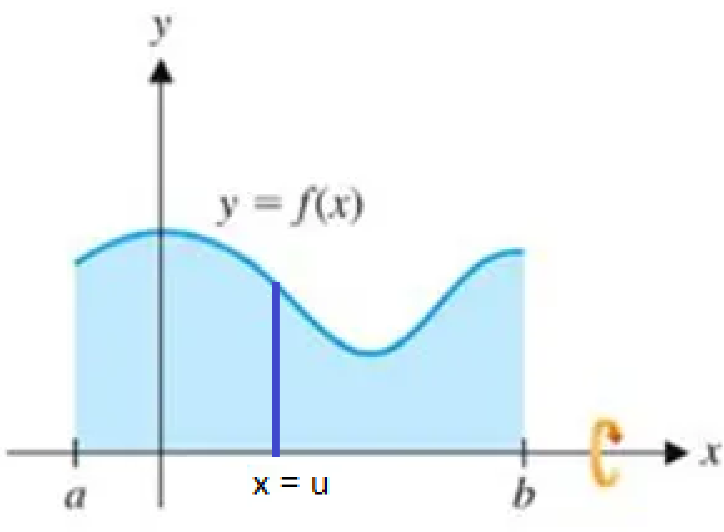
Problem 1: Suppose $y = f(x)$ is a continuous and positive function on $[a, b]$. Let $\mathcal{S}$ be the surface generated when the graph of $f(x)$ is revolved about the x-axis.

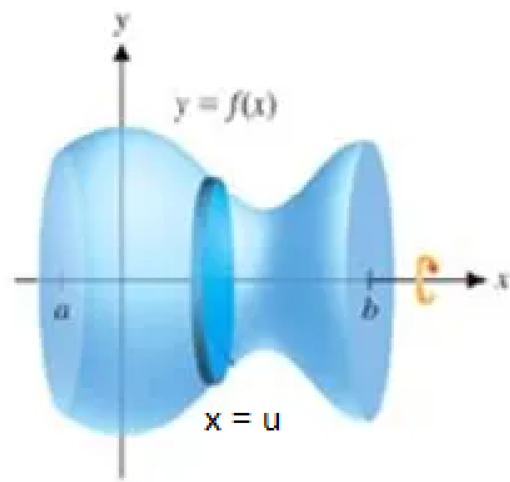
(a) Show that $\mathcal{S}$ is described parametrically by $\vec{r}(u, v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, for $a \le u \le b$, $0 \le v \le 2\pi$.

(b) Find an integral that gives ths surface area of $\mathcal{S}$.

(c) Apply the result of part (b) to the surface $\mathcal{S}_1$ generated with $f(x) = x^3$, for $1 \le x \le 2$.

Solution to (a): We see that for each value of u inbetween a and b, if we rotate the point $(u, f(u))$ about the x-axis then we generate a circle C of radius $f(u)$ in the plane $x = u$ as shown in the picture below.





We see that the x-coordinate at every point of the circle C is u. It now suffices to recall that the parametrization of a circle of radius $f(u)$ in the xy-plane is $\langle f(u)\cos(v), f(u)\sin(v)\rangle$ for $0 \le v \le 2\pi$, but we have a circle in the plane $x = u$ (which is parallel to the yz-plane), so we obtain the parametrization $\vec{r}(u, v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$ for $a \le u \le b$ and $0 \le v \le 2\pi$ as desired.

Solution to (b): We begin by calculating

$$\begin{aligned}\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & f'(u)\cos(v) & f'(u)\sin(v)\\ 0 & -f(u)\sin(v) & f(u)\cos(v)\end{vmatrix} \\ &= \hat{i}(f(u)f'(u)\cos^2(v)+f(u)f'(u)\sin^2(v)) \\ &\quad -\hat{j}(f(u)\cos(v))+\hat{k}(-f(u)\sin(v)) \\ &= f(u)f'(u)\hat{i}-f(u)\cos(v)\hat{j}-f(u)\sin(v)\hat{k}, \text{ hence}\end{aligned} \tag{1}$$

$$\begin{aligned}\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right| &= \sqrt{(f(u)f'(u))^2+(-f(u)\cos(v))^2+(-f(u)\sin(v))^2} \\ &= f(u)\sqrt{(f'(u))^2+1}.\end{aligned} \tag{2}$$

