Generalizations of the Grunwald-Wang Theorem and Applications to Ramsey Theory

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(joint work with Richard Magner)

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The Grunwald-Wang Theorem

Theorem

Let $n \in \mathbb{N}$ be arbitrary and suppose that $x \in \mathbb{Z}$ is such that x is an nth power modulo p for every prime p. x is either an nth power or $8|n$ and $x = 2^{\frac{n}{2}} y^n = 16^{\frac{n}{8}} y^n$.

W. Grunwald in 1933 proved an incorrect version of this theorem since he failed to find the exceptional case when $8|n$.

G. Whaples in 1942 gave another incorrect proof of Grunwald's Theorem.

S. Wang in 1948 found the counter example of 16 and gave a proof of the corrected theorem in his doctoral thesis.

The Exceptional case of $x = 16$

It is clear that $16 = 2^4$ is not an 8th power in $\mathbb{N}$. To see that 16 is an 8th power modulo p for every prime p, we observe that

$$x^8 - 16 = (x^4 - 4)(x^4 + 4) = (x^2 - 2)(x^2 + 2)(x^2 - 2x + 2)(x^2 + 2x + 2).$$

We note that the discriminant of the last 2 factors is -4. Since one of $2, -2$, and -4 will be a square modulo p, we see that $x^8 - 16$ will have a root modulo p.

The Grunwald-Wang Theorem intuitively says that 16 is the only obstruction to a certain local-global principle.

Grunwald-Wang for 3 Variables

Theorem (F., Magner)

Let $n \in \mathbb{N}$ be arbitrary and suppose that $a, b, c \in \mathbb{Z}$ are such that at least one of a, b, and c is an nth power modulo p for every prime p. Then either

1. *n is odd and one of a, b, and c is an nth power.*
2. *n is even and either one of a, b, and c is an $\frac{n}{2}$th power, or $4|n$ and each of a, b, and c is an $\frac{n}{4}$th power.*

In our arxiv paper we also address the situation for a general number field K with ring of integers $\mathcal{O}_K$.

Some Exceptional Cases

It is clear that we still have an exceptional case if $8|n$ and one of a, b, and c is of the form $2^{\frac{n}{2}}y^n$.

A new exceptional case is found with $n = 4$, $a = 3^4 \cdot 4^2 \cdot 5^2$, $b = 3^2 \cdot 4^4 \cdot 5^2$, and $c = a + b = 3^2 \cdot 4^2 \cdot 5^4$.

There are more exceptional cases that actually show up from the 2 variable situation.

Grunwald-Wang for 2 Variables

Theorem (F., Magner, 2022)

Let $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ be such that either

1. $4 \nmid n$ *and neither of* a *and* b *are* n*th powers.*
2. $4 | n$ *and neither of* a *and* b *are* $\frac{n}{2}$*th powers.*

Then there exist infinitely many primes p *modulo which neither of* a *and* b *are an* n*th power.*

Some More Exceptional Cases

Since 3 is a perfect square mod p if $p \equiv 1 \pmod 3$ and every integer is a perfect cube mod p if $p \equiv 2 \pmod 3$, we see that for any $b \in \mathbb{Z}$ one of 3^6 and b^4 will be a 12th power modulo p for any prime p.(Due to Hyde, Lee, and Spearman)

We can break down the Grunwald-Wang exceptional case of 16 by observing that $x^8 - 16 = (x^4 + 4)(x^4 - 4)$, so one of 4 or -4 will be a 4th power modulo p for any prime p.

36 is a 4th power modulo p if $p \not\equiv 13 \pmod{24}$ and 9 is a 4th power modulo p if $p \equiv 13 \pmod{24}$, so one of 36 and 9 will be a 4th power modulo p for any prime p.

Proof Sketch

If $n \in \mathbb{N}$ and $x \in \mathbb{Z}$ is an mth power with $m|n$ maximal, then the Chebotarev Density Theorem tells us that the set S_x of prime ideals $\mathfrak{p}$ in a suitable extension of $\mathbb{Z}$ for which x is not an nth power has density $\frac{m}{n}$.

If n is odd and none of a, b, and c are nth powers, then they are at best $\frac{n}{3}$th powers, so $d(S_a), d(S_b), d(S_c) \leq \frac{1}{3}$.If $d(S_a) = d(S_b) = d(S_c) = \frac{1}{3}$, then we use inclusion exclusion, and in either case we find a positive density of prime ideals for which none of a, b, and c are nth powers.

If n is even then there are more cases (such as $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} = 1$) and more inclusion-exclusion.

