



Canonical Ramsey Theory and Dynamics



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Background

van der Waerden’s theorem([1]): For any $\ell \in \mathbb{N}$ and any finite partition $\mathbb{N} = \bigcup_{i=1}^r C_i$, there exists $a, d \in \mathbb{N}$ and $1 \leq i_0 \leq r$ for which $\{a + jd\}_{j=0}^\ell \subseteq C_{i_0}$.

Topological van der Waerden theorem([2],[3]): If (X, T) is a minimal topological dynamical system, $\emptyset \neq U \subseteq X$ is an open set, and $\ell \in \mathbb{N}$, then there exists $d \in \mathbb{N}$ such that

$$U \cap T^{-d}U \cap \dots \cap T^{-\ell d}U \neq \emptyset \tag{1}$$

Szemerédi’s theorem([4]): If $A \subseteq \mathbb{N}$ has positive upper density ($\limsup_{N \rightarrow \infty} \frac{1}{N} |A \cap [1, N]| > 0$), then A contains arbitrarily long arithmetic progressions.

Furstenberg multiple recurrence theorem([5],[3]): Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \rightarrow X$ a measure preserving transformation. If $\ell \in \mathbb{N}$ and $A \in \mathcal{B}$ is such that $\mu(A) > 0$, then there exists $n \in \mathbb{N}$ for which

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-\ell n}A) > 0. \tag{2}$$

Canonical van der Waerden theorem([6],[7]): For any $\ell \in \mathbb{N}$ and any (not necessarily finite) partition $\mathbb{N} = \bigcup_{i=1}^\infty C_i$, $\exists a, d \in \mathbb{N}$ such that either
(i) $\{a + jd\}_{j=0}^\ell \subseteq C_{i_0}$ for some $i_0 \in \mathbb{N}$, or
(ii) for $0 \leq j_1 < j_2 \leq \ell$, $a + j_1d$ and $a + j_2d$ are not contained in the same cell (set of the form C_k).

Conjectures

Conjecture 1: Let X be a compact Hausdorff space and $T : X \rightarrow X$ a continuous map. For any open $U \subseteq X$ and any $\ell \in \mathbb{N}$, there exists $n \in \mathbb{N}$ with

$$U \cap T^{-n}U \cap T^{-2n}U \cap \dots \cap T^{-\ell n}U \neq \emptyset, \text{ or} \tag{3}$$

$$T^{-in}U \cap T^{-jn}U = \emptyset \ \forall \ 0 \leq i < j \leq \ell. \tag{4}$$

Conjecture 2: Conjecture 1 holds if X is assumed to be a compact metric space.

Conjecture 3: For any $\ell \in \mathbb{N}$ and any (not necessarily finite) partition $\mathbb{N} = \bigcup_{i=1}^\infty C_i$, there exists $d \in \mathbb{N}$ such that either

- (i) for **some** $i_0 \in \mathbb{N}$ we have $C_{i_0} \cap (C_{i_0} - d) \cap (C_{i_0} - 2d) \cap \dots \cap (C_{i_0} - \ell d) \neq \emptyset$, or
- (ii) for **every** $i \in \mathbb{N}$ we have $(C_i - jd) \cap (C_i - kd) = \emptyset$ for all $0 \leq j < k \leq \ell$.

Conjecture 4: Let $(X, \mathcal{B}, \mu)$ be a σ -finite measure space and $T : X \rightarrow X$ a measure preserving transformation. If $A \in \mathcal{B}$, then there exists $n \in \mathbb{N}$ with

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-\ell n}A) > 0, \text{ or} \tag{5}$$

$$\mu(T^{-in}A \cap T^{-jn}A) = 0 \ \forall \ 0 \leq i < j \leq \ell. \tag{6}$$

Conjecture 5(Canonical Szemerédi): For any $A \subseteq \mathbb{N}$ and $\ell \in \mathbb{N}$ there exists $d \in \mathbb{N}$ with

- (i) $A \cap (A - d) \cap (A - 2d) \cap \dots \cap (A - \ell d) \neq \emptyset$, or
- (ii) $(A - id) \cap (A - jd) = \emptyset$ for all $0 \leq i < j \leq \ell$.

Implications

-Conjecture 1 implies Conjectures 2, 3, 5, and Szemerédi’s theorem. While van der Waerden’s theorem is equivalent to its topological analogue, it is not clear whether or not Conjecture 3 implies Conjecture 1.

-Conjecture 2 implies the canonical van der Waerden theorem, but it is not obvious whether or not Conjecture 2 implies Conjecture 1, or whether the canonical van der Waerden theorem implies Conjecture 2.

-Conjecture 4 implies Conjecture 3, which implies Conjecture 5, which implies Szemerédi’s theorem. While the Fursternberg multiple recurrence theorem is equivalent to Szemerédi’s theorem, it is not clear whether or not Conjecture 4 implies Conjectures 2, or whether Conjecture 2 implies Conjecture 5.

-While the Furstenberg multiple recurrence theorem implies the topological van der Waerden Theorem, we do not know if Conjecture 1 implies Conjecture 4, or even if Conjecture 4 implies Conjecture 2.

Theorem: The following are equivalent to Conjecture 4:

- (i) Conjecture 4 when $X = \mathbb{R}$, $\mathcal{B}$ is the Lebesgue σ -algebra, and μ is the Lebesgue measure.
- (ii) Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \rightarrow X$ a non-singular measurable transformation. If $A \in \mathcal{B}$, then there exists $n \in \mathbb{N}$ with

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-\ell n}A) > 0, \text{ or} \tag{7}$$

$$\mu(T^{-in}A \cap T^{-jn}A) = 0 \ \forall \ 0 \leq i < j \leq \ell. \tag{8}$$

- (iii) Item (ii) when $X = [0, 1]$, $\mathcal{B}$ is the Lebesgue σ -algebra, and μ is the Lebesgue measure.

References

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