We now see that

$$\begin{aligned}\text{Surface Area}(\mathcal{S}) &= \iint_{\mathcal{S}} 1dS = \int_a^b\int_0^{2\pi} f(u)\sqrt{(f'(u))^2+1}dvdu \\ &= \boxed{2\pi\int_a^b f(u)\sqrt{(f'(u))^2+1}du}\end{aligned} \tag{3}$$

Solution to (c): From part (b) we see that

$$\begin{aligned}\text{Surface Area}(\mathcal{S}_1) &= 2\pi\int_1^2 u^3\sqrt{(3u^2)^2+1}du = 2\pi\int_1^2 u^3\sqrt{9u^4+1}du \\ &\overset{w=9u^4+1}{=} 2\pi\int_{u=1}^2\sqrt{w}\frac{dw}{36} = \frac{\pi}{18}\cdot\frac{2}{3}w^{\frac{3}{2}}\Big|_{u=1}^2 \\ &= \frac{\pi}{27}(9u^4+1)^{\frac{3}{2}}\Big|_1^2 = \boxed{\frac{\pi}{27}\left(145^{\frac{3}{2}}-10^{\frac{3}{2}}\right)}\end{aligned} \tag{4}$$

Problem 2: Given a sphere of radius R and a length $0 < L \leq 2R$, show that the surface area of the strips of length L on the sphere depend only on L and not on the location of the strip.

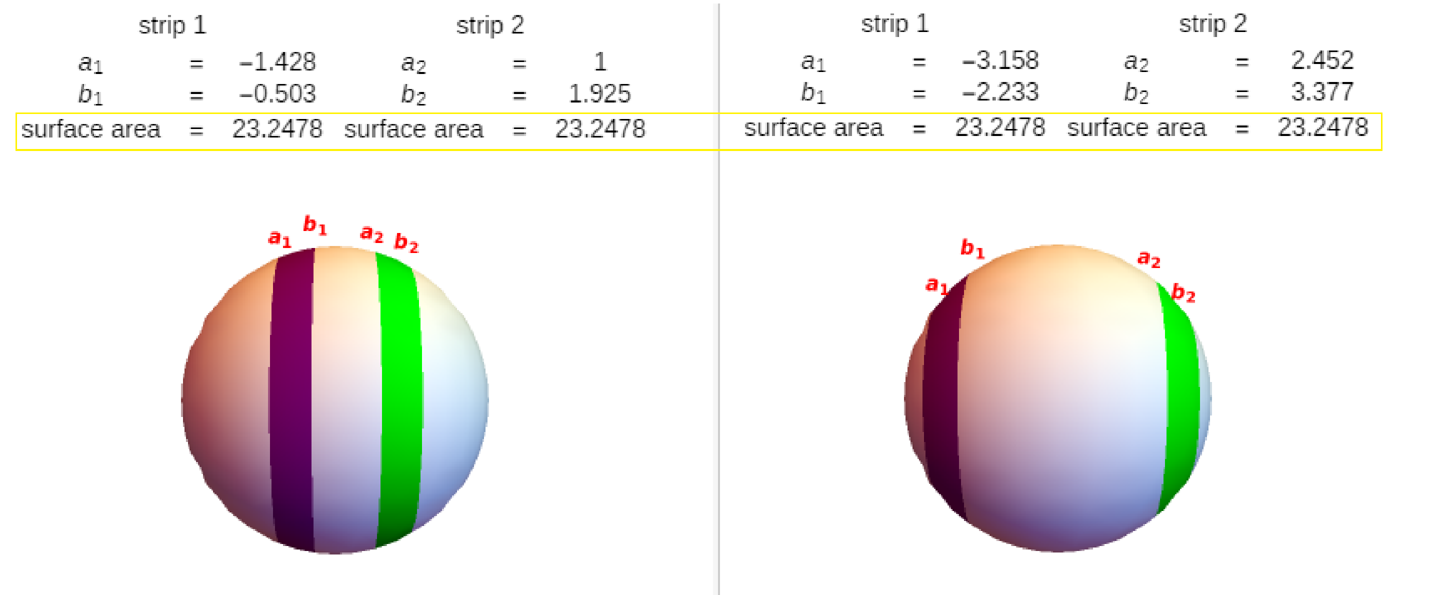


Figure 1: An example of Problem 2 with $L = 0.925$ and $R = 4$.

