Practice Problems and Solutions for Multivariable Calculus, Linear Algebra, and Differential Equations

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1. How to use this document

For Students: As with any other skill, mathematics is best learned from actively doing mathematics instead of passively watching it. This document contains a list of exercises from multivariable calculus, linear algebra, ordinary differential equations, and introductory partial differential equations. These exercises were used in my recitation sessions at OSU for Math 2153, 2173, 2177, and 2255. If you are taking any of these courses or a related course, then you should use the problems in this document as a list of practice problems. After listening to a lecture on a topic and possibly doing the course homework, you can find similar problems in this document for additional practice. If you solve a problem, then you can click on the page number of the solution to jump to that page in the document and compare your answer. In order to ensure that you get the most use out of this document, you should be stuck on a problem (no new ideas generated) for at least 10 minutes before you decide that you don't know how to solve the problem and go read the solution instead. This will ensure that you are actively engaged with the material, and will help you learn more from reading the solution than what you would have learned from reading it earlier. This point is so important that I will repeat it once more. In order to ensure that you get the most use out of this document, you should be stuck on a problem (no new ideas generated) for at least 10 minutes before you decide that you don't know how to solve the problem and go read the solution instead. This will ensure that you are actively engaged with the material, and will help you learn more from reading the solution than what you would have learned from reading it earlier.

For Instructors: Instructors may provide this document to their students as a list of problems for additional practice after the main homework assignments. Alternatively, instructors may create practice exams to give to their students using the problems from this document. Since the solutions are also available, it will be easy for the instructor to release solutions to the practice exams as well. Recitation instructors may present their own solutions to the problems from this list in their recitations, and make the solutions here available to the students as well for review after class and alternative perspectives on the same problem.

Please email me at sohail.farhangi@gmail.com to notify me about any typos or errors that you find in this document.

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2. Problems

2.1. Vectors, Partial Derivatives, Gradient Vectors.

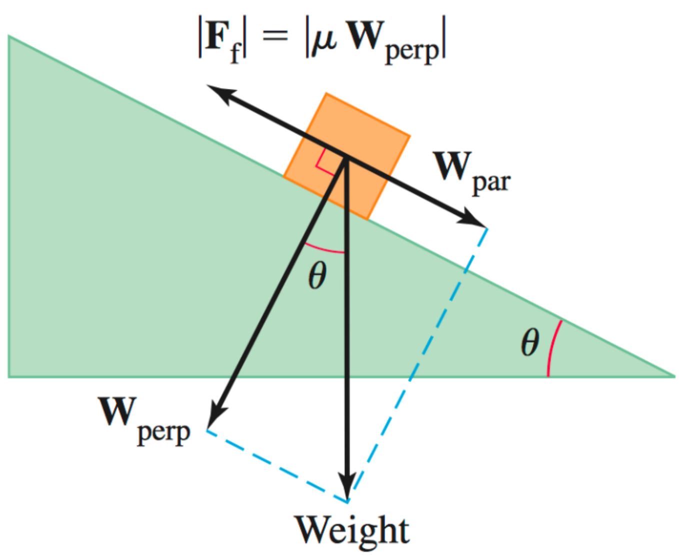
Problem 1.1. *A suitcase is pulled 50ft along a horizontal sidewalk with a constant force of 30lb at an angle of* $30°$ *above the horizontal. How much work is done?*

The solution to Problem 1.1 is on page 43.

Problem 1.2. *A constant force of* $\vec{F} = \langle 2, 4, 1\rangle N$ *moves an object from* $(0, 0, 1)m$ *to* $(2, 4, 6)m$*. How much work is done?*

The solution to Problem 1.2 is on page 44.

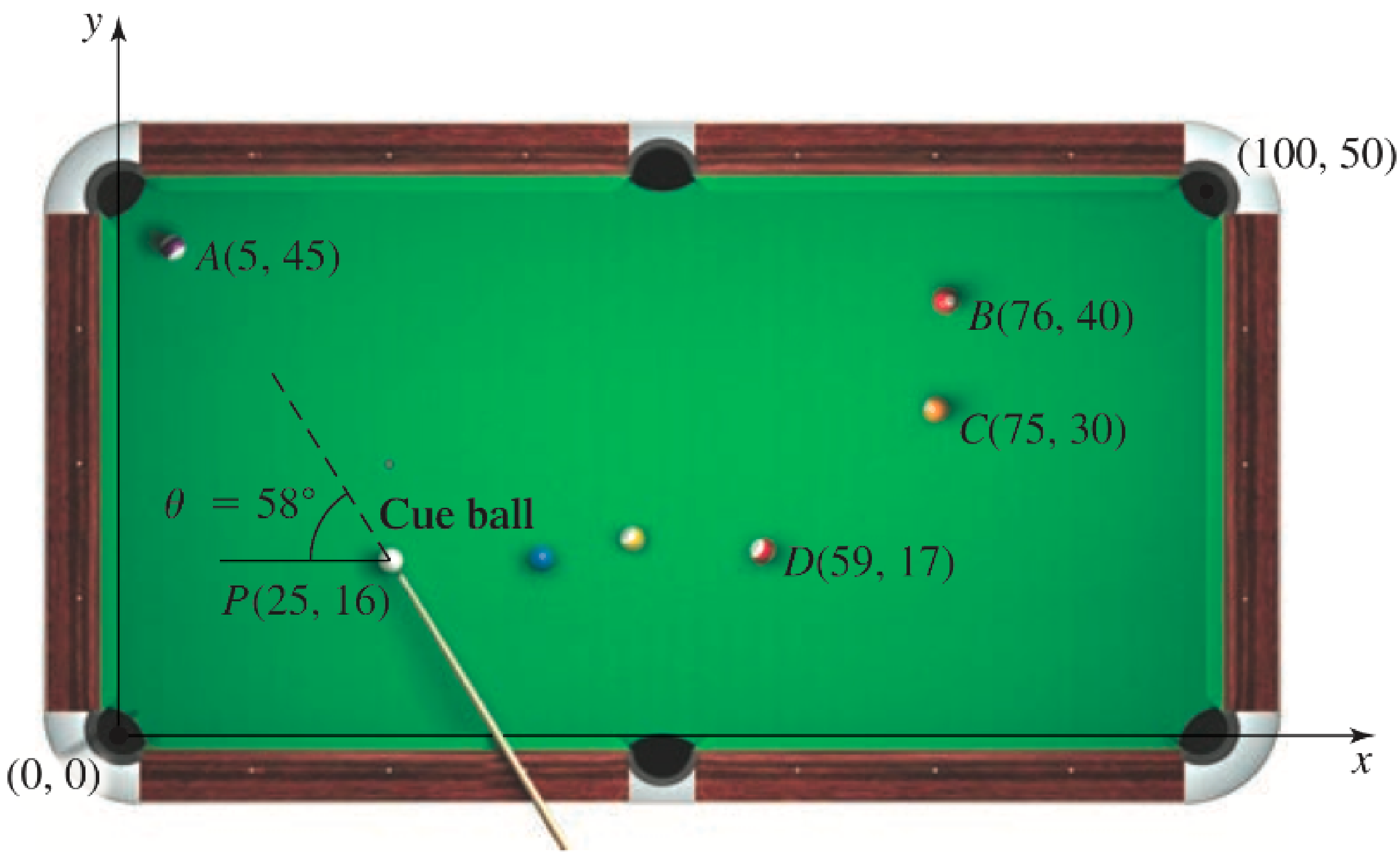
Problem 1.3. *An object on an inclined plane does not slide provided the component of the object's weight parallel to the plane* $|\mathbf{W}_{par}|$ *is less than or equal to the magnitude of the opposing frictional force* $|\mathbf{F}_f|$*. The magnitude of the frictional force, in turn, is proportional to the component of the object's weight perpendicular to the plane* $|\mathbf{W}_{perp}|$*. The constant of proportionality is the coefficient of static friction* $\mu > 0$*. Suppose a 100lb block rests on a plane that is tilted at an angle of* $\theta = 30°$ *to the horizontal. What is the smallest possible value of* μ*?*



The solution to Problem 1.3 is on page 45.

Problem 1.4. *A cue ball in a billiards video game lies at* $P(25, 16)$*. We assume that each ball has a diameter of 2.25 screen units, and pool balls are represented by the point at their center.*

- a. *The cue ball is aimed at an angle of* $58°$ *above the negative* x*-axis toward a target ball at* $A(5, 45)$*. Do the balls collide?*
- b. *The cue ball is aimed at the point* $(50, 25)$ *in an attempt to hit a target ball at* $B(76, 40)$*. Do the balls collide?*
- c. *The cue ball is aimed at an angle* θ *above the* x*-axis in the general direction of a target ball at* $C(75, 30)$*. What range of angles (for* $0 \leq \theta \leq \frac{\pi}{2}$*) will result in a collision? Express your answer in degrees.*



The solution to Problem 1.4 is on page 49.

Problem 1.5. *Determine whether the lines $\vec{r}(t) = \langle 1, 3, 2\rangle + t\langle 6, -7, 1\rangle$ and $R(s) = \langle 10, 6, 14\rangle + s\langle 8, 1, 4\rangle$ are parallel or skew, and find their intersection(s) if any exist.*

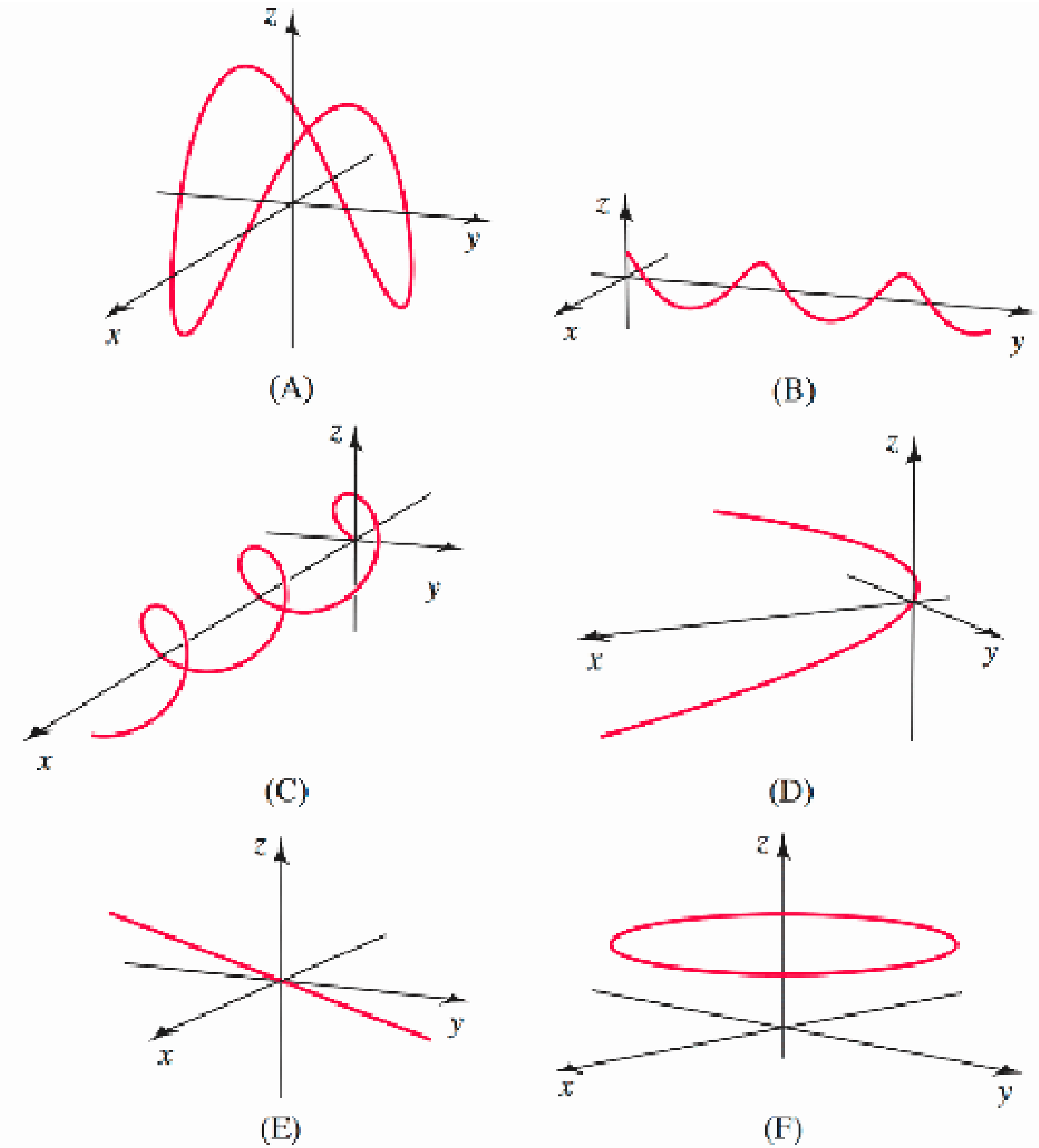
The solution to Problem 1.5 is on page 56.

Problem 1.6. *Find an equation of the plane P through the points $R(5, 3, 7)$, $S(0, 1, 0)$, and $T(1, 2, 1)$.*

The solution to Problem 1.6 is on page 94.

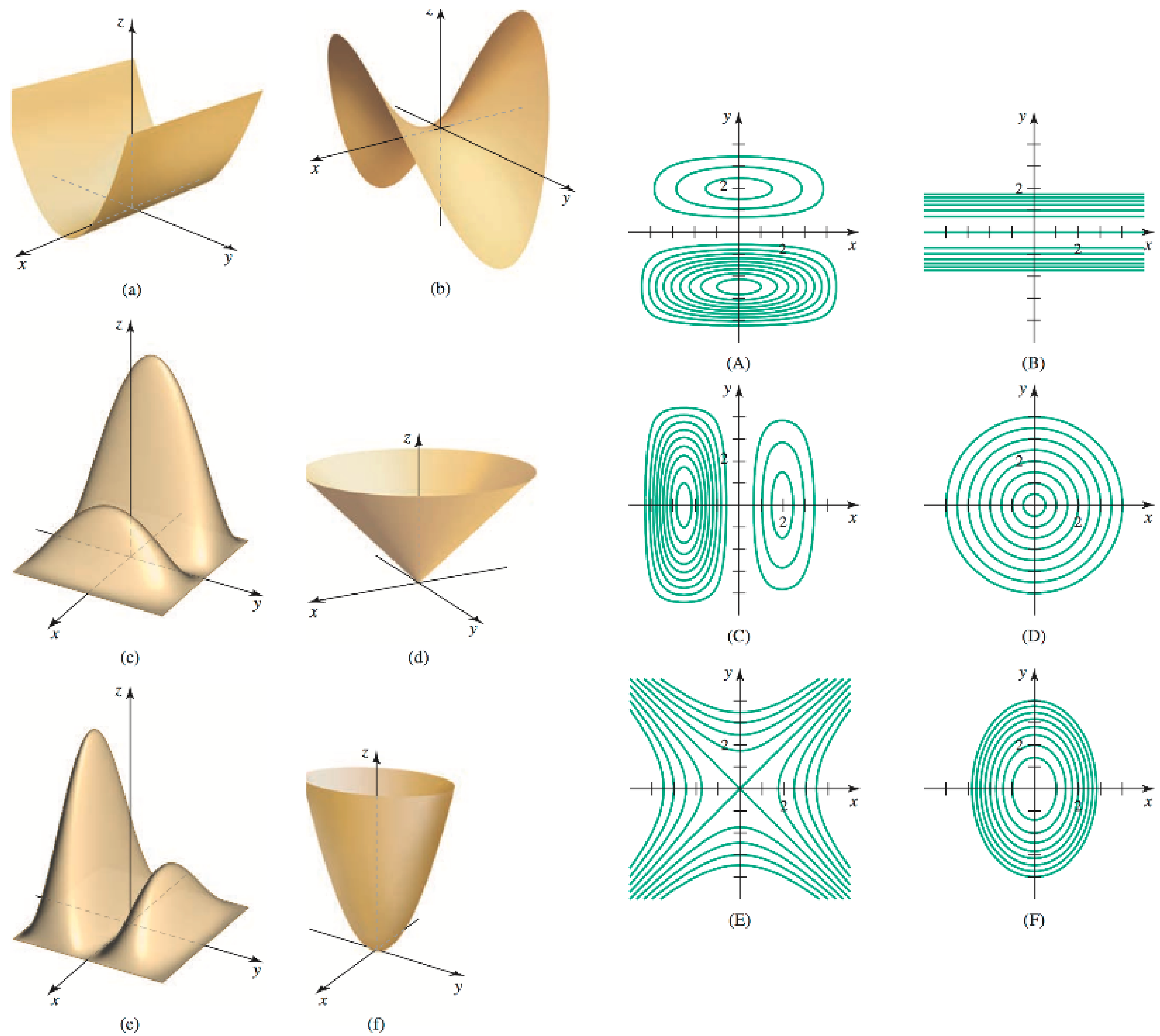
Problem 1.7. *Match function a-f with the appropriate graph A-F.*

a. $\vec{r}(t) = \langle t, -t, t\rangle$.
b. $\vec{r}(t) = \langle t^2, t, t\rangle$.
c. $\vec{r}(t) = \langle 4\cos(t), 4\sin(t), 2\rangle$.
d. $\vec{r}(t) = \langle 2t, \sin(t), \cos(t)\rangle$.
e. $\vec{r}(t) = \langle \sin(t), \cos(t), \sin(2t)\rangle$.
f. $\vec{r}(t) = \langle \sin(t), 2t, \cos(t)\rangle$.



The solution to Problem 1.7 is on page 59.

Problem 1.8. *Match surfaces a-f in the figure below with level curves A-F.*



The solution to Problem 1.8 is on page 60.