Ramsey Theory Preliminaries

Definition

If $p \in \mathbb{Z}[x_1, \cdots, x_n]$ is a polynomial and S is either $\mathbb{N}$ or $\mathbb{Z}$, then the equation

$$p(x_1, \cdots, x_n) = 0 \tag{1}$$

is **partition regular (p.r) over** S if for any partition $S = \sqcup_{i=1}^{r} C_i$ there exists $1 \leq i_0 \leq r$ and $x_1, \cdots, x_n \in C_{i_0}$ satisfying (1).

Polynomial Equations and Partition Regularity

1. $x + y = z$ is p.r. over $\mathbb{N}$ (Schur)
2. $xy = z$ is p.r. over $\mathbb{N}$ (corollary of Schur)
3. $ax + by = dz$ is p.r. over $\mathbb{N}$ if and only if $d \in \{a, b, a + b\}$ (special case of Rado's Theorem)
4. $x + y = wz$ is p.r. over $\mathbb{N}$ (Bergelson-Hindman)
5. $x - y = q(z)$ with $q \in x\mathbb{Z}[x]$ is p.r. over $\mathbb{N}$ (Bergelson)
6. $x + y = z^2$ is not non-trivially p.r. over $\mathbb{N}$ (Csikvári, Gyarmati and Sárkozy)
7. It is open as to whether $x^2 + y^2 = z^2$ is p.r. over $\mathbb{N}$.
8. It is open as to whether $z = xy + x$ is p.r. over $\mathbb{N}$.
9. $z = x^y$ is p.r. over $\mathbb{N}$ bu $z = x^{y+1}$ remains open (Sahasrabudhe).

Our Main Result

Theorem

Let $m, n \in \mathbb{N}$ *and* $a, b, c \in \mathbb{Z} \setminus \{0\}$.

1. *If* $m, n \geq 2$, *then the equation*

$$ax + by = cw^m z^n \tag{2}$$

   is p.r. over $\mathbb{Z}$ *if and only if* $a + b = 0$.

2. *If one of* $\frac{a}{c}$, $\frac{b}{c}$, *or* $\frac{a+b}{c}$ *is a nth power in* $\mathbb{Q}$, *then the equation*

$$ax + by = cwz^n \tag{3}$$

   is p.r. over $\mathbb{Z}$. *If* $\mathbb{Q}$ *is replaced with* $\mathbb{Q}^+$ *then* $\mathbb{Z}$ *can be replaced with* $\mathbb{N}$.

Our Main Result (Continued)

Theorem

3 *Suppose that*

$$ax + by = cwz^n \tag{4}$$

is p.r. over $\mathbb{Q} \setminus \{0\}$.

- (a) *If n is odd then one of* $\frac{a}{c}$, $\frac{b}{c}$, *or* $\frac{a+b}{c}$ *is an nth power in* $\mathbb{Q}$.
- (b) *If* $n \neq 4, 8$ *is even then one of* $\frac{a}{c}$, $\frac{b}{c}$, *or* $\frac{a+b}{c}$ *is a* $\frac{n}{2}$*th power in* $\mathbb{Q}$. *We used Fermat's Last Theorem here!*
- (c) *If n is even, then either one of* $\frac{a}{c}$, $\frac{b}{c}$, *or* $\frac{a+b}{c}$ *is a square in* $\mathbb{Q}$, *or* $(\frac{a}{c})(\frac{b}{c})(\frac{a+b}{c})$ *is a square in* $\mathbb{Q}$.

Proof Sketch of 2

If $\gamma^n \in \{\frac{a}{c}, \frac{b}{c}, \frac{a+b}{c}\}$ for some $\gamma \in \mathbb{Q}$, then

$$ax + by = cwz^n \text{ is p.r. iff } a\gamma x + b\gamma y = c\gamma w(\gamma z)^n \text{ is p.r.} \tag{5}$$

$$\Leftrightarrow ax + by = dwz^n \text{ is p.r. for some } d \in \{a, b, a+b\} \tag{6}$$

$$\Leftarrow ax + by = dw \text{ is p.r. for some } d \in \{a, b, a+b\}. \tag{7}$$

Proof Sketch of 3

For a prime p we may construct the partition $\mathbb{N} = \sqcup_{i=1}^{p-1} C_i$, where C_i is the set of all integers whose first non-zero digit in its base p expansion is i.If p is a prime for which none of ac^{-1}, bc^{-1}, or $(a+b)c^{-1}$ are nth powers modulo p, then this partition contains no solutions to

$$ax + by = cwz^n. \tag{8}$$

It now suffices to apply our generalization of the Grunwald-Wang Theorem. We obtain similar results for rings of integers $\mathcal{O}_K$ of number fields K, and some of these results also have analogues over a general integral domain R.