Hint: Problem 1 can help.

Solution: We begin by recalling that the graph of $f(x) = \sqrt{R^2 - x^2}$ for $-R \leq x \leq R$ is the upper half of the circle of radius R centered at the origin of the xy-plane. We may now use Problem 1(b) to see that for any $-R \leq a \leq R - L$ the surface area obtained by revolving $f(x)$ for $a \leq x \leq a + L$ is

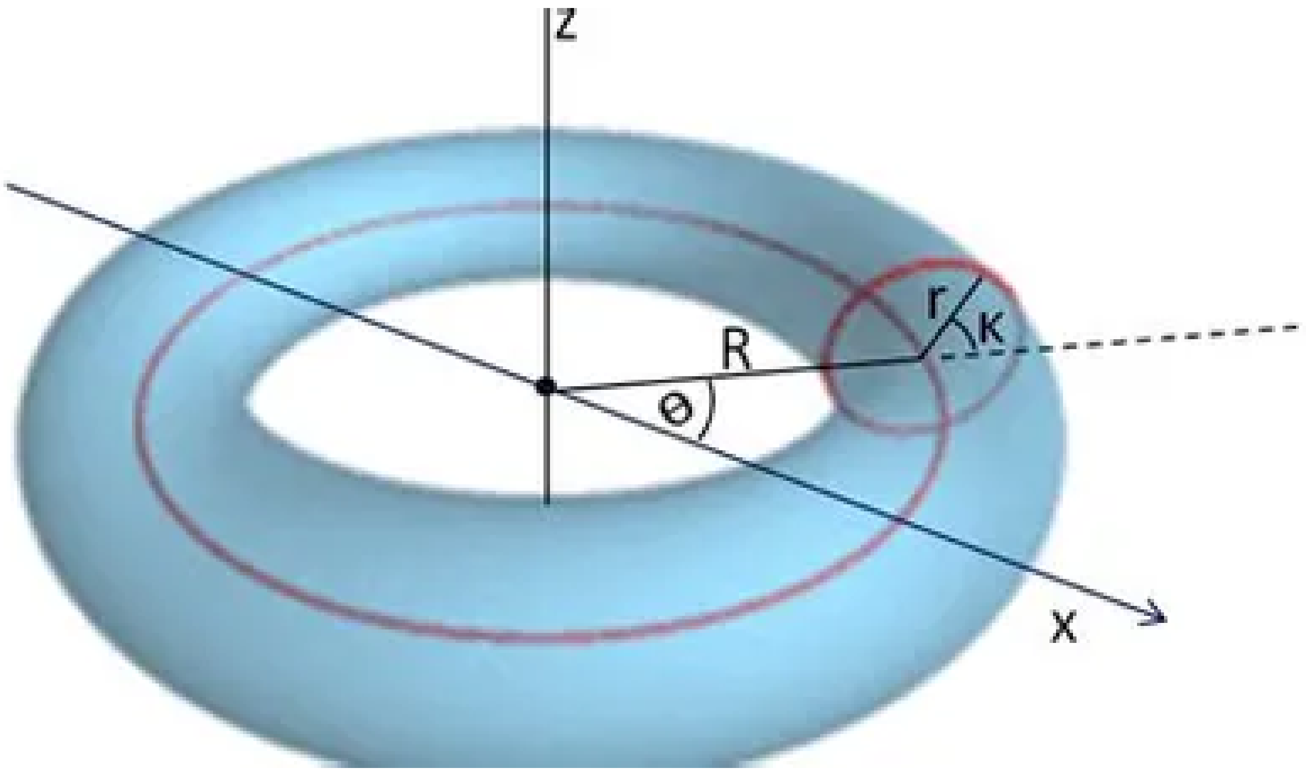
$$\begin{aligned} &2\pi \int_a^{a+L} \sqrt{R^2 - u^2}\sqrt{(\frac{-u}{\sqrt{R^2 - u^2}})^2 + 1}du \\ =&2\pi \int_a^{a+L} \sqrt{R^2 - u^2}\sqrt{\frac{R^2}{R^2 - u^2}}du = 2\pi \int_a^{a+L} Rdu = \boxed{2\pi RL} \end{aligned} \tag{5}$$