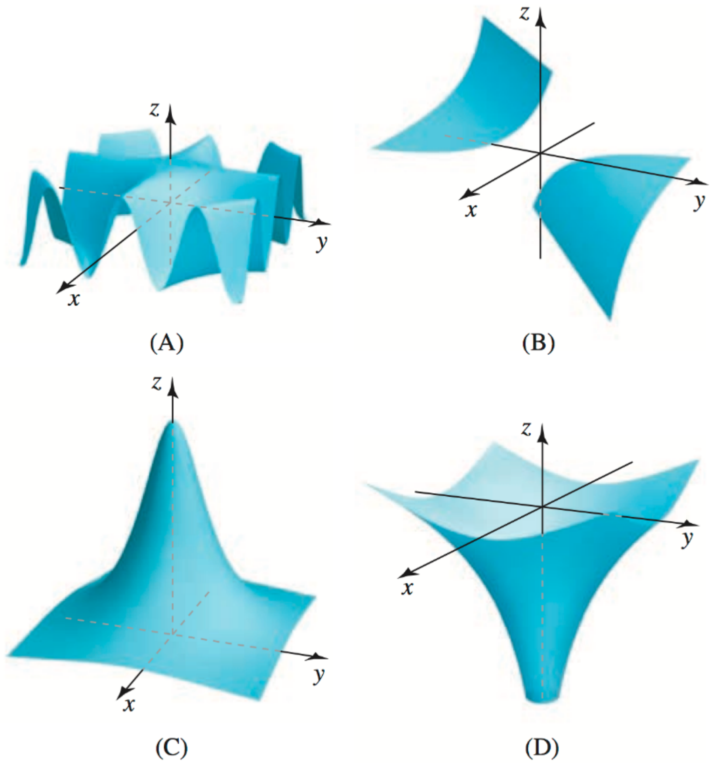
Problem 1.9. *Match functions a-d with surfaces A-D in the figure below.*

a. $f(x, y) = \cos(xy)$

b. $g(x, y) = \ln(x^2 + y^2)$

c. $h(x, y) = \frac{1}{x-y}$

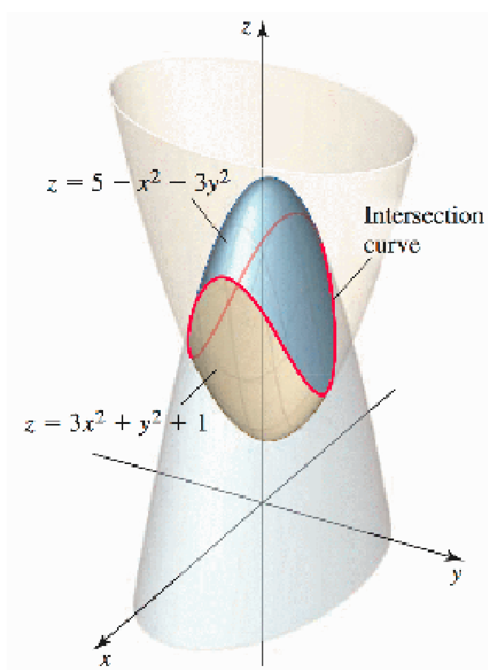
d. $p(x, y) = \frac{1}{1+x^2+y^2}$



The solution to Problem 1.9 is on page 62.

Problem 1.10. *Find a function $\vec{r}(t)$ that describes the curve $\mathcal{C}$ which is the intersection of the surfaces $z = 3x^2 + y^2 + 1$ and $z = 5 - x^2 - 3y^2$. Note that there is not a unique answer to this question since any curve possess infinitely many distinct paramterizations.*

$z = 3x^2 + y^2 + 1; z = 5 - x^2 - 3y^2$



The solution to Problem 1.10 is on page 65.

Problem 1.11. *Suppose that $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector valued functions satisfying $\vec{u}(0) = \langle 0,1,1\rangle, \vec{u}\,'(0) = \langle 0,7,1\rangle, \vec{v}(0) = \langle 0,1,1\rangle$, and $\vec{v}\,'(0) = \langle 1,1,2\rangle$. Evaluate the following expressions.*

a. $\frac{d}{dt}\Big(\vec{u}(t)\cdot\vec{v}(t)\Big)\Big|_{t=0}$

b. $\frac{d}{dt}\Big(\vec{u}(t)\times\vec{v}(t)\Big)\Big|_{t=0}$

c. $\frac{d}{dt}\Big(\cos(t)\vec{u}(t)\Big)\Big|_{t=0}$

d. $\frac{d}{dt}\Big(\vec{u}(\sin(t))\Big)\Big|_{t=0}$

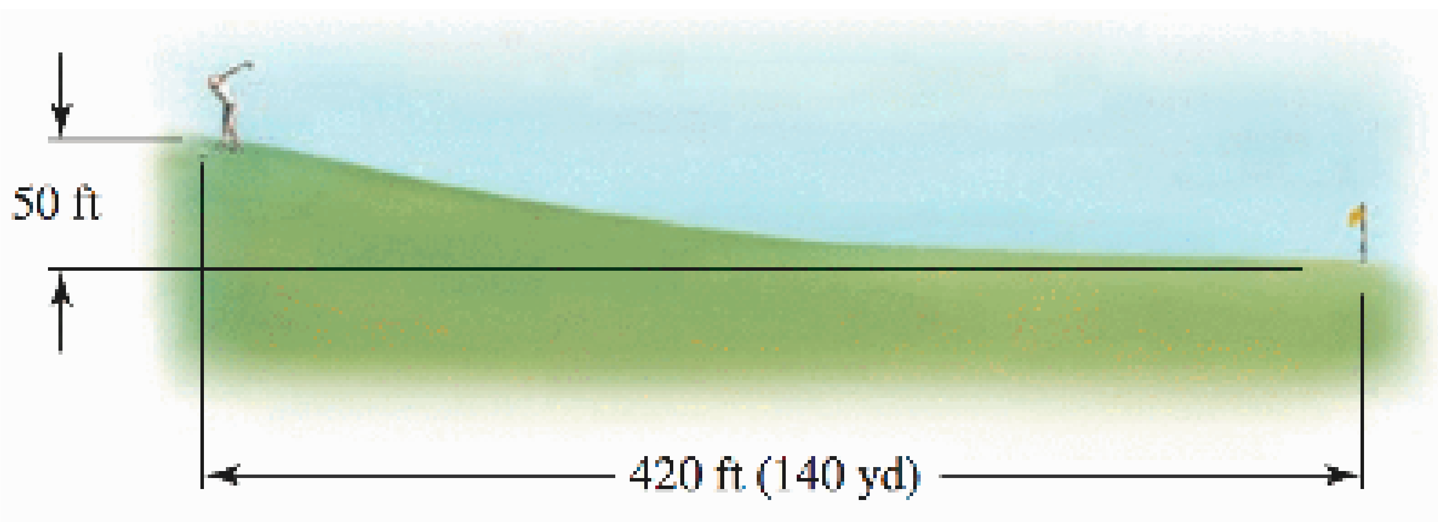
The solution to Problem 1.11 is on page 66.

Problem 1.12. *Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.*

(a) *If the speed of an object is constant, then its velocity components are constant.*
(b) *The functions $\vec{r}(t) = \langle\cos(t), \sin(t)\rangle$ and $\vec{R}(t) = \langle\cos(t^2), \sin(t^2)\rangle$ generate the same set of points for $t \geq 0$. (Bonus: What about for $t \geq \pi^2$?)*
(c) *A velocity vector (vector valued function) of variable magnitude cannot have constant direction.*
(d) *If the acceleration of an object is $\vec{a}(t) = \vec{0}$, for all $t \geq 0$, then the velocity of the object is constant.*
(e) *If you double the initial speed of a projectile, its range also double (assume no forces other than gravity).*
(f) *If you double the initial speed of a projectile, its time of flight also doubles (assume no forces other than gravity).*
(g) *A trajectory with $\vec{v}(t) = \vec{a}(t) \neq \vec{0}$, for all t, is possible.*

The solution to Problem 1.12 is on page 69.

Problem 1.13. *A golfer stands 420ft (140yd) horizontally from the hole and 50ft above the hole (see figure). Assuming the ball is hit with an initial speed of 120ft/s, at what angle(s) should it be hit to land in the hole? Assume the path of the ball lies in a plane. You may approximate earth's gravitational constant by $32ft/s^2$.*



The solution to Problem 1.13 is on page 72.

Problem 1.14. *The electric field due to a point charge of strength Q at the origin has a potential function $V(x, y, z) = kQ/r$, where $r^2 = x^2 + y^2 + z^2$ is the square of the distance between a variable point $P(x, y, z)$ at the charge, and $k > 0$ is a physical constant. The electric field is given by $\mathbf{E}(x, y, z) = -\nabla V(x, y, z)$.*

a. *Show that*

(1)
$$\mathbf{E}(x, y, z) = kQ\langle\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle.$$

b. *Show that $|\mathbf{E}| = kQ/r^2$. Explain why this relationship is called the inverse square law.*

The solution to Problem 1.14 is on page 96.

Problem 1.15. *Consider the function $F(x, y, z) = e^{xyz}$.*

a. *Write F as a composite function $f \circ g$, where f is a function of one variable and g is a function of three variables.*
b. *Calculate $\nabla F(x, y, z)$ as well as $\nabla g(x, y, z)$. Find a relationship between $\nabla F(x, y, z)$ and $\nabla g(x, y, z)$.*

The solution to Problem 1.15 is on page 98.

Problem 1.16. *Consider the function $f(x, y) = \ln(1 + 4x^2 + 3y^2)$ and the point $P = (\frac{3}{4}, -\sqrt{3})$.*

a. *Find the gradient field $\nabla f(x, y)$ of $f(x, y)$ and then evaluate it at P.*
b. *Find the angles θ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.*
c. *Write the directional derivative at P as a function of θ; call this function $g(\theta)$.*
d. *Find the value of θ that maximizes $g(\theta)$ and find the maximum value.*
e. *Verify that the value of θ that maximizes g corresponds to the direction of the gradient vector at P. Verify that the maximum value of g equals the magnitude of the gradient vector at P.*

The solution to Problem 1.16 is on page 99.

Problem 1.17. *Find the gradient field $\vec{F} = \nabla\varphi$ for the potential function*

$$\varphi(x, y) = \sqrt{x^2 + y^2}, \quad \text{for } x^2 + y^2 \leq 9, (x, y) \neq (0, 0). \tag{2}$$

Sketch two level curves of φ and two vectors of $\vec{F}$ of your choice.

The solution to Problem 1.17 is on page 102.

Problem 1.18. *Below is a contour plot of some function $z = f(x, y)$ along with 4 vectors.*

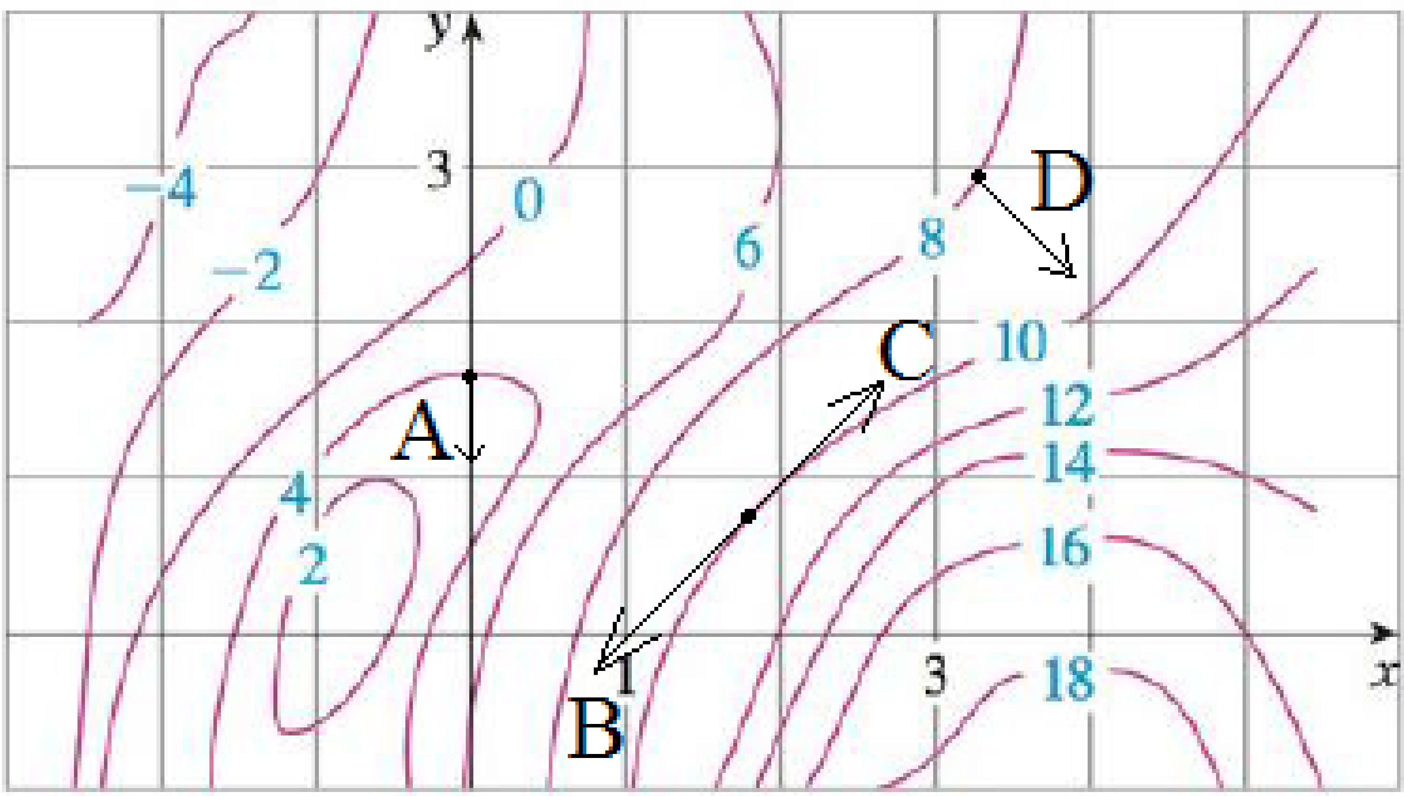


FIGURE 1. Contour plot of $z = f(x, y)$.

Which of the vectors in the above plot could possibly be a gradient vector of the function $f(x, y)$? Please circle all that apply.

(A) (B) (C) (D) (E) None of the given vectors

The solution to Problem 1.18 is on page 103.

Problem 1.19. *Consider the function $f(x, y) = x^2 + y^2$ and and the point $P = (2, 3)$.*

(a) *Find the unit vector that points in direction of maximum decrease of the function f at the point P.*

(b) *Calculate the directional derivative of f at the point P in the direction of the vector $\vec{u} = \langle 3, 2\rangle$.*

The solution to Problem 1.19 is on page 111.

Problem 1.20. *Imagine a string that is fixed at both ends (for example, a guitar string). When plucked, the string forms a standing wave. The displacement u of the string varies with position x and with time t. Suppose it is given by $u = f(x,t) = 2\sin(\pi x)\sin(\frac{\pi}{2}t)$, for $0 \le x \le 1$ and $t \ge 0$ (see figure 12). At a fixed point in time, the string forms a wave on $[0,1]$. Alternatively, if you focus on a point on the string (fix a value of x), that point oscillates up and down in time.*

(a) *What is the period of the motion in time?*
(b) *Find the rate of change of the displacement with respect to time at a constant position (which is the vertical velocity of a point on the string).*
(c) *At a fixed time, what point on the string is moving fastest?*
(d) *At a fixed position on the string, when is the strong moving fastest?*
(e) *Find the rate of change of the displacement with respect to position at a constant time (which is the slope of the string).*
(f) *At a fixed time, where is the slope of the string greatest?*

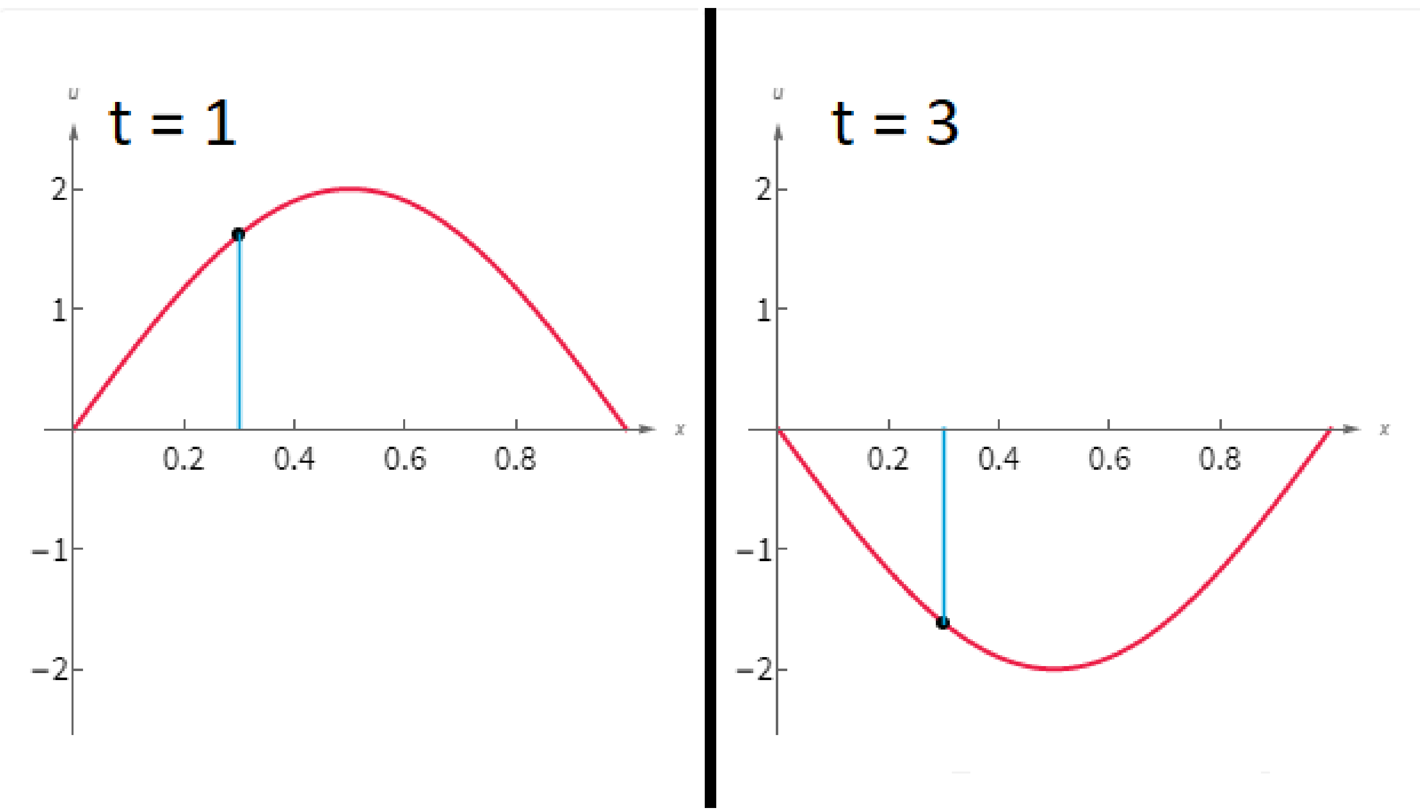


FIGURE 2. Snapshots of the wave at times $t = 1$ and $t = 3$.

