1}^{\infty}, \vec{v} \rangle_{\mathscr{H}}|^2 \leq ||\vec{v}||_{\mathscr{H}}^2 < \infty.$$

van der Corput and Mixing

MvdC2: Let $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq \mathcal{H}$ be a bounded sequence. Let $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ be such that $\mathscr{H} := \mathcal{H}((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ is well defined. If

$$\overline{\lim_{h\to\infty}} |\langle (x_{n+h})_{n=1}^\infty, (x_n)_{n=1}^\infty \rangle_{\mathscr{H}}| = 0,$$

then $\{(x_{n+h})_{n=1}^\infty\}_{h=1}^\infty = \{U^h(x_n)_{n=1}^\infty\}_{h=1}^\infty$ is a strongly mixing sequence in $\mathscr{H}$. In particular, for any over $\vec{v} \in \mathscr{H}$, we have

$$\lim_{h\to\infty} \langle (x_{n+h})_{n=1}^\infty, \vec{v} \rangle_{\mathscr{H}} = \lim_{h\to\infty} \left| \langle U^h(x_n)_{n=1}^\infty, \vec{v} \rangle_{\mathscr{H}} \right| = 0.$$

van der Corput and Mixing

MvdC3: Let $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq \mathcal{H}$ be a bounded sequence. Let $(N_q)_{q=1}^\infty \subseteq \mathbb{N}$ be such that $\mathscr{H} := \mathcal{H}((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty, (N_q)_{q=1}^\infty)$ is well defined. If

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} |\langle (x_{n+h})_{n=1}^\infty, (x_n)_{n=1}^\infty\rangle_{\mathscr{H}}| = 0,$$

then $\{(x_{n+h})_{n=1}^\infty\}_{h=1}^\infty = \{U^h(x_n)_{n=1}^\infty\}_{h=1}^\infty$ is a weakly mixing sequence in $\mathscr{H}$. In particular, for any over $\vec{v} \in \mathscr{H}$, we have

$$\lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} |\langle (x_{n+h})_{n=1}^\infty, \vec{v}\rangle_{\mathscr{H}}| = \lim_{H\to\infty} \frac{1}{H}\sum_{h=1}^{H} \left|\langle U^h(x_n)_{n=1}^\infty, \vec{v}\rangle_{\mathscr{H}}\right| = 0.$$

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