Problem 3(Surface Area and Volume of a Torus)**:**

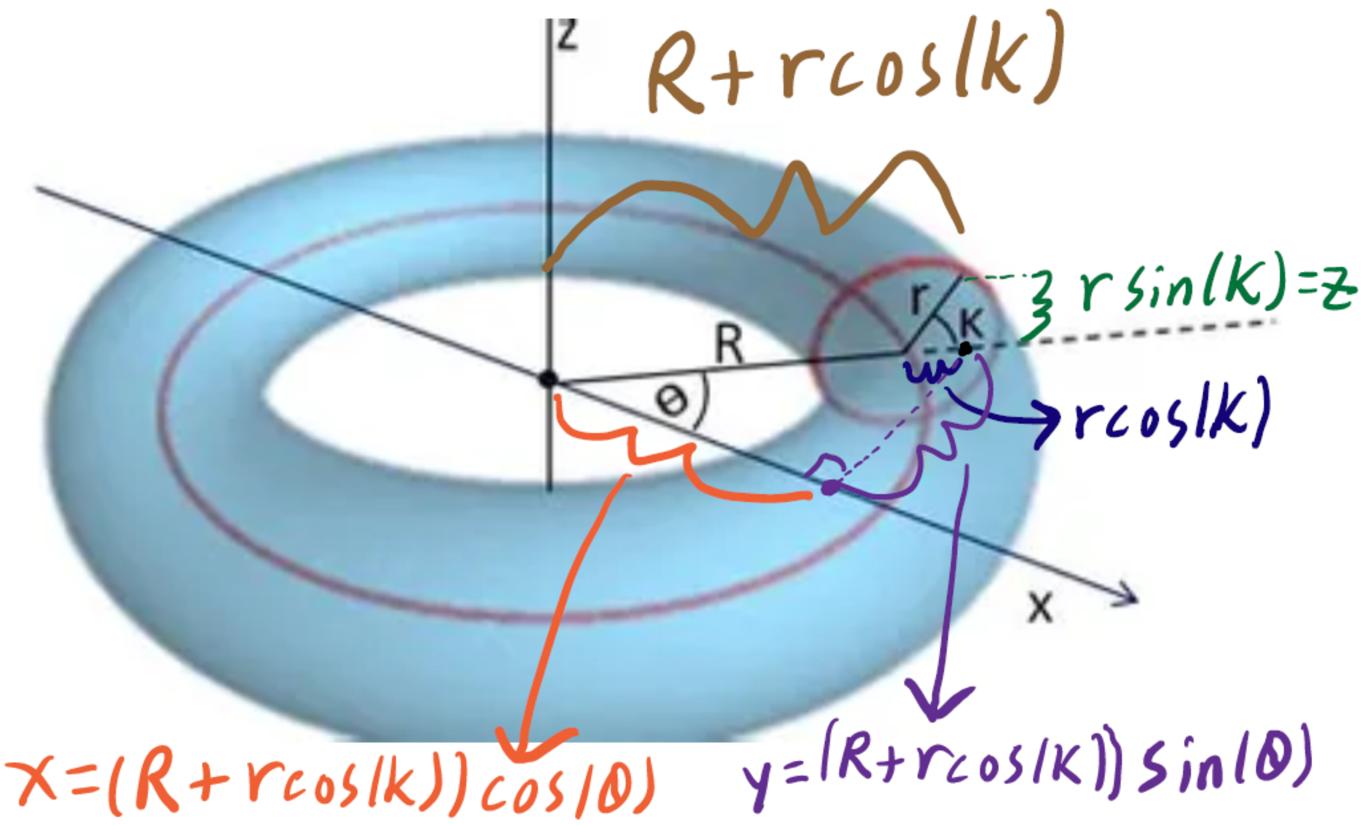
(a) Show that a torus T with radii $R > r$ (See figure) may be described parametrically by $r(K,\theta) = \langle (R + r\cos(K))\cos(\theta), (R + r\cos(K))\sin(\theta), r\sin(K)\rangle$, for $0 \le K \le 2\pi$, $0 \le \theta \le 2\pi$.