The solution to Problem 1.20 is on page 104.

Problem 1.21. *Let $w = f(x,y,z) = 2x + 3y + 4z$, which is defined for all $(x,y,z) \in \mathbb{R}^3$. Suppose we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.*

(a) *We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = w(x,y) = f(x,y,z(x,y))$. Show that $\frac{\partial}{\partial x}w(x,y) = 18$.*
(b) *Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = w(x,z) = f(x,y(x,z),z)$. Show that $\frac{\partial}{\partial x}w(x,z) = 8$.*
(c) *Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.*

The solution to Problem 1.21 is on page 109.

2.2. Unit Tangent/Normal/Binormal Vectors, Arclength, Curvature, Limits and Differentiability in Higher Dimensions.

Problem 2.1. *Let $\vec{r}(t) = \langle t, 2, \frac{2}{t}\rangle$ for $t > 1$. Find the unit tangent vector $\hat{T}(t)$ at all points of the curve $\vec{r}(t)$.*

The solution to Problem 2.1 is on page 68.

Problem 2.2. *Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.*

(a) *If an object moves on a trajectory with constant speed S over a time interval $a \leq t \leq b$, then the length of the trajectory is $S(b-a)$.*
(b) *The curves defined by*

(3) $$\vec{r}(t) = \langle f(t), g(t)\rangle \text{ and } \vec{R}(t) = \langle g(t), f(t)\rangle$$

have the same length over the interval $[a,b]$.
(c) *The curve $\vec{r}(t) = \langle f(t), g(t)\rangle$, for $0 \leq a \leq t \leq b$, and the curve $\vec{R}(t) = \langle f(t^2), g(t^2)\rangle$, for $\sqrt{a} \leq t \leq \sqrt{b}$, have the same length.*
(d) *The curve $\vec{r}(t) = \langle t, t^2, 3t^2\rangle$, for $1 \leq t \leq 4$, is parameterized by arclength.*

The solution to Problem 2.2 is on page 75.

Problem 2.3. *Consider the curve $\mathcal{C}$ that is described by the parameterization $\vec{r}(t) = \langle t^m, t^m, t^{\frac{3}{2}m}\rangle$ where $0 \leq a \leq t \leq b$ and $m \neq 0$.*

(a) *Find the arclength function $s(t)$. Note that your answer may include a, b, and m in it.*
(b) *Find the parameterization by arclength for $\mathcal{C}$ when $a = \sqrt{\frac{28}{9}}, b = 4$, and $m = 2$.*

The solution to Problem 2.3 is on page 77.

Problem 2.4. *Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.*

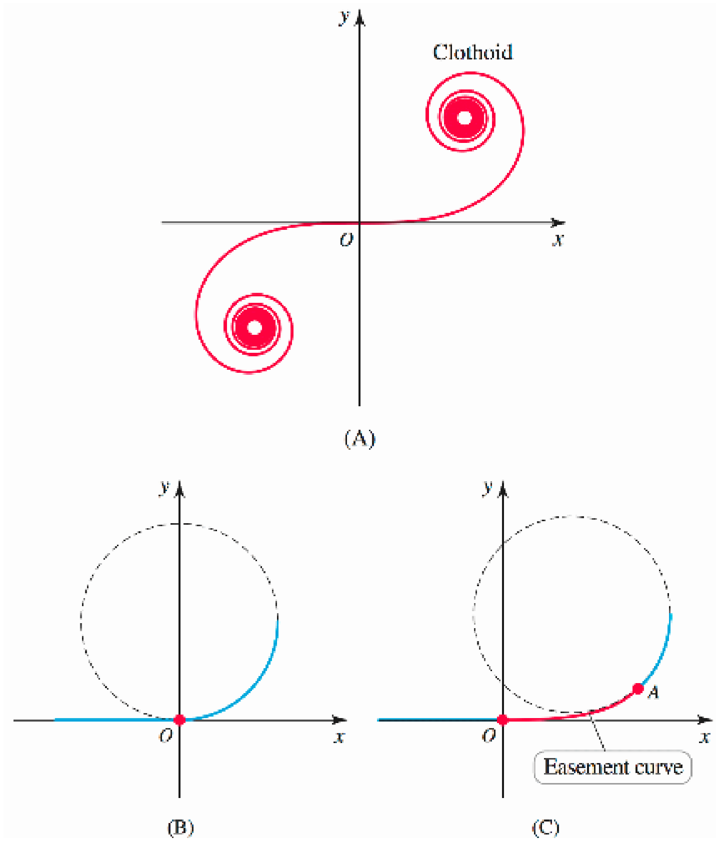
(a) *The position, unit tangent, and principal unit normal vectors ($\vec{r}, \hat{T}$, and $\hat{N}$) at a point lie in the same plane.*
(b) *The vectors $\hat{T}$ and $\hat{N}$ at a point depend on the orientation of a curve.*
(c) *The curvature at a point depends on the orientation of a curve.*
(d) *An object with unit speed ($|\vec{v}| = 1$) on a circle of radius R has an acceleration of $\vec{a} = \frac{1}{R}\hat{N}$.*
(e) *If the speedometer of a car reads a constant 60 mi/hr, the car is not accelerating.*
(f) *A curve in the xy-plane that is concave up at all points has positive torsion.*
(g) *A curve with large curvature also has large torsion.*

The solution to Problem 2.4 is on page 79.

Problem 2.5. *Compute the unit binormal vector $\hat{B}$ and torsion τ of the curve parameterized by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), -t\rangle, t \in \mathbb{R}(-\infty < t < \infty)$.*

The solution to Problem 2.5 is on page 82.

Problem 2.6. *The function $\vec{r}(t) = \langle \int_0^t \cos(\frac{1}{2}u^2)du, \int_0^t \sin(\frac{1}{2}u^2)du\rangle, t \in \mathbb{R}$ whose graph is called a* **clothoid** *or* **Euler Spiral**, *has applications in the design of railroad tracks, rollercoasters, and highways.*



(a) *A car moves from left to right on a straight highway, approaching a curve at the origin (Figure B). Sudden changes in curvature at the start of the curve may cause the driver to jerk the steering wheel. Suppose the curve starting at the origin is a segment of a circle of radius a. Explain why there is a sudden change in the curvature of the road at the origin.*

(b) *A better approach is to use a segment of a clothoid as an easement curve, in between the straight highway and a circle, to avoid sudden changes in curvature (Figure C). Assume the easement curve corresponds to the clothoid $\vec{r}(t)$, for $0 \leq t \leq 1.2$. Find the curvature of the easement curve as a function of t and explain why this curve eliminates the sudden change in curvature at the origin.*

(c) *Find the radius of a circle connected to the easement curve at point A (that corresponds to $t = 1.2$ on the curve $\vec{r}(t)$) so that the curvature of the circle matches the curvature of the easement curve at point A.*

The solution to Problem 2.6 is on page 92.

Problem 2.7. *Verify that*

(4)
$$\lim_{(x,y)\to(0,0)} \frac{\sin(x) + \sin(y)}{x + y} = 1.$$

The solution to Problem 2.7 is on page 84.

Problem 2.8. *Consider the function*

(5)
$$f(x, y) = \frac{xy^2}{x^2 + y^4}.$$

(a) *Show that if L is a line that passes through the origin, then*

(6)
$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in L}} f(x, y) = 0.$$

(b) *Show that*

(7)
$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist.

The solution to Problem 2.8 is on page 86.

Problem 2.9. *Consider the function* $f(x,y) = \sqrt{|xy|}$.

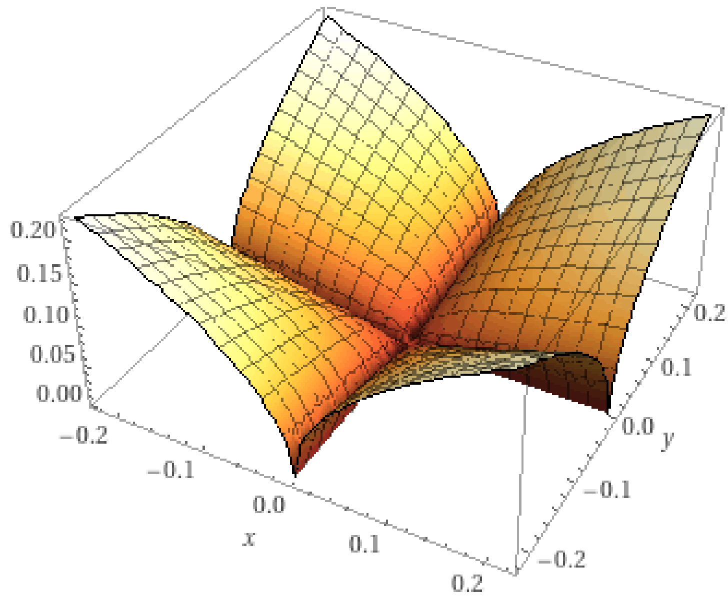


FIGURE 3. A graph of $z = \sqrt{|xy|}$.

(a) *Is* f *continuous at* $(0,0)$*?*
(b) *Show that* $f_x(0,0)$ *and* $f_y(0,0)$ *exist by calculating their values.*
(c) *Determine whether* f_x *and* f_y *are continuous at* $(0,0)$.
(d) *Is* f *differentiable at* $(0,0)$*?*

The solution to Problem 2.9 is on page 88.

2.3. **Optimization: Second Derivative Test, Lagrange Multipliers.**

Problem 3.1. *Determine all critical points of the function* $f(x,y) = x^3 - y^3 + xy$*, then classify each of the critical points as a local maximum, local minimum, or saddle point.*

The solution to Problem 3.1 is on page 112.

Problem 3.2. *A lidless cardboard box is to be made with a volume of 4* m^3*. Find the dimensions of the box that require the least cardboard.*

The solution to Problem 3.2 is on page 113.

Problem 3.3. *Consider the function* $f(x,y) = 3 + x^4 + 3y^4$*. Show that* $(0,0)$ *is a critical point for* $f(x,y)$ *and show that the second derivative test is inconclusive at* $(0,0)$*. Then describe the behavior of* $f(x,y)$ *at* $(0,0)$.
Hint: The product of 2 negative numbers is positive.

The solution to Problem 3.3 is on page 117.

Problem 3.4. *Show that the second derivative test is inconclusive when applied to the function* $f(x,y) = x^4y^2$ *at the point* $(0,0)$*. Show that* $f(x,y)$ *has a local minimum at* $(0,0)$ *by direct analysis.*
Hint: The product of 2 negative numbers is positive.

The solution to problem 3.4 is on page 118.

Problem 3.5. *Find the absolute minimum and absolute maximum values of the function* $f(x,y) = xy$ *over the region* $R = \{(x,y) \mid (x-1)^2 + y^2 \leq 1\}$.

The solution to Problem 3.5 is on page 119.

Problem 3.6. *Find the absolute minimum and maximum value of the function*

$$f(x, y) = 2x^2 - 4x + 3y^2 + 2 = 2(x-1)^2 + 3y^2 \tag{8}$$

over the region

$$R := \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}. \tag{9}$$

The solution to Problem 3.6 is on page 121.

Problem 3.7. *Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function*

$$f(x, y, z) = xyz \tag{10}$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 9. \tag{11}$$

The solution to Problem 3.7 is on page 124.

Problem 3.8. *What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing that you have found an absolute minimum of the distance function.*

The solution to Problem 3.8 is on page 174.

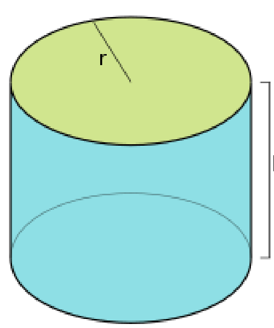
Problem 3.9. *Find the point on the plane $x + y + z = 4$ nearest the point $P(0, 3, 6)$. Remember to justify why your answer is a global minimum and not just a local minimum.*

The solution to Problem 3.9 is on page 177.

Problem 3.10. *Find the point on the plane $2x + 3y + 6z - 10 = 0$ closest to the point $(-2, 5, 1)$ by using the method of Lagrange Multipliers. Can you justify that your answer is a global minimum and not just a local minimum?*

The solution to Problem 3.10 is on page 180.

Problem 3.11. *Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of 32π in^3.*



The solution to problem 3.11 is on page 182.

Problem 3.12. *Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K, L) = CK^aL^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $pK + qL = B$. Find the values of K and L that maximize the production function*

$$P = f(K, L) = 10K^{\frac{1}{3}}L^{\frac{2}{3}} \tag{12}$$

subject to

$$30K + 60L = 360, \tag{13}$$

assuming $K \geq 0$ *and* $L \geq 0$.

The solution to Problem 3.12 is on page 183.

Problem 3.13. *Given the production function* $P = f(K, L) = K^a L^{1-a}$ *and the budget constraint* $pK + qL = B$, *where* $a, p, q,$ *and* B *are given, show that* P *is maximized when* $K = aB/p$ *and* $L = (1-a)B/q$. *(Recall that* $p, q, K, L \geq 0$ *and* $0 < a < 1$ *in order for the model to make sense in the real world and for the production function* f *to be well defined.)*

The solution to Problem 3.13 is on page 185.

Problem 3.14. *Find the absolute minimum and absolute maximum values of the function*

$$f(x, y) = x^2 + 4y^2 + 1 \tag{14}$$

over the region

$$R = \{(x, y) : x^2 + 4y^2 \leq 1\}. \tag{15}$$

You should know how to solve this type of problem using lagrange multipliers, but you can avoid using lagrange multipliers (and even avoid parameterization of the boundary) in this particular problem if you think about it carefully.

The solution to Problem 3.14 is on page 187.

Problem 3.15. *Show that each of the following functions* $f(x, y)$ *have exactly 1 critical point that is a local extrema, but not a global extrema.*[1]

(i) $f(x, y) = e^{3x} + y^3 - 3ye^x$.

(ii) $f(x, y) = x^2 + y^2(1 + x)^3$.

Remark: A continuously differentiable single variable function $f(x)$ *that has exactly 1 critical point that is a local extrema will also have that critical point be a global extrema. This problem shows that the same phenomena does not hold for functions of 2 or more variables.*

The solution to Problem 3.15 is on page 189.

2.4. Double Integrals, Polar Coordinates.

Problem 4.1. *Evaluate*

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^1 yx \sin(x^2) dy dx, \tag{16}$$

$$\int_0^1 \int_0^{\sqrt{\frac{\pi}{2}}} yx \sin(x^2) dx dy, \text{ and} \tag{17}$$

$$\left(\int_0^{\sqrt{\frac{\pi}{2}}} x \sin(x^2) dx \right) \left(\int_0^1 y dy \right) \tag{18}$$

Note that all 3 integrals should result in the same value once evaluated. Please show your work for the calculations of each of the 3 integrals separately.

The solution to Problem 4.1 is on page 191.

Problem 4.2. *Suppose that the second partial derivative of* f *are continuous on* $R = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$. *Show that*

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y) dA = f(a, b) - f(a, 0) - f(0, b) + f(0, 0). \tag{19}$$

Hint: Think about the fundamental theorem of calculus.

The solution to problem 4.2 is on page 193.

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[1] I took item (i) from Tom Vogel of Texas A& M and item (ii) from Henry Wente of University of Toledo.

Problem 4.3. *Let* $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.

a. *Evaluate* $\iint_R \cos(x\sqrt{y})dA$.
b. *Evaluate* $\iint_R x^3 y \cos(x^2y^2)dA$.

Hint: Choose a convenient order of integration.

The solution to problem 4.3 is on page 195.

Problem 4.4. *Let* $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$. *Let* F *be an antiderivative of* f *satisfying* $F(0) = 0$, *and let* G *be an antiderivative of* F. *Show that if* f *and* F *are integrable, and* $r, s \ge 1$ *are real numbers, then*

$$\iint_R x^{2r-1}y^{s-1}f(x^ry^s)dA = \frac{G(1) - G(0)}{rs}. \tag{20}$$

Hint: Pick a convenient order of integration, then apply u-substition. It also helps if you do problem 14.1.60 before doing this problem.

The solution to problem 4.4 is on page 197.

Problem 4.5. *Let* R *be the region in quadrants 1 and 4 bounded by the semicircle of radius 4 centered at* $(0,0)$. *Sketch a picture of* R, *then evaluate*

$$\iint_R x^2 y dA. \tag{21}$$

The solution to problem 4.5 is on page 199.

Problem 4.6. *Let* R *be the region that is bounded by both branches of* $y = \frac{1}{x}$, *the line* $y = x + \frac{3}{2}$, *and the line* $y = x - \frac{3}{2}$.

(a) *Find the area of* R.
(b) *Evaluate*

$$\iint_R xy dA. \tag{22}$$

The solution to Problem 4.6 is on page 131.

Problem 4.7. *Let* R *be the region inside of the ellipse* $\frac{x^2}{18} + \frac{y^2}{36} = 1$ *for which we also have* $y \le \frac{4}{3}x$.

(a) *Find the area of* R.
(b) *Evaluate*

$$\iint_R xy dA. \tag{23}$$

The solution to Problem 4.7 is on page 134.

