On certain aspects of the Möbius randomness principle

Sarnak Conjecture Seminar
at the Ohio State University

Based on the work of Davit Karagulyan

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Important Conjectures I

Let μ denote the Möbius function.

Conjecture (Riemann Hypothesis)

For every $\epsilon > 0$ we have

$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{2}+\epsilon}} \sum_{n=1}^{N} \mu(n) = 0. \tag{1}$$

Conjecture (Sarnak)

If $(c_n)_{n=1}^{\infty}$ is a bounded deterministic sequence of complex numbers then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n\mu(n) = 0. \tag{2}$$

Important Conjectures I

Conjecture (Chowla)

For any $k \in \mathbb{N}$, *any* $a_1, \cdots, a_k \in \mathbb{N}$ *and any* $(\epsilon_0, \epsilon_1, \cdots, \epsilon_k) \in \{1,2\}^{k+1} \setminus (2,2,\cdots,2)$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu^{\epsilon_0}(n)\mu^{\epsilon_1}(n+a_1)\cdots\mu^{\epsilon_k}(n+a_k) = 0. \tag{3}$$

Properties (R), (S), and (Chw)

Definition

Suppose that $(z_n)_{n=1}^{\infty} \in \{-1, 0, 1\}$.

1. $(z_n)_{n=1}^{\infty}$ satisfies property (R) if for every $\epsilon > 0$ we have
$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{2}+\epsilon}} \sum_{n=1}^{N} z_n = 0. \tag{4}$$
2. $(z_n)_{n=1}^{\infty}$ satisfies property (S) if for every bounded deterministic sequence of complex numbers $(c_n)_{n=1}^{\infty}$ we have
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n z_n = 0. \tag{5}$$

Properties (R), (S), and (Chw)

Definition

3 $(z_n)_{n=1}^{\infty}$ satisfies property (Chw) if for any $k \in \mathbb{N}$, any $a_1, \cdots, a_k \in \mathbb{N}$ and any $(\epsilon_0, \epsilon_1, \cdots, \epsilon_k) \in \{1,2\}^{k+1} \setminus (2, 2, \cdots, 2)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} z_n^{\epsilon_0} z_{n+a_1}^{\epsilon_1} \cdots z_{n+a_k}^{\epsilon_k} = 0. \quad (6)$$

It is well known that property (Chw) implies property (S).

Property (Chw) does not imply Property (R)

Proposition

Let $(x_n)_{n=1}^{\infty}$ be a sequence of independent random variables taking values in $\{-1, 0, 1\}$, with $\mathbb{E}[X] \to 0$ as $n \to \infty$. Then the sequence $(X_n(\omega))_{n=1}^{\infty}$ satisfies properties (Chw) and (S) for almost all $\omega \in \Omega$.

Corollary

Properties (Chw) and (S) do not imply property (R).

Proof Idea

Consider the sequence of independent random variables $(X_n)_{n=1}^{\infty}$ given by

$$\mathbb{P}(X_k = 1) = \frac{1}{2}(1+\frac{1}{\log(k+1)}) \text{ and } \mathbb{P}(X_k = -1) = \frac{1}{2}(1-\frac{1}{\log(k+1)})$$

For $\epsilon < \frac{1}{4}$ we have

$$\lim_{N\to\infty} \frac{1}{N^{\frac{1}{2}+\epsilon}} \sum_{n=1}^{N} X_n(\omega) = \infty \tag{7}$$

for almost all $\omega \in \Omega$.

The Randomized Chowla and Sarnak Conjectures are True

Theorem (Abdalaoui, Disertori 2013)

Let $(\epsilon_n)_{n=1}^{\infty}$ be an i.i.d. sequence of random variables satisfying $\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = \frac{1}{2}$. The function

$$\mu_{rand}(n) = \begin{cases} \epsilon_n & \text{if } n \text{ is square free} \\ 0 & \text{else} \end{cases}. \tag{8}$$

satisfies property (S) almost surely.

In fact, property (Chw) is satisfied almost surely as a consequence of Karagulyan's proposition.