(b) Show that the surface area of the torus T is $4\pi^2 Rr$.
Interestingly, the arclength of the small circle is $2\pi r$ and the arclength of the large circle inside the torus is $2\pi R$, so the surface area of the torus happens to be the product of the arclengths of the 2 circles from which it is created.

(c) Use part (a) to find a parametrization $\vec{s}(K,\theta,r)$ for the solid torus $\mathcal{T}$ (T from part (a) as well as its interior), then use $\vec{s}$ and a change of variables to show that the volume of $\mathcal{T}$ is $\pi r^2 R$.



Solution to (a): The justification that $\vec{r}(K,\theta)$ is indeed a parametrization for T is given by the diagram below.



Solution to (b): We begin by calculating

$$\frac{\partial \vec{r}}{\partial K} \times \frac{\partial \vec{r}}{\partial \theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin(K)\cos(\theta) & -r\sin(K)\sin(\theta) & r\cos(K) \\ -(R+r\cos(K))\sin(\theta) & (R+r\cos(K))\cos(\theta) & 0 \end{vmatrix} \tag{6}$$

$$\begin{aligned} =&\hat{i}(-r\cos(K)(R+r\cos(K))\cos(\theta) \\ &-\hat{j}(-r\cos(K)(R+r\cos(K))\sin(\theta)) \\ &\hat{k}\Big(-r\sin(K)\cos(\theta)(R+r\cos(K))\cos(\theta) \\ &-r\sin(K)\sin(\theta)(R+r\cos(K))\sin(\theta)\Big) \end{aligned}$$

$$\begin{aligned} =&-r\cos(K)\cos(\theta)(R+r\cos(K))\hat{i} \\ &+r\cos(K)\sin(\theta)(R+r\cos(K))\hat{j} \\ &-r\sin(K)(R+r\cos(K))\hat{k}, \text{ hence} \end{aligned}$$

$$\left|\frac{\partial\vec{r}}{\partial K}\times\frac{\partial\vec{r}}{\partial\theta}\right| = (R+r\cos(K))\sqrt{(r\cos(K)\cos(\theta))^2+(r\cos(K)\sin(\theta))^2+(-r\sin(K))^2} \tag{7}$$
$$=(R+r\cos(K))\sqrt{r^2\cos^2(K)+r^2\sin^2(K)}$$
$$=r(R+r\cos(K))$$

We now see that

$$\text{Surface Area}(T) = \iint_T 1dS = \int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial\vec{r}}{\partial K}\times\frac{\partial\vec{r}}{\partial\theta}\right| dKd\theta \tag{8}$$
$$= \int_0^{2\pi}\int_0^{2\pi}(rR+r^2\cos(K))dKd\theta$$
$$= \int_0^{2\pi}\left(rRK+r^2\sin(K)\Big|_{K=0}^{2\pi}\right)d\theta$$
$$= \int_0^{2\pi} 2\pi rRd\theta = \boxed{4\pi^2 rR}.$$

Solution to (c): We only need to replace the radius r with a new radius $0\le\rho\le r$ in order to get toroidal shells within the original torus, so we obtain the parametrization

$$\vec{s}(K,\theta,\rho) = \langle(R+\rho\cos(K))\cos(\theta),(R+\rho\cos(K))\sin(\theta),\rho\sin(K)\rangle \tag{9}$$
$$\text{for } 0\le K\le 2\pi, 0\le\theta\le 2\pi, \text{ and } 0\le\rho\le r.$$

Now that we have found $\vec{s}$, we can we can calculate the Jacobian of the transformation $(x,y,z)=\vec{s}(K,\theta,r)$. We see that

$$J(K,\theta,\rho) = \begin{vmatrix}\frac{\partial x}{\partial K} & \frac{\partial y}{\partial K} & \frac{\partial z}{\partial K}\\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta}\\ \frac{\partial x}{\partial\rho} & \frac{\partial y}{\partial\rho} & \frac{\partial z}{\partial\rho}\end{vmatrix} \tag{10}$$
$$= \begin{vmatrix}-\rho\sin(K)\cos(\theta) & -\rho\sin(K)\sin(\theta) & \rho\cos(K)\\ -(R+\rho\cos(K))\sin(\theta) & (R+\rho\cos(K))\cos(\theta) & 0\\ \cos(K)\cos(\theta) & \cos(K)\sin(\theta) & \sin(K)\end{vmatrix}$$