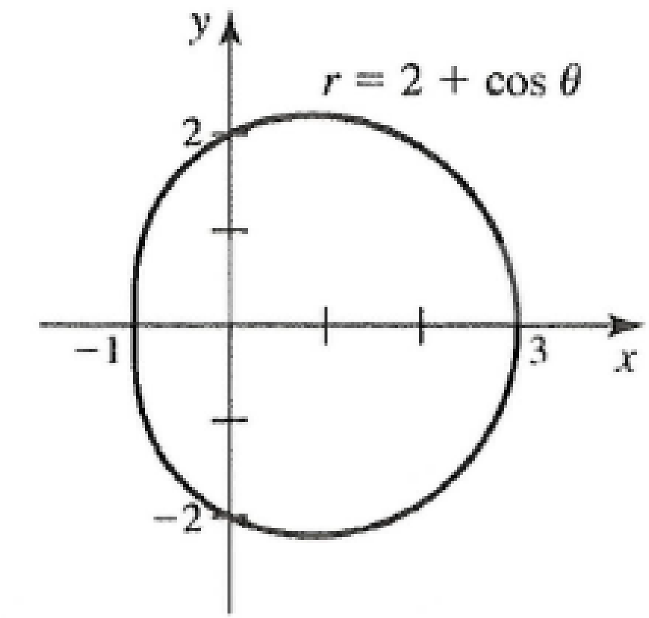
Problem 4.8. *Find the volume of the solid bounded by the planes* $x = 0, x = 5, z = y - 1, z = -2y - 1, z = 0$, *and* $z = 2$.

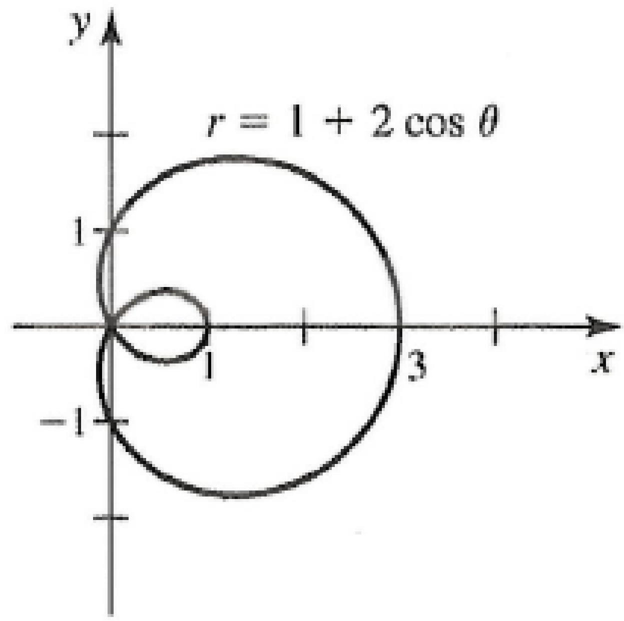
The solution to Problem 4.8 is on page 139.

Problem 4.9. *Let* R *be the region in the xy-plane that is bounded by the spiral* $r = \theta$ *for* $0 \le \theta \le \pi$ *and the x-axis. Find the volume of the 3-dimensional solid* S *that lies above the region* R *and underneath the surface* $z = x^2 + y^2$.

The solution to Problem 4.9 is on page 142.

Problem 4.10. *The limaçon* $r = b + a\cos(\theta)$ *has an inner loop if* $b < a$ *and no inner loop if* $b > a$.





(a) *Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.*
(b) *Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.*
(c) *Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.*

The solution to Problem 4.10 is on page 143.

Problem 4.11. *Let R be the region inside both the cardiod $r = 1 + \sin(\theta)$ and the cardiod $r = 1 + \cos(\theta)$. Sketch a picture of the region R, or create an image of the region R using a graphing program, then use double integration to find the area of R.*

The solution to problem 4.11 is on page 145.

Problem 4.12. *Evaluate*

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5 + 1} dydx \tag{24}$$

by changing the order of integration.

Hint: Start by drawing a picture of the region of integration.

The solution to problem 4.12 is on page 147.

Problem 4.13. *Find the volume of the solid S bounded by the paraboloid $z = 8 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.*

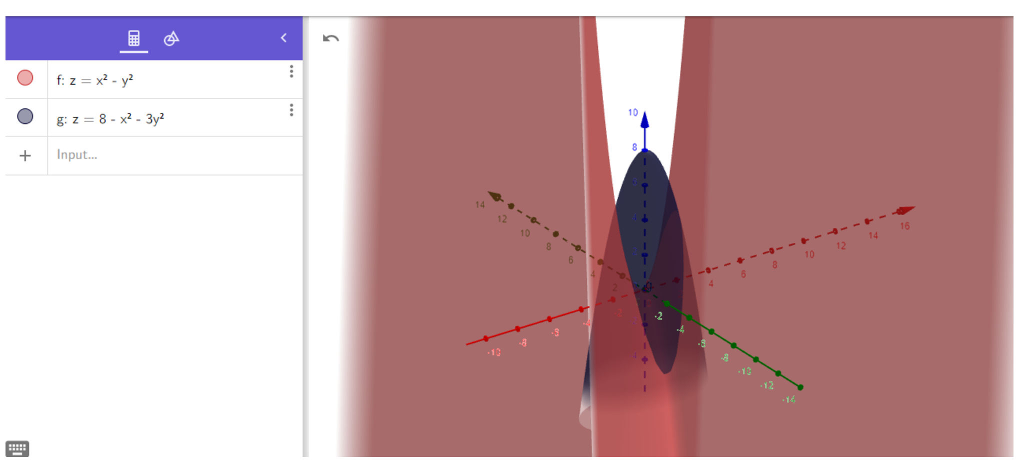


FIGURE 4. A view of the solid S whose volume we are calculating.

The solution to problem 4.13 is on page 154.

2.5. **Triple Integrals, Spherical Coordinates, Cylindrical Coordinates.**

Problem 5.1. *Write an iterated integral for* $\iiint_D f(x,y,z)dV$*, where* D *is a sphere of radius* 9 *centered at* $(0,0,1)$*. Use the order* $dV = dzdydx$*.*

Hint: Start by finding the equation of the the surface of the sphere of radius 9 *centered at* $(0,0,1)$*.*

The solution to Problem 5.1 is on page 156.

Problem 5.2. *Sketch by hand or graph with a computer program the region of integration for the integral*

(25) $$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} f(x,y,z)dxdydz.$$

Note: You may also describe the region of integration in writing instead. If you choose to do this, please write complete sentences and provide a thorough description.

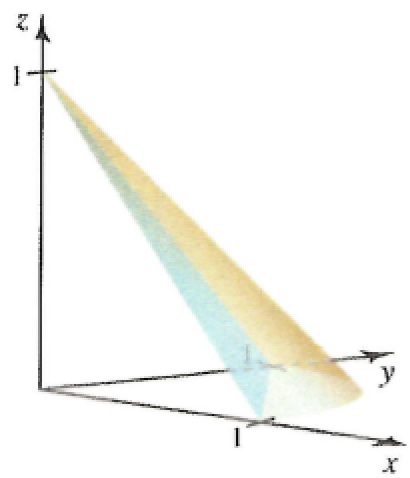
The solution to Problem 5.2 is on page 158.

Problem 5.3. *Evaluate*

(26) $$\int_1^{\ln(8)} \int_1^{\sqrt{z}} \int_{\ln(y)}^{\ln(2y)} e^{x+y^2-z}dxdydz.$$

The solution to Problem 5.3 is on page 159.

Problem 5.4. *Find the volume of the solid* S *in the first octant that is bounded by the cone* $z = 1-\sqrt{x^2+y^2}$ *and the plane* $x+y+z=1$*.*



The solution to Problem 5.4 is on page 160.

Problem 5.5. *Use triple integration in Cartesian coordinates to find the volume of the tetrahedron* S *that has its vertices at* $(0,0,0)$*,* $(a,0,0)$*,* $(0,b,0)$*, and* $(0,0,c)$*, where* $a,b,c>0$*.*
Hint: One of the faces of the tetrahedron lies on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$*.*

The solution to Problem 5.5 is on page 164.

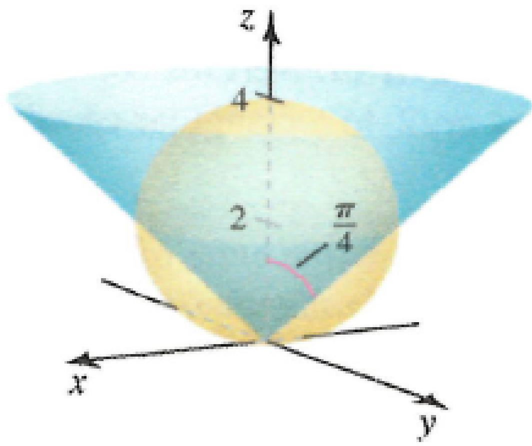
Problem 5.6. *Evaluate*

(27) $$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}}dydxdz.$$

Hint: A different order of integration can make the problem easier, even though it is not necessary.

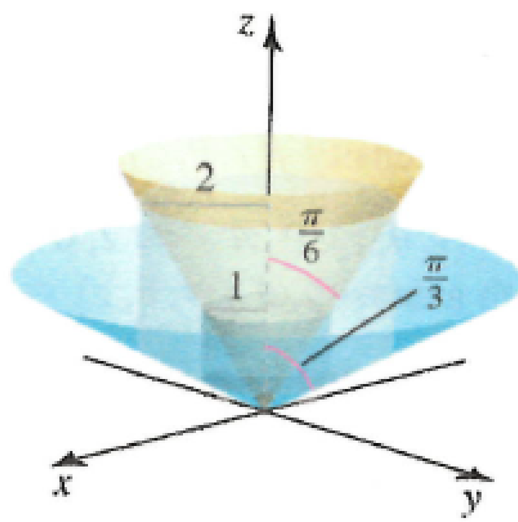
The solution to Problem 5.6 is on page 165.

Problem 5.7. *Find the volume of the solid region S outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.*



The solution to Problem 5.7 is on page 168.

Problem 5.8. *Find the volume of the solid region S that is bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.*



The solution to problem 5.8 is on page 171.

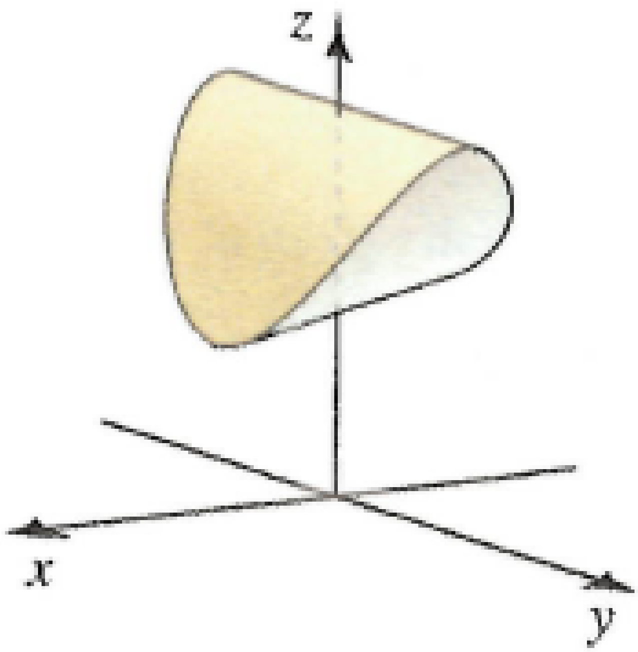
Problem 5.9. *Rewrite the the triple integral*

$$\int_0^2 \int_0^{9-x^2} \int_0^x f(x,y,z)dydzdx \tag{28}$$

using the order dzdxdy.

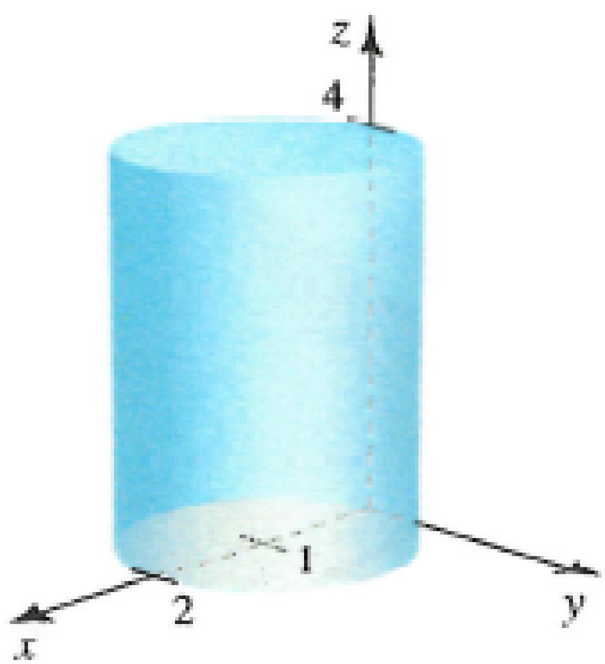
The solution to problem 5.9 is on page 202.

Problem 5.10. *Find the volume of the solid S that is bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$.*



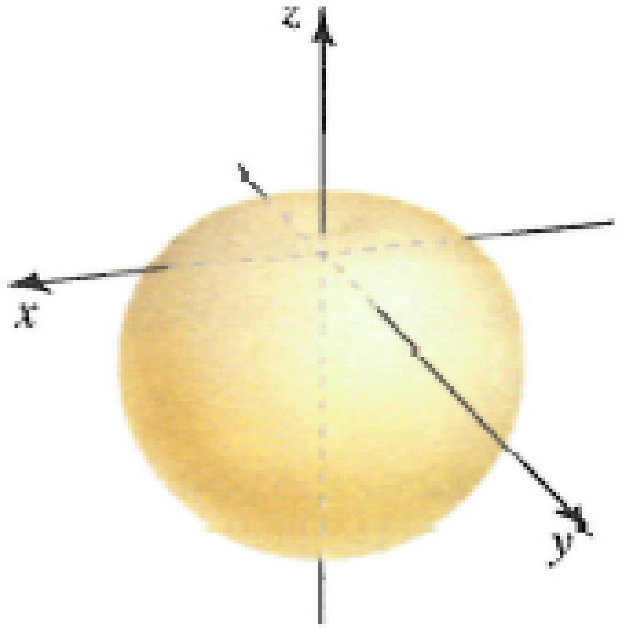
The solution to problem 5.10 is on page 205.

Problem 5.11. *Find the volume of the solid cylinder E whose height is 4 and whose base is the disk* $\{(r,\theta) : 0 \leq r \leq 2\cos(\theta)\}$.



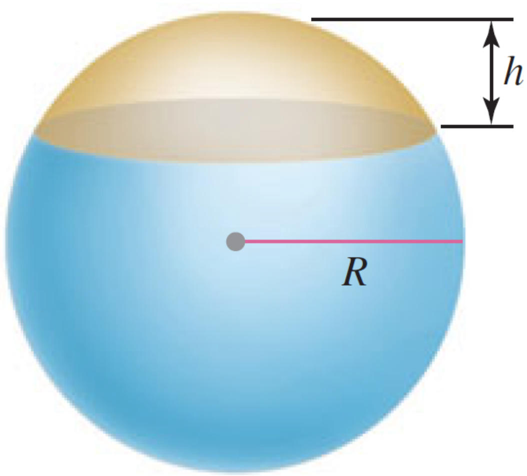
The solution to Problem 5.11 is on page 240.

Problem 5.12. *Find the volume of the solid cardiod of revolution* $D = \{(\rho,\varphi,\theta) : 0 \leq \rho \leq \frac{1}{2}(1-\cos(\varphi)), 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\}$.



The solution to problem 5.12 is on page 242.

Problem 5.13. *Find the volume of S, the cap of a sphere of radius R with thickness h.*



The solution to problem 5.13 is on page 243.

2.6. **Change of Variables in Double and Triple Integrals.**

Problem 6.1. *Let R be the region bounded by the lines $y-x=0, y-x=2, y+x=0, y+x=2$. Use a change of variables to evaluate*

(29) $$\iint_R \sqrt{y^2-x^2}\,dA.$$

The solution to Problem 6.1 is on page 248.

Problem 6.2. *Let R be the region in the first quadrant bounded by the hyperbolas $xy=1$ and $xy=4$ and the lines $y=x$ and $y=3x$. Evaluate*

(30) $$\iint_R y^4\,dA.$$

Note that you can also solve this problem in Cartesian coordinates and polar coordinates, not just a change of variables. Try solving it with all three methods and compare their difficulties!

The solution to Problem 6.2 is on page 149.