Randomized Riemann Hypothesis

Theorem (Wintner, 1944)

Let $(\epsilon_n)_{n=1}^{\infty}$ be an i.i.d. sequence of random variables satisfying $\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = \frac{1}{2}$. The function

$$f(n) = \begin{cases} \prod_{p|n} \epsilon_p & \text{if } n \text{ is square free} \\ 0 & \text{else} \end{cases}. \tag{9}$$

satisfies property (R) almost surely.

Property (R) does not imply Property (S)

Let us consider the **Rudin-Shapiro** given by $s_n = (-1)^{b_n}$ where b_n counts the number of occurrences of '11' appearing in the binary expansion of $n \geq 0$. It is well known that

$$\sqrt{\frac{3}{5}} < \frac{1}{\sqrt{N}} \sum_{n=1}^{N} s_n < \sqrt{6}. \tag{10}$$

The generating function $S(X) = \sum_{n \geq 0} s_n X^n$ satisfies

$$(1+X)^5 S(X)^2 + (1+X)^4 S(X) + X^3 = 0. \tag{11}$$

The Rudin-Shapiro sequence is deterministic.

Properties (S) and (R) do not imply Property (Chw)

Theorem

There exists a sequence $(z_n)_{n=1}^{\infty} \in \{-1,0,1\}^{\mathbb{N}}$ *satisfying properties (S) and (R) such that for any* $k \in \mathbb{N}$*, any* $a_1, \cdots, a_k \in \mathbb{N}$ *and any* $(\epsilon_0, \epsilon_1, \cdots, \epsilon_k) \in \{1,2\}^{k+1} \setminus (2,2,\cdots,2)$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} z_n^{\epsilon_0} z_{n+a_1}^{\epsilon_1} \cdots z_{n+a_k}^{\epsilon_k} \neq 0. \tag{12}$$

Proof idea

Let $(\epsilon_n)_{n=1}^{\infty}$ be an i.i.d. sequence of random variables satisfying $\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = \frac{1}{2}$. $(\epsilon_n(\omega))_{n=1}^{\infty}$ satisfies property (Chw) (and hence property (S)) for a.e. $\omega \in \Omega$. However, the sequence

$$(z_n)_{n=1}^{\infty} := \epsilon_1, \epsilon_1, \epsilon_2, \epsilon_2, \cdots, \epsilon_n, \epsilon_n, \epsilon_{n+1}, \epsilon_{n+1}, \cdots \tag{13}$$

does not satisfy property (Chw) while still satisfying properties (S) and (R). In particular, we have created a correlation of $(z_n)_{n=1}^{\infty}$ with $(z_{n+1})_{n=1}^{\infty}$, so we only need to repeat this procedure countably many times and use a diagonalization argument.

Comparing Variants of difference theorems

Theorem (van der Corput)

If $(c_n)_{n=1}^{\infty}$ *is a bounded sequence of complex numbers for which*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{n+h}\overline{c_n} = 0 \tag{14}$$

for all $h \in \mathbb{N}$*, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n = 0. \tag{15}$$

In fact, for any $a, b \in \mathbb{N}$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{an+b} = 0. \tag{16}$$

Comparing Variants of difference theorems

Theorem (Kátai)

If $(c_n)_{n=1}^{\infty}$ *is a bounded sequence of complex numbers for which*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{pn}\overline{c_{qn}} = 0 \tag{17}$$

for all distinct primes p *and* q*, then for any bounded multiplicative function* $f : \mathbb{N} \to \mathbb{C}$ *we (uniformly) have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n)c_n = 0. \tag{18}$$

Comparing Variants of difference theorems

For any irrational $\alpha \in \mathbb{R}$ the sequence $(e^{2\pi i n\alpha})_{n=1}^{\infty}$ (or $(e^{2\pi i\sqrt{n}\alpha})_{n=1}^{\infty}$) satisfies Kátai's Orthogonality Criterion but not van der Corput's Difference Theorem.

There exists a bounded sequence of complex numbers $(c_n)_{n=1}^{\infty}$ satisfying van der Corput's Difference Theorem but not Kátai's Orthogonality Criterion. In fact, for any distinct primes p and q we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} c_{pn}\overline{c_{qn}} \neq 0. \tag{19}$$