$$
\begin{aligned}
&= -\rho\sin(K)\cos(\theta)(R+\rho\cos(K))\cos(\theta)\sin(K) \\
&\quad - (-\rho\sin(K)\sin(\theta))(-(R+\rho\cos(K))\sin(\theta))\sin(K) \\
&\quad + \rho\cos(K)\Big(-(R+\rho\cos(K))\sin(\theta)\cos(K)\sin(\theta) \\
&\qquad\qquad - \cos(K)\cos(\theta)(R+\rho\cos(K))\cos(\theta)\Big) \\
&= (R+\rho\cos(K))\Big(-\rho\sin^2(K)\cos^2(\theta) - \rho\sin^2(K)\sin^2(\theta) \\
&\qquad\qquad + \rho\cos^2(K)(-\sin^2(\theta)-\cos^2(\theta))\Big) \\
&= (R+\rho\cos(K))(-\rho\sin^2(K)-\rho\cos^2(K)) = -\rho(R+\rho\cos(K)).
\end{aligned}
$$

Recalling that $dV = dxdydz = |J(K,\theta,\rho)|dKd\theta d\rho$, we see that

$$
\begin{aligned}
\text{Volume}(\mathcal{T}) &= \iiint_T 1dV = \int_0^{2\pi}\int_0^{2\pi}\int_0^r |J(K,\theta,\rho)|dKd\theta d\rho \\
&= \int_0^r\int_0^{2\pi}\int_0^{2\pi} \rho(R+\rho\cos(K))dKd\theta d\rho \\
&= \int_0^{2\pi}\int_0^{2\pi}\left(\rho R + \rho^2\sin(K)\Big|_{K=0}^{2\pi}\right)d\theta d\rho \\
&= \int_0^r\int_0^{2\pi}\rho R d\theta d\rho = 2\pi\int_0^r \rho R d\rho \\
&= 2\pi\left(\frac{1}{2}\rho^2 R\Big|_{\rho=0}^r\right) = \boxed{\pi r^2 R}.
\end{aligned} \tag{11}
$$

Problem 4: Let $z = s(x, y)$ define the surface $\mathcal{S}$ over a region R in the xy-plane, where $z \geq 0$ on R. Show that the downward flux of the vertical vector field $\vec{F} = \langle 0, 0, -1\rangle$ across $\mathcal{S}$ equals the area of R. Interpret the result physically.

Solution: We see that the surface $\mathcal{S}$ can be parametrized by $\vec{r}(x, y) = \langle x, y, s(x, y)\rangle$ for $(x, y) \in R$. We now proceed to calculate $\hat{n}dS$, the vector normal to the surface whose length is proportional to the differential area at each point.

$$\begin{aligned}\hat{n}dS = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & s_x(x, y) \\ 0 & 1 & s_y(x, y)\end{vmatrix} \\ &= \hat{i}(-s_x(x, y)) - \hat{j}s_y(x, y) + \hat{k}(1) \\ &= -s_x(x, y)\hat{i} - s_y(x, y)\hat{j} + \hat{k}.\end{aligned} \tag{12}$$

We now see that the downward flux of the vector field $\vec{F}$ is given by

$$\iint_{\mathcal{S}} \vec{F} \cdot \hat{n}dS = \iint_R \langle 0, 0, -1\rangle \cdot \langle -s_x(x, y), -s_y(x, y), 1\rangle dA \tag{13}$$
$$= \iint_R -1dA = \boxed{-\text{Area}(R)}.$$

One way in which to physically interpret this result is the following. If $\mathcal{S}$ is modeling the roof of a house built over the region R, and $\vec{F}$ represents the force of rain drops that are falling straight down, then the downward flux of the rain on the roof (the force imparted by the rain onto the roof) of the house depends only on the area of the base of the house, not the shape of the roof.