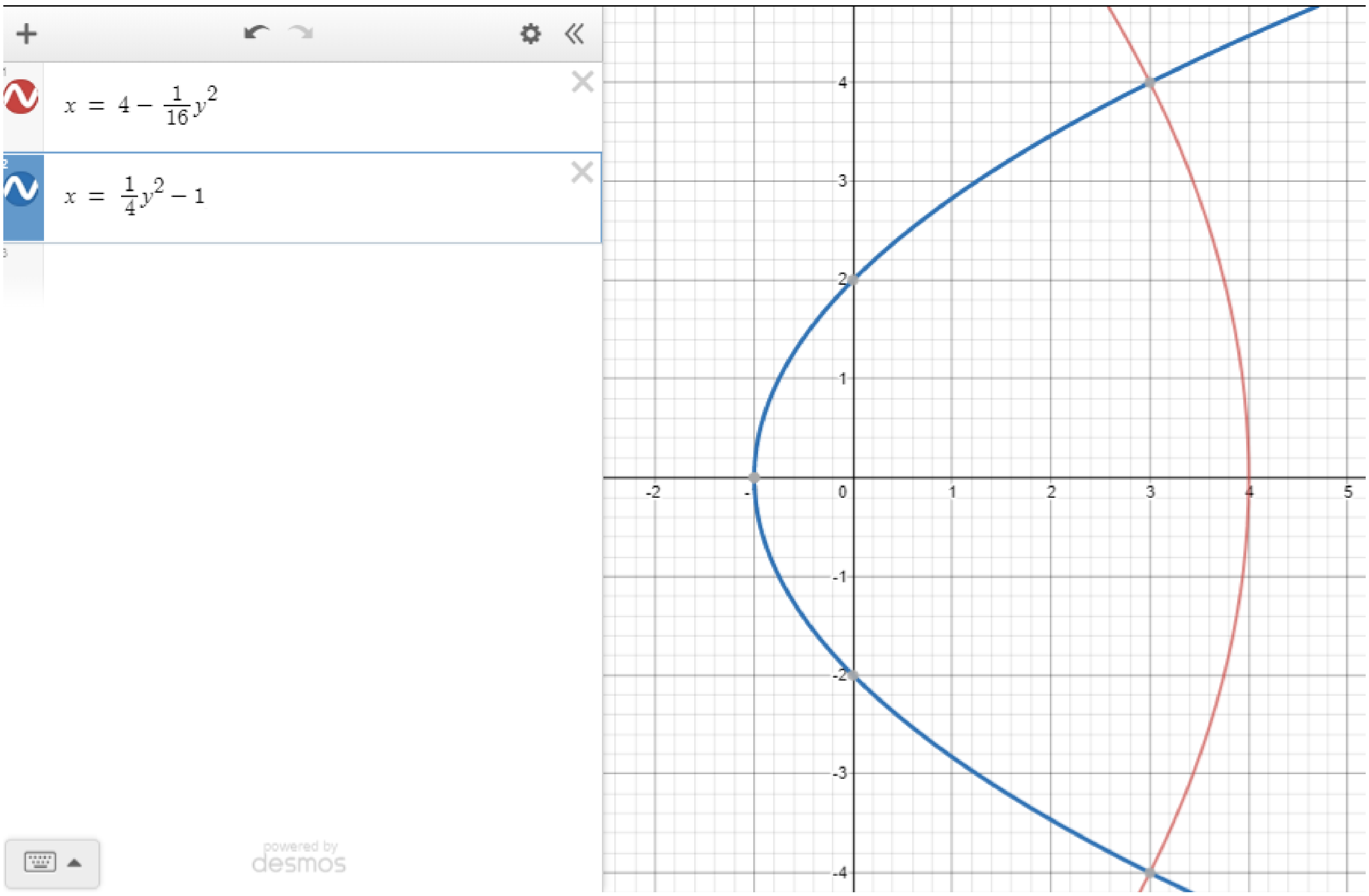
Problem 6.3. *Find the volume of the solid D that is bounded by the planes $y-2x=0, y-2x=1, z-3y=0, z-3y=1, z-4x=0,$ and $z-4x=3$.*

The solution to Problem 6.3 is on page 250.

Problem 6.4. *This problem has parts* **a.-g.** *spread out across the following pages. Your solutions to parts* **a, b,** *and* **f** *should include (hand drawn or computer generated) pictures.*

Consider the Transformation T from the uv-plane to the xy-plane given by $T(u,v)=(u^2-v^2,2uv)$.

- **a.** *Show that the lines $u=a$ in the uv-plane map to parabolas in the xy-plane that open in the negative x-direction with vertices[2] on the positive x-axis.[3] Compare the images of the lines $u=a$ and $u=-a$ under T.*
- **b.** *Show that the lines $v=b$ in the uv-plane map to parabolas in the xy-plane that open in the positive x-direction with vertices on the negative x-axis.[4] Compare the images of the lines $v=b$ and $v=-b$ under T.*
- **c.** *Evaluate $J(u,v)$.*
- **d.** *Use a change of variables into parabolic coordinates to find the area of the region R in the xy-plane bounded by the curves $x=4-\frac{1}{16}y^2$ and $x=\frac{1}{4}y^2-1$. Sketch a picture of the new region of integration as well.*



- **e.** *Use a change of variables into parabolic coordinates to find the area of the curved rectangle R above the x-axis bounded by $x=4-\frac{1}{16}y^2, x=9-\frac{1}{36}y^2, x=\frac{1}{4}y^2-1,$ and $x=\frac{1}{64}y^2-16$. Sketch a picture of the new region of integration as well.*
- **f.** *Describe the effect of the transformation $(u,v)\mapsto(2uv,u^2-v^2)$ on horizontal and vertical lines in the uv-plane.[5]*

[2]The vertex of the parabola $y=x^2$ is the point $(0,0)$ and the vertex of the parabola $x=y^2$ is also $(0,0)$.
[3]You have to show that the curve $\vec{r_1}(v)=(a^2-v^2,2av)$ represents the same curve as $x+c=dy^2$ for some negative numbers c and d.
[4]You have to show that the curve $\vec{r_2}(u)=(u^2-b^2,2ub)$ represents the same curve as $x+c=dy^2$ for some positive numbers c and d.
[5]Remember that the transformation $(x,y)\mapsto(y,x)$ reflects points in the xy-plane across the line $y=x$. It will also help to use the results of parts **a.** and **b.** of this problem.

g. *Show that the parabolas that are the images of the lines $u = a$ and $v = b$ under $T(u, v) = (u^2 - v^2, 2uv)$ are orthogonal to eachother.*

The solution to Problem 6.4 is on page 253.

Parts (a) and (c) of Problm 8.6 can be done without knowledge about surface integrals and give more practice with change of variables.

2.7. Line Integrals, Vector Fields, Conservativity, Flux, Circulation.

Problem 7.1. *Use a scalar line integral to find the length of the curve*

(31) $$\vec{r}(t) = \langle 20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2}\rangle, \text{ for } 0 \le t \le 2.$$

The solution to problem 7.1 is on page 208.

Problem 7.2. *Find the work required to move an object along the line segment from $(1, 1, 1)$ to $(8, 4, 2)$ through the forcefield $\vec{F}$ given by*

(32) $$\vec{F} = \frac{\langle x, y, z\rangle}{x^2 + y^2 + z^2}.$$

The solution to problem 7.2 is on page 209.

Problem 7.3. *Determine whether the vector field $\vec{F}$ given by*

(33) $$\vec{F} = \langle y - e^{x+y}, x - e^{x+y} + 1, \frac{1}{z}\rangle$$

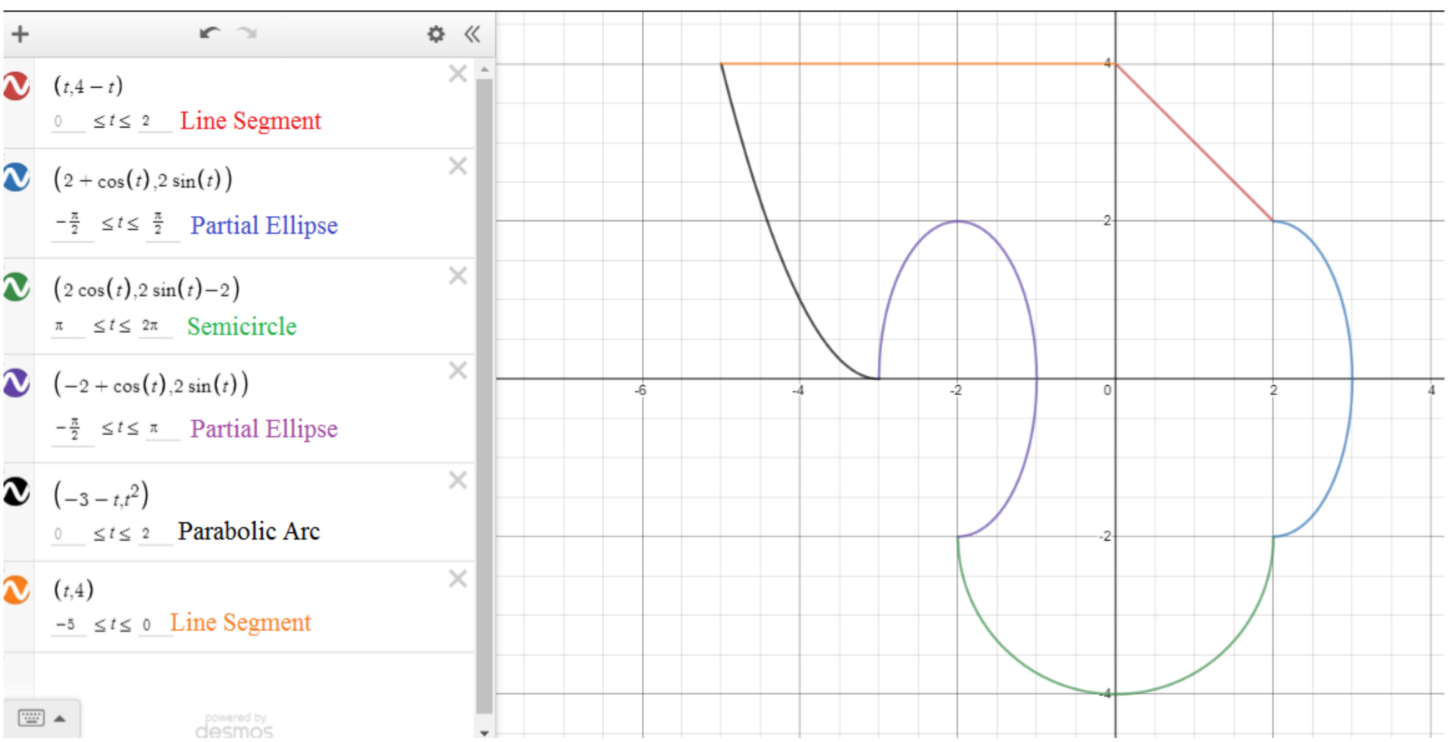
is a conservative vector field. If $\vec{F}$ is conservative, then find a potential function φ.

The solution to problem 7.3 is on page 211.

Problem 7.4. *Evaluate*

(34) $$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 5 + x))) - 1, y^3 + 2 + e^{y^2}\rangle \cdot d\vec{r},$$

where C is the curve that is shown in the picture below.



The solution to problem 7.4 is on page 212.

Problem 7.5. *Consider the vector field $\vec{F} = \langle x, -y\rangle$ and the curve C which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.*

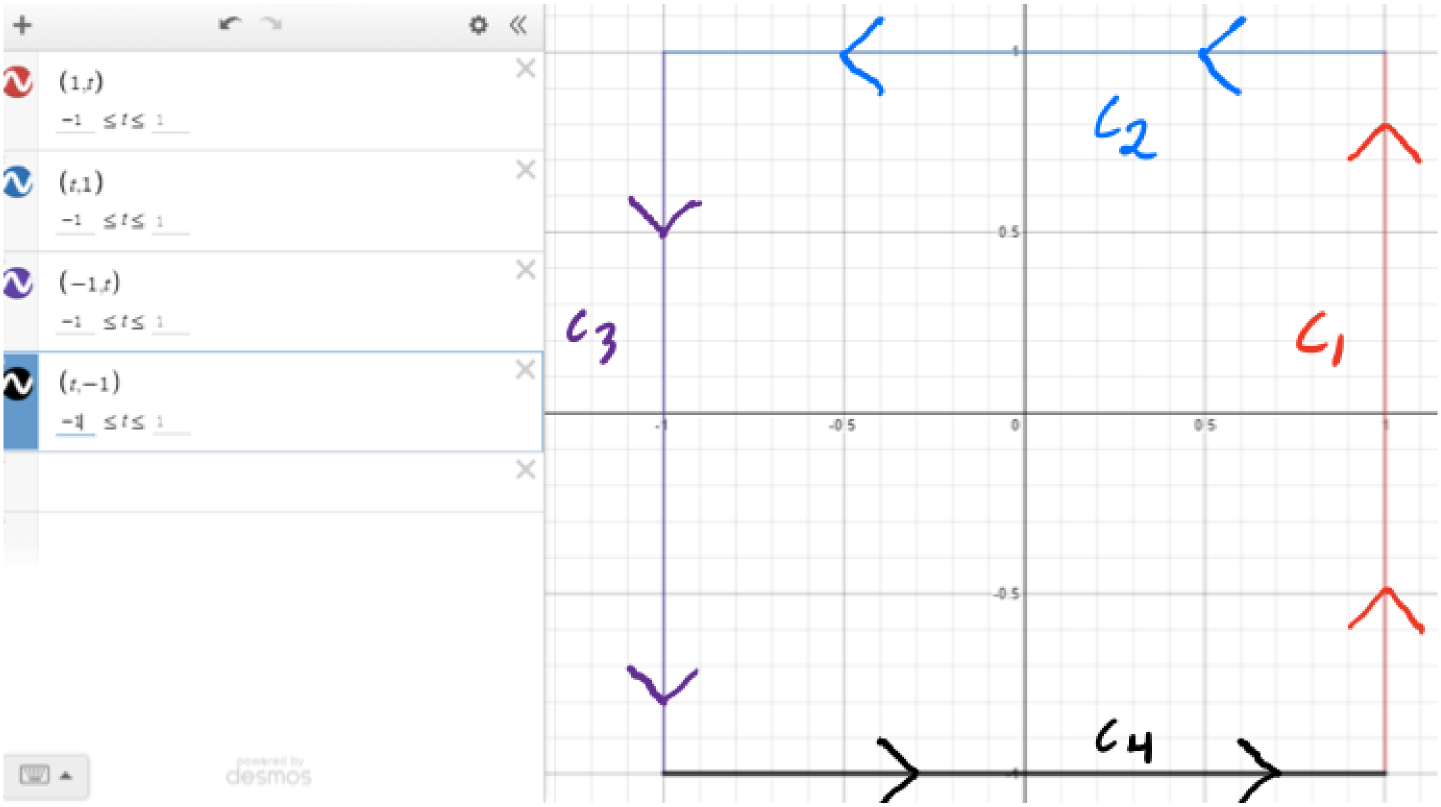


FIGURE 5. The curve C.

(a) *Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by finding a parametrization $\vec{r}(t)$ for the curve C.*
(b) *Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by using the Fundamental Theorem for Line Integrals.*

The solution to Problem 7.5 is on page 266.

Problem 7.6. *Find the average value of*

$$f(x, y) = \sqrt{4 + 9y^{2/3}} \tag{35}$$

on the curve $y = x^{3/2}$, for $0 \le x \le 5$.

The solution to Problem 7.6 is on page 269.

Problem 7.7. *Consider*

$$\int_{\mathcal{C}} (x^2 + y^2) ds,$$

where $\mathcal{C}$ is the line segment from $(0, 0)$ to $(5, 5)$.

(1) *Find a parametric description for $\mathcal{C}$ in the form $\vec{r}(t) = \langle x(t), y(t)\rangle$. (Remember to state the domain of the parameter.)*
(2) *Evaluate $|\vec{r'}(t)|$.*
(3) *Convert the line integral to an ordinary integral with respect to the parameter and evaluate it.*

The solution to Problem 7.7 is on page 271.

Problem 7.8. *Compute*

$$\int_{\mathcal{C}} xe^{yz} ds, \tag{36}$$

where $\mathcal{C}$ is $\vec{r}(t) = \langle t, 2t, -4t\rangle$ for $1 \le t \le 2$.

The solution to Problem 7.8 is on page 272.

Problem 7.9. *Compute*

(37) $$\int_{\mathcal{C}} \frac{xy}{z} ds,$$

where $\mathcal{C}$ *is the line segment from* $(1,4,1)$ *to* $(3,6,3)$.

The solution to Problem 7.9 is on page 273.

Problem 7.10. *Let* $f(x,y) = x$ *and consider the segment of the parabola* $y = x^2$ *joining* $O(0,0)$ *and* $P(1,1)$.

(1) Let $\mathcal{C}_1$ *be the segment from* O *to* P*. Find a parametrization of* $\mathcal{C}_1$*, then evaluate* $\int_{\mathcal{C}_1} f ds$.
(2) Let $\mathcal{C}_2$ *be the segment from* P *to* O*. Find a parametrization of* $\mathcal{C}_2$*, then evaluate* $\int_{\mathcal{C}_2} f ds$.
(3) Compare the results of (1) and (2).

The solution to Problem 7.10 is on page 290.

Problem 7.11. *Find the average value of the function* $f(x,y) = x + 2y$ *on the line segment from* $(1,1)$ *to* $(2,5)$.

The solution to Problem 7.11 is on page 292.

Problem 7.12. *Find the average value of the function* $f(x,y,z) = x$ *over the curve* $\mathcal{C}$ *that is paramterized by*

(38) $$\vec{r}(t) = \langle 20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2}\rangle, 0 \le t \le 4\pi.$$

The solution to Problem 7.12 is on page 293.