Problem 5: Let $\mathcal{S}$ be the upper half of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$ and let $\vec{F}=\langle z,x,y\rangle$. Use Stoke's theorem to evaluate

$$\iint_{\mathcal{S}}(\nabla\times\vec{F})\cdot\hat{n}dS. \tag{14}$$

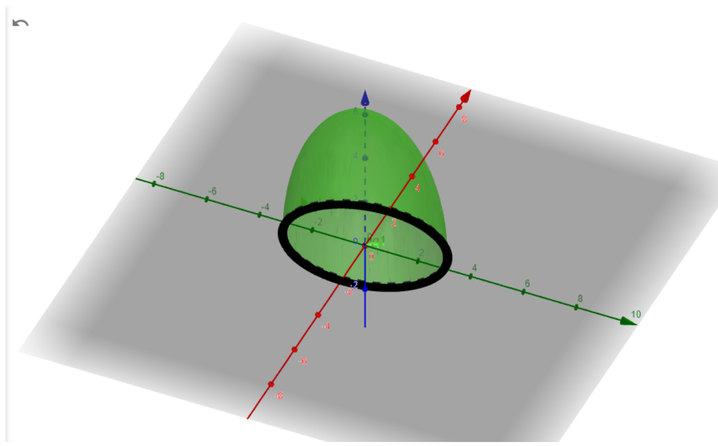


Figure 2: A view of $\mathcal{S}$ and $\partial\mathcal{S}$.

Solution: We see that the boundary $\partial\mathcal{S}$ of $\mathcal{S}$ is obtained when $z=0$, so it is given by the equation $\frac{x^2}{4}+\frac{y^2}{9}=1$. Since $\partial\mathcal{S}$ is an ellipse in the $xy-plane$, we see that it can be parametrized by $\vec{r}(t)=\langle 2\cos(t),3\sin(t),0\rangle$ for $0\le t\le 2\pi$. We observe that

$$\vec{F}(\vec{r}(t))=\langle 0,2\cos(t),3\sin(t)\rangle \text{ and } \vec{r}\,'(t)=\langle -2\sin(t),3\cos(t),0\rangle. \tag{15}$$

We now use Stoke's theorem to see that

$$\begin{aligned}\iint_{\mathcal{S}}(\nabla\times\vec{F})\cdot\hat{n}dS &= \int_{\partial\mathcal{S}}\vec{F}\cdot d\vec{r}=\int_0^{2\pi}\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt \\ &= \int_0^{2\pi}\langle 0,2\cos(t),3\sin(t)\rangle\cdot\langle -2\sin(t),3\cos(t),0\rangle dt \\ &= \int_0^{2\pi}(0+6\cos^2(t)+0)dt \ \left(\cos(2t)=2\cos^2(t)-1\right) \\ &= \int_0^{2\pi}(3\cos(2t)+3)dt=\frac{3}{2}\sin(2t)+3t\Big|_0^{2\pi}=\boxed{6\pi}.\end{aligned} \tag{16}$$

Problem 6: Let C be the circle $x^2 + y^2 = 12$ in the plane $z = 0$ (as a subset of $\mathbb{R}^3$) and let $\vec{F} = \langle (x+4)^x, y\ln(y+4), e^{z^2+\sqrt{z}}\rangle$. Use Stoke's theorem to evaluate

$$\oint_C \vec{F}\cdot d\vec{r}. \tag{17}$$

Solution: It is clear that the line integral in equation (17) is very difficult to evaluate directly, and the formulation of the problem suggests that the surfaces integral arising from Stoke's theorem will be easier to evaluate. To this end, we begin by verifying that $\nabla\times\vec{F} = \vec{0}$ whenever $\vec{F}$ is of the form $\vec{F} = \langle f_1(x), f_2(y), f_3(z)\rangle$.

$$\begin{aligned}\nabla\times\vec{F} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1(x) & f_2(y) & f_3(z)\end{vmatrix} \\ &= \hat{i}\left(\frac{\partial f_3(z)}{\partial y} - \frac{\partial f_2(y)}{\partial z}\right) - \hat{j}\left(\frac{\partial f_3(z)}{\partial x} - \frac{\partial f_1(x)}{\partial z}\right) \\ &\quad + \hat{k}\left(\frac{\partial f_2(y)}{\partial x} - \frac{\partial f_1(x)}{\partial y}\right) \\ &= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}.\end{aligned} \tag{18}$$

We may now view C as the boundary ∂S of the upper half of the sphere of radius $2\sqrt{3}$ $\mathcal{S}$ so that we may apply Stoke's Theorem.