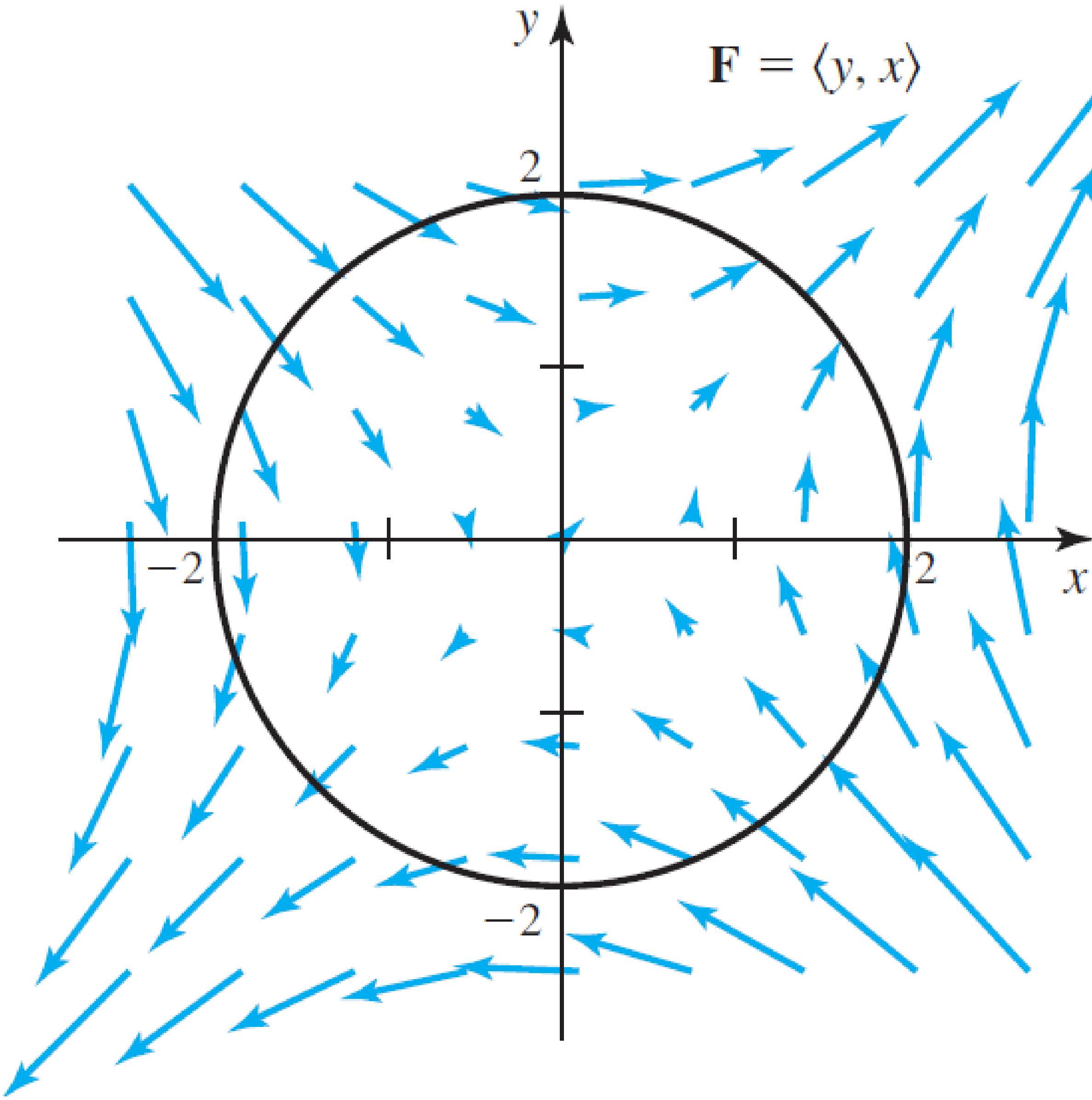
Problem 7.13. *Compute the circulation of* $\vec{F} = \langle y - x, x\rangle$ *on the curve* $\mathcal{C}$ *which is given by* $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ *for* $0 \le t \le 2\pi$.

The solution to Problem 7.13 is on page 274.

Problem 7.14. *Let* a *be a positive number. Consider the vector field* $\vec{F} = \langle y, x\rangle$ *and the curve* $\mathcal{C}$ *given by* $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ *for* $0 \le t \le 2\pi$*. Compute the flux of* $\vec{F}$ *across* $\mathcal{C}$*. (Your answer should be in terms of* a*.)*

The solution to Problem 7.14 is on page 275.

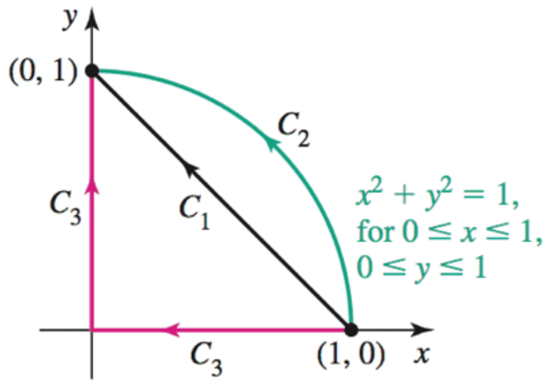
Problem 7.15. *Consider the flow field* $\mathbf{F} = \langle y, x\rangle$ *shown in the figure below.*



(a) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$, $0 \le t \le \frac{\pi}{2}$.
(b) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$, $\frac{\pi}{2} \le t \le \pi$.
(c) Explain why the flux across the quarter circle in the third quadrant equals the flux computed in part **a**.
(d) Explain why the flux across the quarter circle in the fourth quadrant equals the flux computed in part **b**.
(e) What is the outward flux across the full circle?

The solution to Problem 7.15 is on page 276.

Problem 7.16. *Consider the rotation field* $\vec{F} = \langle -y, x\rangle$*, and the three paths shown in the figure.*



(1) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_1$.
(2) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_2$.
(3) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_3$.
(4) Does it appear that the line integral $\int_{\mathcal{C}} \vec{F} \cdot \vec{T} ds$ is independent of the path, where $\mathcal{C}$ is any path from $(1,0)$ to $(0,1)$?

The solution to Problem 7.16 is on page 280.

Problem 7.17. *Find the work required to move an object along the line segment from $(1,1,1)$ to $(8,4,2)$ through the force field $\vec{F}$ given by*

$$\vec{F} = \frac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}.$$

The solution to Problem 7.17 is on page 283.

Problem 7.18. *Given the force field $\mathbf{F} = \langle x, y, z \rangle$, find the work required to move an object around the tilted ellipse that is parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 4\cos(t) \rangle$, $0 \le t \le 2\pi$.*

The solution to Problem 7.18 is on page 285.

Problem 7.19. *Evaluate the line integral $\int_C \nabla\phi \cdot d\vec{r}$ for $\phi(x,y) = xy$ and $C : \vec{r}(t) = \langle \cos(t), \sin(t) \rangle$, for $0 \le t \le \pi$ in two ways.*

(a) Use a parametric description of C and evaluate the integral directly
(b) Use the Fundamental Theorem for line integrals.

The solution to Problem 7.19 is on page 286.

Problem 7.20. *Let $\vec{F}$ be the vector field*

$$\vec{F} = \langle f(x,y), g(x,y) \rangle = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle.$$

It is a rotational vector field with the graph below

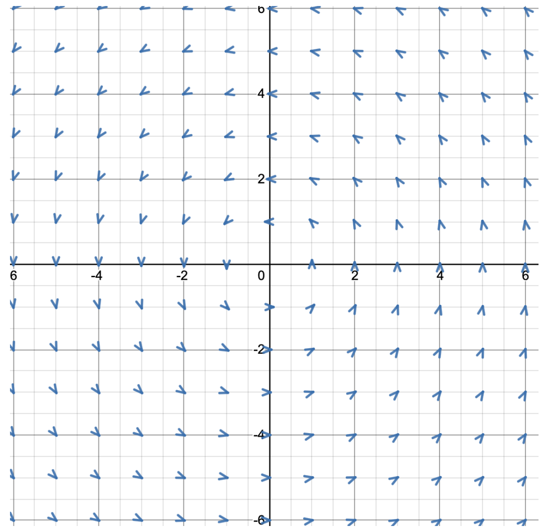


FIGURE 6. vector field $\vec{F}$

(1) Find the domain R of $\vec{F}$.
(2) Is the domain R connected? Is R simply connected?
(3) Show that $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$.
(4) Let C_a be the parameterized circle $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$, $0 \le t < 2\pi$ of radius $a > 0$. Show that the integral

$$\int_{C_a} \vec{F} \cdot d\vec{r} = 2\pi.$$

(5) Is $\vec{F}$ a conservative vector field on R? If so, please explain. Otherwise, please explain why it doesn't contradict the result in (3).
(6) Let R_1 be the region $R_1 = \{1 \le x \le 2, 1 \le y \le 2\}$. Is $\vec{F}$ a conservative vector field on R_1? Please explain.

The solution to Problem 7.20 is on page 287.

2.8. Surface Integrals, Green's Theorem, Stoke's Theorem, Divergence Theorem.

Problem 8.1. *An idealized two-dimensional ocean is modeled by the square region $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. with boundary $\mathcal{C}$. Consider the stream function $\Psi(x, y) = 4\cos(x)\cos(y)$ defined on R. Some of the level curves of Ψ are shown in the figure below.*

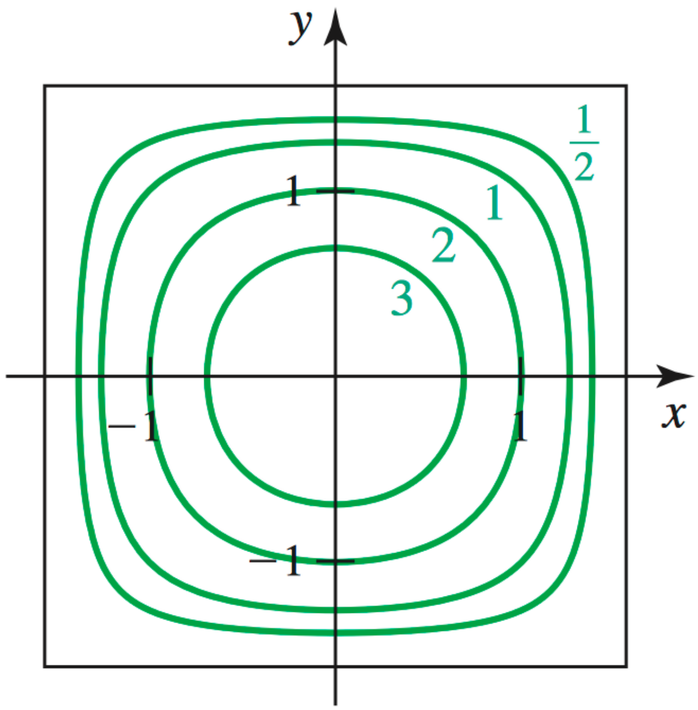
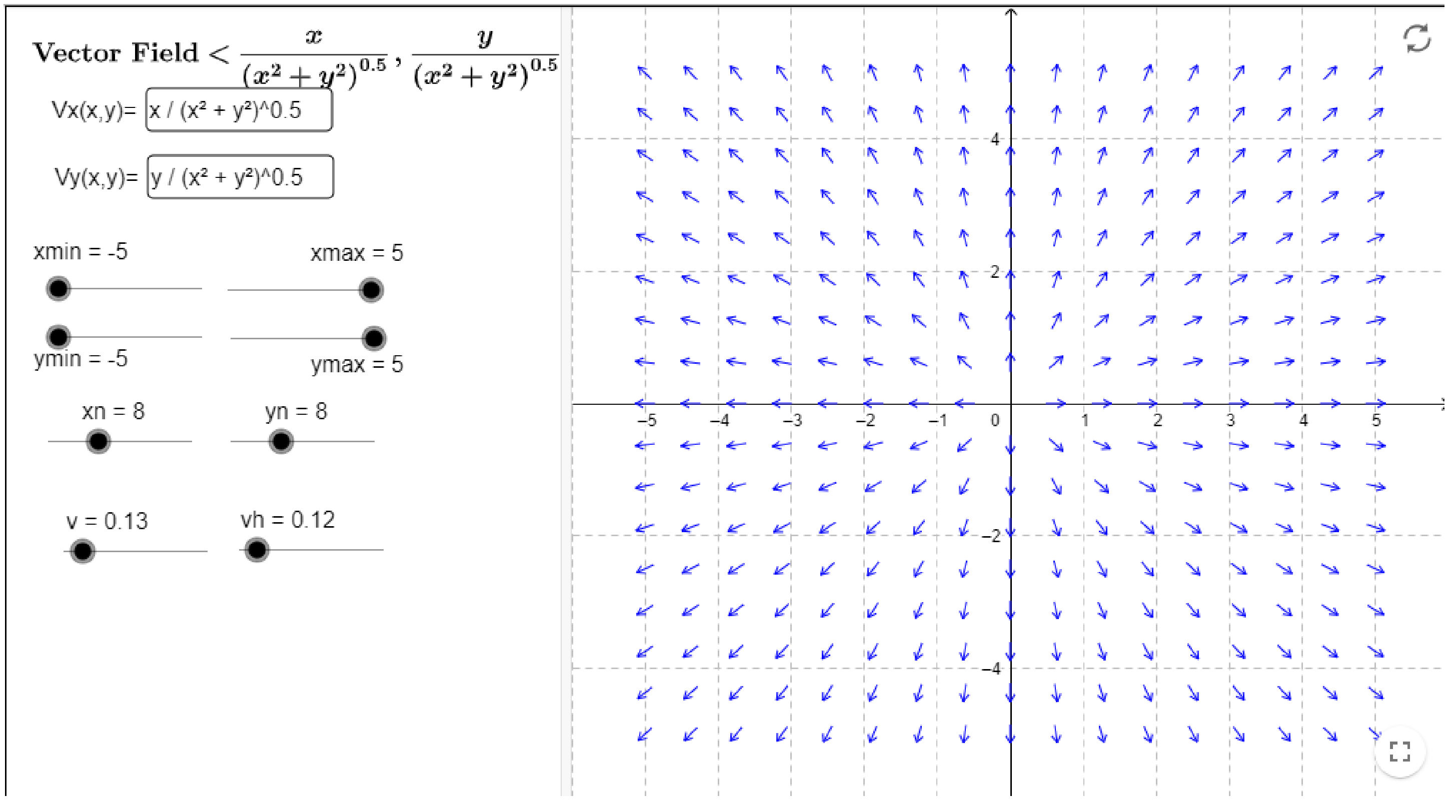


FIGURE 7. Some level curves of the stream function $\Psi(x, y)$.

(a) *The horizontal (east-west) component of the velocity is* $u = \Psi_y$ *and the vertical (north-south) component of the velocity is* $v = -\Psi_x$. *Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.*
(b) *Is the velocity field source free? Explain.*
(c) *Is the velocity field irrotational? Explain.*
(d) *Find the total outward flux across* $\mathcal{C}$.
(e) *Find the circulation on* $\mathcal{C}$ *assuming counterclockwise orientation.*

The solution to Problem 8.1 is on page 214.

Problem 8.2. *Consider the radial field* $\vec{F}(x, y) = \frac{\langle x, y \rangle}{\sqrt{x^2+y^2}} = \frac{\vec{r}}{|\vec{r}|}$ *shown below.*



(a) *Explain why the conditions of Green's Theorem do not apply to $\vec{F}$ on a region R containing the origin.*
(b) *Let R be the unit disk centered at the origin and compute*

(39)
$$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA.$$

(c) *Evaluate the line integral in the flux form of Green's Theorem applied to the region R and the vector field $\vec{F}$.*
(d) *Do the results of parts (b) and (c) agree? Explain.*

The solution to Problem 8.2 is on page 217.

Problem 8.3. *Suppose $y = f(x)$ is a continuous and positive function on $[a, b]$. Let $\mathcal{S}$ be the surface generated when the graph of $f(x)$ is revolved about the x-axis.*

(a) *Show that $\mathcal{S}$ is described parametrically by $\vec{r}(u, v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, for $a \le u \le b$, $0 \le v \le 2\pi$.*
(b) *Find an integral that gives ths surface area of $\mathcal{S}$.*
(c) *Apply the result of part (b) to the surface $\mathcal{S}_1$ generated with $f(x) = x^3$, for $1 \le x \le 2$.*

The solution to Problem 8.3 is on page 220.

Problem 8.4. *Given a sphere of radius R and a length $0 < L \le 2R$, show that the surface area of the strips of length L on the sphere depend only on L and not on the location of the strip.*

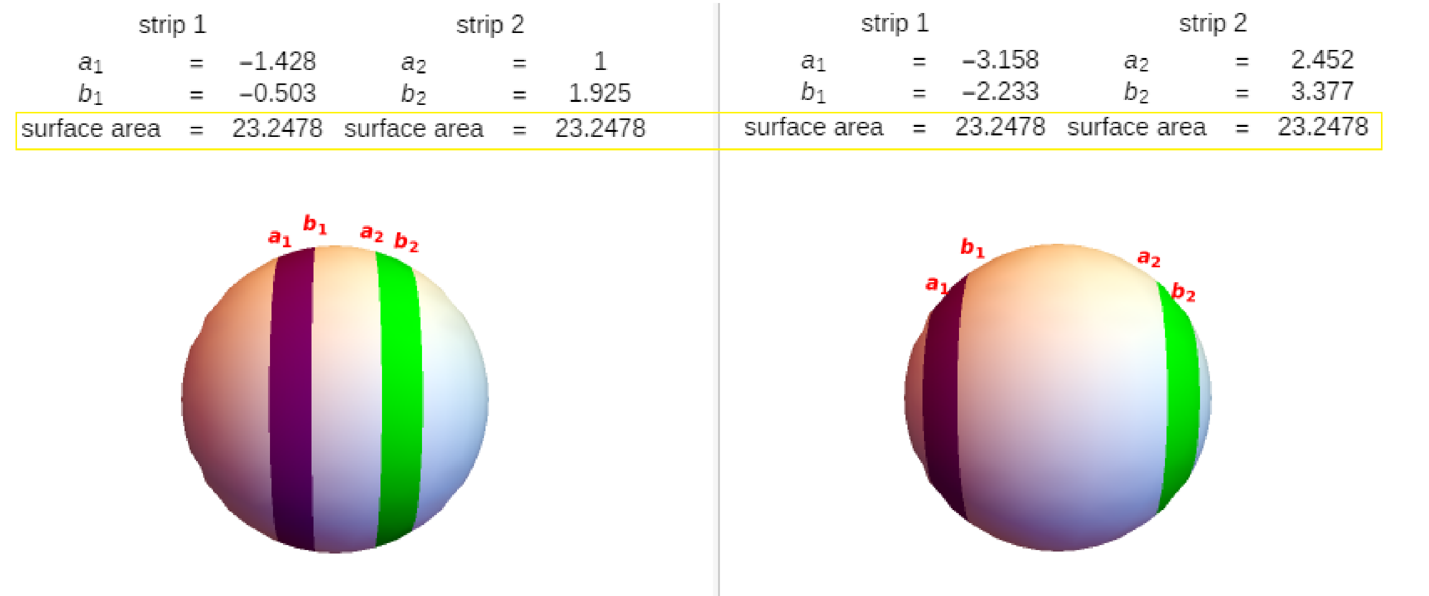


FIGURE 8. An example of Problem 8.4 with $L = 0.925$ and $R = 4$.

Hint: Problem 8.3 can help.