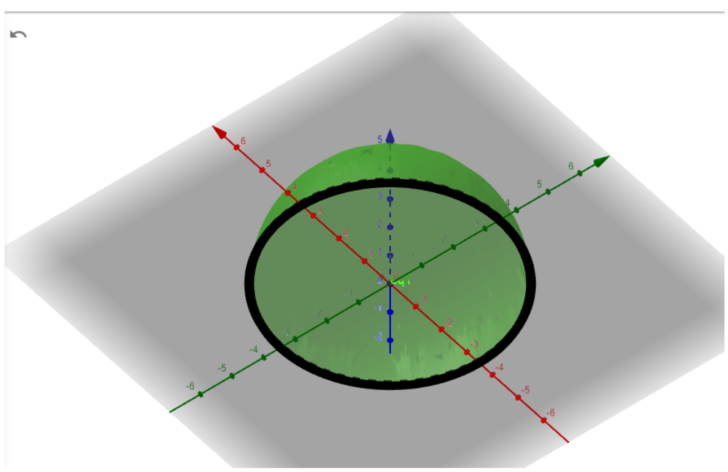


Figure 3: A view of C and $\mathcal{S}$.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_{\mathcal{S}} (\nabla\times\vec{F})dS = \iint_{\mathcal{S}} 0dS = \boxed{0}. \tag{19}$$

Remark: We could have applied the same procedure for the vector field $\vec{F} = \langle (x +$

$4)^x, y\ln(y+4)\rangle$ by identifying it with the vector field $\vec{F} = \langle (x+4)^x, y\ln(y+4), 0\rangle$. In particular, a 2-dimensional circulation integral may become easier by viewing it as a circulation integral in 3-dimensions and using Stoke's theorem.

Problem 7: Let $\mathcal{S}$ be the surface of the cube cut from the first octant by the planes $x = 1, y = 1$, and $z = 1$. Let $\vec{F} = \langle x^2, 2xz, y^2 \rangle$. Use the divergence theorem to evaluate the net outward flux of $\vec{F}$ across $\mathcal{S}$.

Solution: We begin by observing that

$$\text{Div}(\vec{F}) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}(y^2) = 2x + 0 + 0 = 2x. \tag{20}$$

We may now apply the Diverence theorem to see that

$$\begin{aligned} \text{Flux}(\vec{F}, \mathcal{S}) &= \iint_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = \iiint_{\text{int}(\mathcal{S})} \text{Div}(\vec{F}) dV \\ &= \int_0^1 \int_0^1 \int_0^1 2x dx dy dz = \int_0^1 \int_0^1 (x^2 \Big|_{x=0}^1) dy dz \\ &= \int_0^1 \int_0^1 1 dy dz = \boxed{1}. \end{aligned} \tag{21}$$

Remark: One of the benefits to calculating the divergence in this problem with the divergence theorem rather than by direct calculation is that it is easier to evaluate 1 triple integral than a sum of 6 surface integrals.

Problem 8: Let $\mathcal{S}$ be the boundary of the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$ and let $\vec{F} = \langle x^2e^y\cos(z), -4xe^y\cos(z), 2xe^y\sin(z)\rangle$. Evaluate the outward flux of $\vec{F}$ across $\mathcal{S}$.

Solution: We begin by observing that

$$\begin{aligned}\text{Div}(\vec{F}) &= \frac{\partial}{\partial x}(x^2e^y\cos(z)) + \frac{\partial}{\partial y}(-4xe^y\cos(z)) + \frac{\partial}{\partial z}(2xe^y\sin(z)) \\ &= 2xe^y\cos(z) - 4xe^y\cos(z) + 2xe^y\cos(z) = 0.\end{aligned} \tag{22}$$

We may now apply the Diverence theorem to see that

$$\begin{aligned}\text{Flux}(\vec{F}, \mathcal{S}) &= \iint_{\mathcal{S}} \vec{F}\cdot\hat{n}dS = \iiint_{\text{int}(\mathcal{S})} \text{Div}(\vec{F})dV \\ &= \iiint_{\text{int}(\mathcal{S})} 0dV = \boxed{0}.\end{aligned} \tag{23}$$