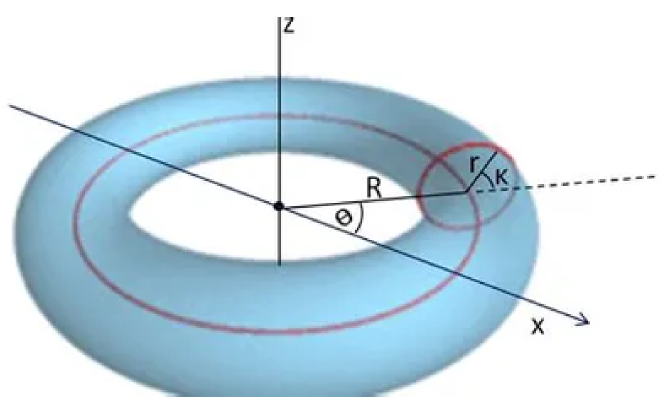
The solution to Problem 8.4 is on page 222.

Problem 8.5 (Rain on roofs)**.** *Let $z = s(x, y)$ define the surface $\mathcal{S}$ over a region R in the xy-plane, where $z \geq 0$ on R. Show that the downward flux of the vertical vector field $\vec{F} = \langle 0, 0, -1\rangle$ across $\mathcal{S}$ equals the area of R. Interpret the result physically.*

The solution to Problem 8.5 is on page 223.

Problem 8.6 (Surface Area and Volume of a Torus)**.**

(a) *Show that a torus T with radii $R > r$ (See figure) may be described parametrically by $r(K, \theta) = \langle (R+r\cos(K))\cos(\theta), (R+r\cos(K))\sin(\theta), r\sin(K)\rangle$, for $0 \leq K \leq 2\pi$, $0 \leq \theta \leq 2\pi$.*
(b) *Show that the surface area of the torus T is $4\pi^2 Rr$.*
Interestingly, the arclength of the small circle is $2\pi r$ and the arclength of the large circle inside the torus is $2\pi R$, so the surface area of the torus happens to be the product of the arclengths of the 2 circles from which it is created.
(c) *Use part (a) to find a parametrization $\vec{s}(K, \theta, r)$ for the solid torus $\mathcal{T}$ (T from part (a) as well as its interior), then use $\vec{s}$ and a change of variables to show that the volume of $\mathcal{T}$ is $\pi r^2 R$.*



The solution to Problem 8.6 is on page 224.

Problem 8.7. *Let $\mathcal{S}$ be the upper half of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ and let $\vec{F} = \langle z, x, y\rangle$. Use Stoke's theorem to evaluate*

$$\iint_{\mathcal{S}} (\nabla \times \vec{F}) \cdot \hat{n} dS. \quad (40)$$

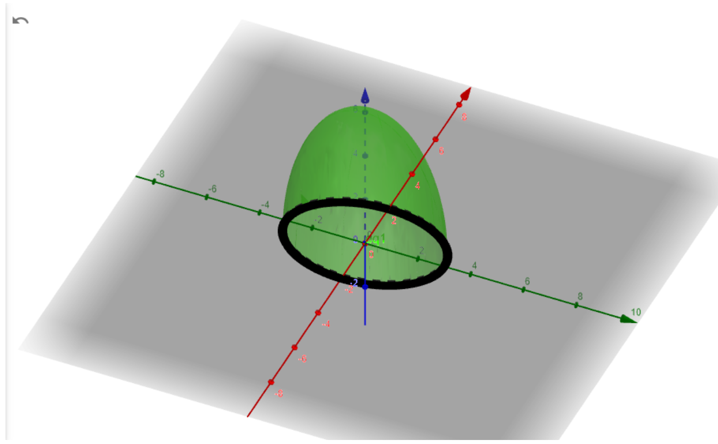


FIGURE 9. A view of $\mathcal{S}$ and $\partial\mathcal{S}$.

The solution to Problem 8.7 is on page 229.

Problem 8.8. *Let C be the circle $x^2+y^2=12$ in the plane $z=0$ (as a subset of $\mathbb{R}^3$) and let $\vec{F}=\langle (x+4)^x, y\ln(y+4), e^{z^2+\sqrt{z}}\rangle$. Use Stoke's theorem to evaluate*

$$\oint_C \vec{F}\cdot d\vec{r}. \tag{41}$$

The solution to Problem 8.8 is on page 230.

Problem 8.9. *Let $\mathcal{S}$ be the surface of the cube cut from the first octant by the planes $x=1, y=1$, and $z=1$. Let $\vec{F}=\langle x^2, 2xz, y^2\rangle$. Use the Divergence theorem to evaluate the net outward flux of $\vec{F}$ across $\mathcal{S}$.*

The solution to Problem 8.9 is on page 232.

Problem 8.10. *Let $\mathcal{S}$ be the boundary of the ellipsoid $\frac{x^2}{4}+y^2+z^2=1$ and let $\vec{F}=\langle x^2e^y\cos(z), -4xe^y\cos(z), 2xe^y\sin(z)\rangle$. Evaluate the outward flux of $\vec{F}$ across $\mathcal{S}$.*

The solution to Problem 8.10 is on page 233.

2.9. **Linear Algebra.**

Problem 9.1. *Three people play a game in which there are always 2 winners and 1 loser. They have the understanding that the loser always gives each winner an amount equal to what the winner already has. After 3 games, each has lost once and each has \$24. With how much money did each begin?*

The solution to problem 9.1 is on page 234.

Problem 9.2. *For the following problems, determine all possiblities for the solution set (from among infinitely many solutions, a unique solution, or no solution) of the system of linear equations described. After determining the possibilities for the solution set create concrete examples of systems corresponding to each possibility.*

(1) A homogeneous system of 4 equations in 5 unknowns.
(2) A system of 4 equations in 3 unknowns.
(3) A system of 3 equations in 4 unknowns that has $x_1=-1$, $x_2=0$, $x_3=2$, $x_4=-3$ as a solution.
(4) A homogeneous system of 3 equations in 3 unknowns.
(5) A homogeneous system of 3 equations in 3 unknowns that has solution $x_1=1$, $x_2=3$, $x_3=-1$.
(6) A system of 2 equations in 3 unknowns.

You are free to make use of the following facts.

(1) *Any homogeneous system of equations is consistent.*
   - *This is seen by the fact that the trivial solution (the solution in which all variables are equal to 0) is always a solution to a homogeneous system of equations.*

(2) *If a consistent system of equations (a system of equations with at least 1 solution) has more than 1 solution, then it has infinitely many solutions.*

(3) *If a consistent system of equations has more variables than equations, then it has infinitely many solutions.*

The solution to problem 9.2 is on page 236.

Problem 9.3. *For what value(s) of a does the following system have nontrivial solutions?*

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -x_1 & + & ax_2 & + & x_3 & = & 0 \\ 3x_1 & + & 4x_2 & - & x_3 & = & 0 \end{array}. \tag{42}$$

The solution to problem 9.3 is on page 239.

Problem 9.4. *Let*

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -3 & 1 & -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{43}$$

a) *Determine conditions on b_1, b_2, and b_3 that are necessary and sufficient for the system of equations $A\vec{x} = \vec{b}$ to be consistent.*
b) *For each of the following choices of $\vec{b}$, either show that the system $A\vec{x} = \vec{b}$ is inconsistent or exhibit the solution.*

$$\textbf{i)}\ \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \textbf{ii)}\ \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \quad \textbf{iii)}\ \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \quad \textbf{iv)}\ \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The solution to Problem 9.4 is on page 295.

Problem 9.5. *Find the inverse of*

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix} \tag{44}$$

The solution to Problem 9.5 is on page 298.

Problem 9.6. *Consider the matrices A, D ane E given by*

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } E = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}. \tag{45}$$

Find matrices B and C for which $AB = D$ and $CA = E$.

The solution to Problem 9.6 is on page 299.

Problem 9.7. *Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$, and let I_n denote the $(n \times n)$ identity matrix. Let $A = I_n + \vec{u}\vec{v}^T$, and suppose that $\vec{v}^T\vec{u} \neq -1$. Show that*

$$A^{-1} = I_n - a\vec{u}\vec{v}^T, \text{ where } a = \frac{1}{1 + \vec{v}^T\vec{u}}. \tag{46}$$

This result is known as the Sherman-Woodberry formula.

On page 300 you can find a concrete example of what you are being asked to show in Problem 9.7 when $n = 3$. After the example is the solution.

2.10. **Complex Numbers.**

Problem 10.1. *Plot $z = -1 - \frac{1}{\sqrt{3}}i$ in the complex plane. Then find the modulus and argument of z, and express z in the form $z = re^{i\theta}$.*

The solution to Problem 10.1 is on page 303.

Problem 10.2. *For $z = -1 + 4i$ and $w = 5 + 2i$ evaluate $\left|\frac{z}{2w}\right|$.*

The solution to Problem 10.2 is on page 304.

Problem 10.3. *Evaluate* $i(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}})$.

The solution to Problem 10.3 is on page 305.

Problem 10.4. *Let* $z = -1 + i$ *and* $w = 1 + i\sqrt{3}$ *be the two complex numbers.*
(1) Compute directly $z \cdot w$ *and* $\frac{z}{w}$ *and express the answer in Cartesian form, i.e., the form* $x + iy$*, where* x *and* y *are real numbers.*
(2) Express z *and* w *in polar form. Compute* $z \cdot w$ *and* $\frac{z}{w}$ *in polar forms. Compare your answer with part (1).*
(3) Draw the four complex numbers w*,* z*,* $z \cdot w$ *and* $\frac{z}{w}$ *in the following coordinate. Explain what multiplication by* w *and division by* w *do to the complex number* z *in terms of argument and modulus.*

The solution to Problem 10.4 is on page 306.

Problem 10.5. *(1) Equate the real and imaginary parts of both sides of the identity*

$$e^{i(a-b)} = e^{ia}e^{-ib}$$

to prove that

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b);$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b).$$

(2) Equate the real and imaginary parts of both sides of the identity

$$e^{i2\theta} = e^{i\theta} \cdot e^{i\theta}$$

to prove that

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \text{ and } \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

The solution to Problem 10.5 is on page 309.

Problem 10.6. *Find all possible fourth roots of* -16*. Equivalently, find all possible values of* $(-16)^{\frac{1}{4}}$*.*

The solution to Problem 10.6 is on page 311.

Problem 10.7. *Determine* A, ω*, and* φ *for which*

$$-3\sin(4t) + 3\cos(4t) = A\sin(\omega t + \varphi). \tag{47}$$

The solution to Problem 10.7 is on page 312.

Problem 10.8. *Determine* R, δ*, and* ω_0 *for which*

$$-2\cos(\pi t) - 3\sin(\pi t) = R\cos(\omega_0 t - \delta). \tag{48}$$

The solution to Problem 3 is on page 33.

2.11. Ordinary Differential Equations.

Problem 11.1. *Solve the following initial value problem.*

$$y'' - 3y' - 18y = 0; \quad y(0) = 0, y'(0) = 4. \tag{49}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

The solution to Problem 11.1 is on page 316.

Problem 11.2. *Solve the following initial value problem.*

$$y'' - y' + \frac{1}{4}y = 0; \quad y(0) = 1, y'(0) = 2. \tag{50}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

The solution to Problem 11.2 is on page 319.

Problem 11.3. *Solve the following initial value problem.*

(51) $$y'' + 6y' + 10y = 0; \quad y(0) = 0, y'(0) = 6.$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

The solution to Problem 11.3 is on page 321.

Problem 11.4. *Let a be a real number.*

(a) *Find the general solution to equation (52) in terms of a.*

(52) $$y'' - (a+2)y' + 2ay = 0.$$

(b) *Solve the initial value problem given in (53).*

(53) $$y'' - 25y' + 46y; \quad y(0) = 0,\ y'(0) = 21.$$

The solution to Problem 11.4 is on page 318.

Problem 11.5. *Solve the following initial value problem.*

(54) $$t^2y'' + 6ty' + 6y = 0;\ y(1) = 0, y'(1) = -4.$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

The solution to Problem 11.5 is on page 323.

Problem 11.6. *Find the general solution of the equation*

(55) $$y''y' = 1.$$

The solution to Problem 11.6 is on page 326.

Problem 11.7. *Solve the differential equation*

(56) $$y'' = e^{-y'}.$$

The solution to Problem 11.7 is on page 327.

Problem 11.8. *For the following differential equations use the method of undetermined coefficients in order to find the general form of the solution.*

(57) $$y'' + y = \cos(2t) + t^3.$$

(58) $$y'' + 4y = \cos(2t).$$

(59) $$2y'' - 8y' + 8y = 4e^{2t}.$$

(60) $$y'' - y = 25te^{-t}\sin(3t).$$

(61) $$y^{(4)} - 3y'' + 2y = 6te^{2t}.$$

The solution to Problem 11.8 is on page 328.

Problem 11.9. *Find a particular solution of the following equation.*

(62) $$y'' - y' - 6y = \sin(t) + 3\cos(t).$$

The solution to Problem 11.9 is on page 332.

Problem 11.10. *Find a particular solution of the following equation.*

(63) $$y'' + y = \cos(2t) + t^3.$$

The solution to Problem 11.10 is on page 333.

Problem 11.11. *Find a particular solution of the following equation.*

$$y'' + 4y = \cos(2t). \tag{64}$$

The solution to Problem 11.11 is on page 335.

Problem 11.12. *Find a particular solution to equation (65).*

$$y'' + 4y = t\sin(2t). \tag{65}$$

The solution to Problem 11.12 is on page 337.

Problem 11.13. *Find the general solution of the following equation and solve the given initial value problem.*

$$y'' + y = 4\sin(2t); \quad y(0) = 1, y'(0) = 0. \tag{66}$$

Draw the graph of the solution and determine the period of the function. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

The solution to Problem 11.13 is on page 339.

Problem 11.14. *Use the method of undetermined coefficients to find the general solution to the differential equation*

$$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t). \tag{67}$$

The solution to Problem 11.14 is on page 341.

Problem 11.15. *Solve the initial value problem*

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}, \quad y(\pi) = 0,\ t > 0. \tag{68}$$

The solution to Problem 11.15 is on page 346.

Problem 11.16. *Show that if a and λ are positive constants and b is any real number, then every solution of the equation*

$$y' + ay = be^{-\lambda t} \tag{69}$$

has the property that $y \to 0$ as $t \to \infty$.

The solution to Problem 11.16 is on page 347.

Problem 11.17. *Solve the initial value problem*

$$y' = \frac{3x^2 - e^x}{2y - 5}, \quad y(0) = 1. \tag{70}$$

The solution to Problem 11.17 is on page 349.

Problem 11.18. Part a: *Verify that $y_1(t) = 1 - t$ and $y_2(t) = -\frac{t^2}{4}$ are both solutions of the initial value problem*

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1. \tag{71}$$

Where are these solutions valid?

Part b: *Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2 of the 10th edition of 'Elementary Differential Equations' by W.E. Boyce and R.C. DiPrima.*

Part c: *Show that $y(t) = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, then the initial condition is also satisfied and the solution $y = y_1(t)$ is obtained. Show that no other choice of c gives a second solution. Note that no choice of c gives the solution $y = y_2(t)$.*

The solution to Problem 11.18 is on page 351.

Problem 11.19. *Solve the differential equation*

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}. \tag{72}$$

The solution to Problem 11.19 is on page 353.

Problem 11.20. *Find and classify (stable, unstable, semistable) the equilibrium points of the differential equation*

$$\frac{dy}{dt} = y(1 - y^2), \quad -\infty < y_0 < \infty. \tag{73}$$

The solution to Problem 11.20 is on page 355.

Problem 11.21. *Find and classify (stable, unstable semistable) the equilibrium points of the differential equation*

$$\frac{dy}{dt} = y^2(4 - y^2), \quad -\infty < y_0 < \infty. \tag{74}$$

The solution to Problem 11.21 is on page 358.

Problem 11.22. *Find and classify (stable, unstable semistable) the equilibrium points of the differential equation*

$$\frac{dy}{dt} = y^2(1 - y)^2, \quad -\infty < y_0 < \infty. \tag{75}$$

The solution to Problem 11.22 is on page 361.

Problem 11.23. *Solve the following initial value problem and find* **an** *interval on which the solution is valid.*

$$(2x - y) + (2y - x)y' = 0, \quad y(1) = 3. \tag{76}$$

The solution to Problem 11.23 is on page 364.

Problem 11.24. *Find the general solution of the differential equation*

$$1 + \left(\frac{x}{y} - \sin(y)\right) y' = 0. \tag{77}$$

The solution to Problem 11.24 is on page 366.

Problem 11.25. *Use Euler's method to approximate values of the solution of the given initial value problem at* $t = 0.1, 0.2, 0.3,$ *and* 0.4 *with* $h = 0.1$.

$$y' = 0.5 - t + 2y, \quad y(0) = 1. \tag{78}$$

The solution to Problem 11.25 is on page 368.

Problem 11.26. *A homebuyer takes out a mortgage of* $100,000 *with an interest rate of* 9%. *What monthly payment is required to pay off the loan in 30 years? In 20 years? What is the total amount paid during the term of the loan in each of these cases?*

The solution to Problem 11.26 is on page 369.

Problem 11.27. *Consider the differential equation*

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0. \tag{79}$$

Find all values of α *(if any) for which all solutions of equation* (1221) *tend to zero as* $t \to \infty$. *Also find all values of* α *(if any) for which all nonzero solutions become unbounded as* $t \to \infty$.

The solution to Problem 11.27 is on page 373.

Problem 11.28. *Consider the differential equation*

(80) $$y'' + (3-\alpha)y' - 2(\alpha-1)y = 0.$$

Find all values of α (if any) for which all solutions of equation (1212) tend to zero as $t \to \infty$. Also find all values of α (if any) for which all nonzero solutions become unbounded as $t \to \infty$.

The solution to Problem 11.28 is on page 374.

Problem 11.29. *As will be shown in Section 16.4, the equation $y'' + py' + qy = f(t)$, where p and q are constants and f is a specified function, is used to model both the mechanical oscillators and electrical circuits. Depending on the values of p and q, the solutions to this equation display a wide variety of behavior.*
Consider the equation

$$y'' + 9y = 8\sin(t).$$

(a). Verify that the following equations have the given general solutions

$$y = c_1 \sin(3t) + c_2 \cos(3t) + \sin t.$$

(b). Solve the initial value problem with the given initial conditions $y(0) = 0$, $y'(0) = 2$.
(c). Graph the solutions to the initial value problem, for $t \geq 0$.

The solution to Problem 11.29 is on page 375.

Problem 11.30. *Find the Wronskian of the differential equation*

(81) $$t^2y'' - t(t+2)y' + (t+2)y = 0$$

without solving the equation.

The solution to Problem 11.30 is on page 377.

Problem 11.31. *Given that $y_1(t) = t$ is a solution to equation (81), use the Wronskian $W(t)$ to find another independent solution $y_2(t)$. (Compare with problem 11.35)*

The solution to Problem 11.31 is on page 378.

Problem 11.32. *Solve the initial value problem*

(82) $$y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 2,$$

then sketch the graph of the solution and describe the behavior as $t \to \infty$.

The solution to Problem 11.32 is on page 379.

Problem 11.33. *Solve the differential equation*

(83) $$t^2y'' - ty' + 5y = 0, \quad t > 0.$$

The solution to Problem 11.33 is on page 381.

Problem 11.34. *Given $a \in \mathbb{R}$, solve the differential equation*

(84) $$y'' + 2ay' + a^2y = 0.$$

Hint: *It helps to consider the Wronskian.*

The solution to Problem 11.34 is on page 383.

Problem 11.35. *Given that $y_1(t) = t$ is a solution to the differential equation*

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0, \tag{85}$$

use the method of reduction of order to find a second solution. (Compare with Problem 11.31)

The solution to Problem 11.35 is on page 385.

Problem 11.36. *Use the method of variation of parameters to find the general solution to the differential equation*

$$(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1, \tag{86}$$

given that $y_1(t) = e^t$ and $y_2(t) = t$ are solutions to the corresponding homogeneous equation.

The solution to Problem 11.36 is on page 386.

Problem 11.37. *Use the method of reduction of order to find the general solution to the differential equation*

$$(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1, \tag{87}$$

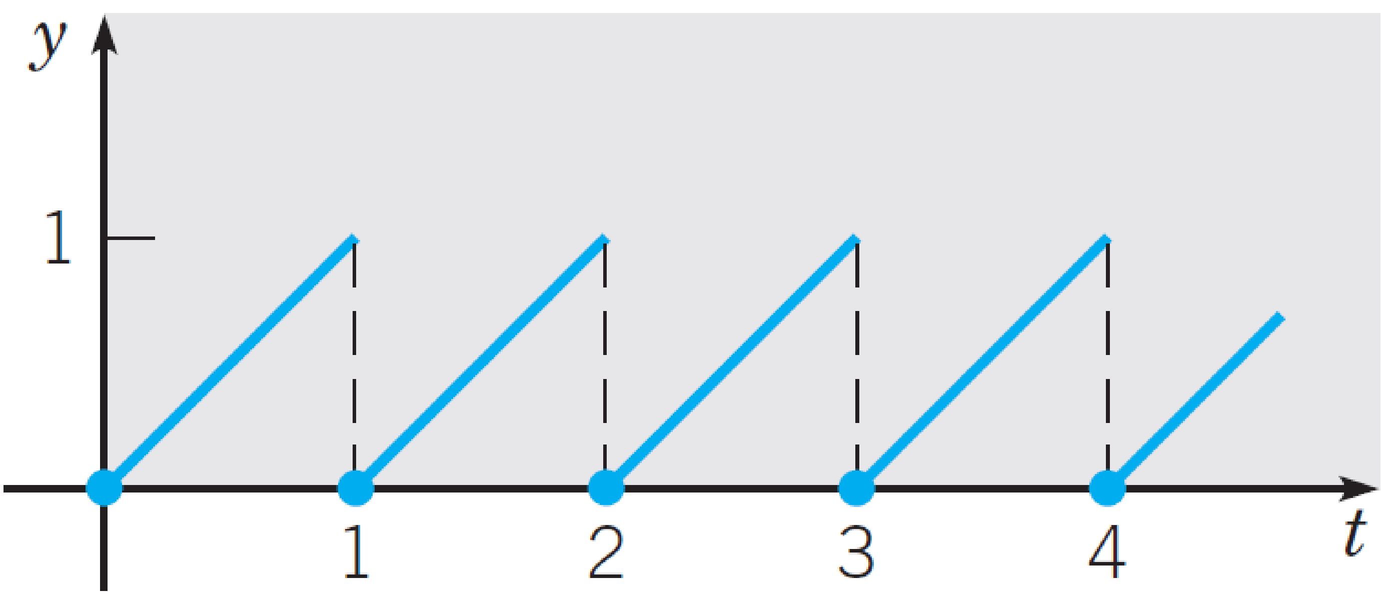
given that $y_1(t) = e^t$ is a solution to the corresponding homogeneous equation.

The solution to Problem 11.37 is on page 388.

Problem 11.38. *A spring–mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos(3t) - 2\sin(3t))$ N, determine the steady state response. Express your answer in the form $R\cos(\omega t - \delta)$.*

The solution to Problem 11.38 is on page 390.

Problem 11.39. *Find the Laplace transform of the function $f : [0,\infty) \to [0,1)$ that is defined by $f(t) = t$ when $0 \le t < 1$ and $f(t+1) = f(t)$.*



The solution to Problem 11.39 is on page 393.

Problem 11.40. *Solve the initial value problem*

$$y'' + 4y = \sin(t) - u_{2\pi}(t)\sin(t-2\pi); \quad y(0) = 0,\ y'(0) = 0. \tag{88}$$

The solution to Problem 11.40 is on page 397.

Problem 11.41. *Solve the initial value problem*

(89) $$y^{(4)} + 5y'' + 4y = 1 - u_{2\pi}(t); y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 0.$$

Hint: It may help to do Problem 11.40 first.

The solution to Problem 11.41 is on page 400.

Problem 11.42. *Solve the initial value problem*

(90) $$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t); \quad y(0) = 0, y'(0) = 0.$$

The solution to Problem 11.42 is on page 403.

Problem 11.43. *Solve the initial value problem*

(91) $$y'' + 3y' + 2y = \cos(\alpha t); \quad y(0) = 1, y'(0) = 0$$

by using the Laplace transform and convolution integrals.

The solution to Problem 11.43 is on page 406.

Problem 11.44. *Show that $W(5, \sin^2(t), \cos(2t)) = 0$. Can this also be shown without directly computing the Wronskian?*

The solution to Problem 11.44 is on page 409.

Problem 11.45. *Find the general solution to the differential equation*

(92) $$y''' + y' = \sec(t).$$

The solution to Problem 11.45 is on page 411.

Problem 11.46. *Let $y = \phi(x)$ be a solution to the initial value problem*

(93) $$y'' + x^2 y' + \sin(x)y = 0; \quad y(0) = a_0, y'(0) = a_1.$$

Find $\phi''(0), \phi'''(0)$, and $\phi^{(4)}(0)$.

The solution to Problem 11.46 is on page 413.

Problem 11.47. *Solve the differential equation*

(94) $$y' + (x+1)y = x + 1$$

by finding a series solution and by using an integrating factor, then compare your answers.

The solution to Problem 11.47 is on page 415.

Problem 11.48. *Determine a lower bound for the radii of convergence r_1 and r_2 of the series solution to the differential equation*

(95) $$(1 + x^3)y'' + 4xy' + y = 0,$$

centered at $x_1 = 0$ and $x_2 = 2$. Then find the series solution to equation (95) centered at $x_2 = 2$.

The solution to Problem 11.48 is on page 418.

2.12. **Partial Differential Equations, Fourier Series, Eigenvalues of ODEs.**

Problem 12.1. *Consider the partial differential equation*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{96}$$

Show that for a solution $u(r,\theta) = R(r)\Theta(\theta)$ having separated variables, we must have

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \text{ and} \tag{97}$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \tag{98}$$

where λ is some constant.

The solution to Problem 12.1 is on page 422.

Problem 12.2. *Consider the partial differential equation*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{99}$$

Show that for a solution $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ having separated variables, we must have

$$\Theta''(\theta) + \mu\Theta(\theta) = 0, \tag{100}$$

$$Z''(z) + \lambda Z(z) = 0, \text{ and} \tag{101}$$

$$r^2 R''(r) + rR'(r) - (r^2\lambda + \mu)R(r) = 0, \tag{102}$$

where μ and λ are constants.

The solution to Problem 12.2 is on page 40.

Problem 12.3. *Find the values of λ (eigenvalues) for which the following problem has a nontrivial solution. Also determine the corresponding nontrivial solutions (eigenfunctions).*

$$y'' + \lambda y = 0; \quad 0 < x < \pi, \quad y(0) - y'(0) = 0, \ y(\pi) = 0. \tag{103}$$

The solution to Problem 12.3 is on page 426.

Problem 12.4. *Find the values of λ for which the initial value problem given by*

$$y'' - 2y' + \lambda y = 0; \quad 0 < x < \pi \tag{104}$$

$$y(0) = y(\pi) = 0 \tag{105}$$

has nontrivial solutions. Then, for each such λ, find the nontrivial solutions.

The solution to Problem 12.4 is on page 430.

Problem 12.5. *Find the fourier series of the function*

$$f(x) = \begin{cases} 1 & \text{if } -2 < x < 0 \\ x & \text{if } 0 < x < 2 \end{cases}, \tag{106}$$

over the interval $[-2, 2]$.

The solution to Problem 12.5 is on page 433.

Problem 12.6. *Find the Fourier sine series for*

$$f(x) = e^x, \quad 0 < x < 1. \tag{107}$$

The solution to Problem 12.6 is on page 436.

Problem 12.7. *Find the Fourier cosine series for*

$$f(x) = 1 + x, \quad 0 < x < \pi. \tag{108}$$

The solution to Problem 12.7 is on page 438.

Problem 12.8. *Determine the function to which the Fourier series of*

$$f(x) = |x|, \quad -\pi < x < \pi \tag{109}$$

converges pointwise.

The solution to Problem 12.8 is on page 440.

Problem 12.9. *Determine the function to which the Fourier series of*

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ x^2 & \text{if } 0 < x < \pi \end{cases} \tag{110}$$

converges pointwise.

The solution to Problem 12.9 is on page 442.

Problem 12.10. *Find the solution $u(x,t)$ to the heat flow problem*

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{111}$$

(112) $$\mu(0,t) = \mu(L,t) = 0, \quad t > 0$$

(113) $$u(x,0) = f(x), \quad 0 < x < L,$$

with $\beta = 5, L = \pi$, *and the initial value function*

(114) $$f(x) = 1 - \cos(2x).$$

The solution to Problem 12.10 is on page 444.

Problem 12.11. *Formally solve the vibrating string problem*

(115) $$\frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

(116) $$u(0,t) = u(L,t) = 0, \quad t > 0,$$

(117) $$u(x,0) = f(x), \quad 0 \le x \le L,$$

(118) $$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le L,$$

with $\alpha = 4$, $L = \pi$, *and the initial value functions*

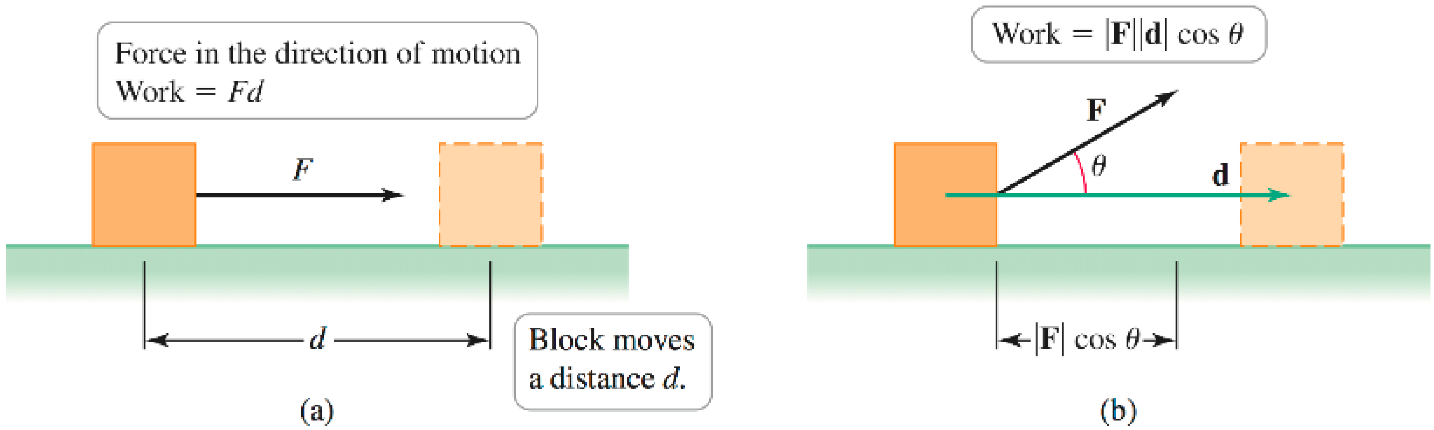
(119) $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx),$$

(120) $$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

The solution to Problem 12.11 is on page 447.

Problem 1.1: A suitcase is pulled 50ft along a horizontal sidewalk with a constant force of 30lb at an angle of 30° above the horizontal. How much work is done?

Solution: For this problem it suffices to use the formula for work that is shown in the diagram below.



The only thing that we need to be careful of is to remember that the standard unit of measure for work is Joules (J) which is given by J = kg·m/s^2 = N·m, where N represents Newtons. To this end, we recall that 1lb≈4.4482N and 1ft≈0.3048m. It follows that the total amount of work done is given by

$$\text{Work} = 30\text{lb} \cdot 50\text{ft} \cdot \cos(30^\circ) \approx 30 \cdot 4.4482\text{N} \cdot 50 \cdot 0.3048\text{m} \cdot \frac{\sqrt{3}}{2} \approx \boxed{761.2506\text{J}}. \tag{121}$$

Problem 1.2: A constant force of $\vec{F} = \langle 2, 4, 1\rangle$N moves an object from $(0, 0, 1)$m to $(2, 4, 6)$m. How much work is done?

Solution: For this problem it helps to use the formula

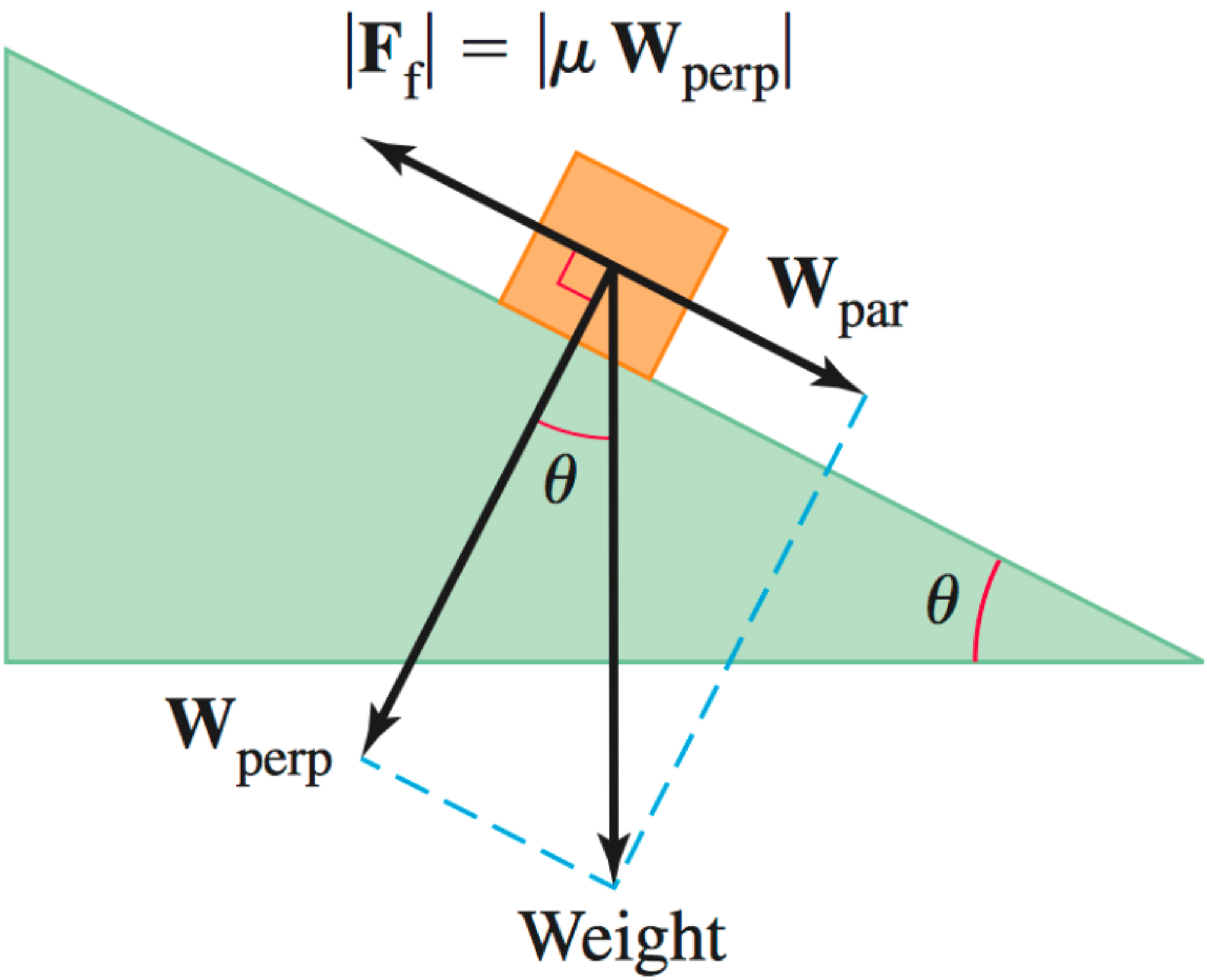
$$\text{Work} = |\vec{F}| \cdot |\vec{d}\,| \cos(\theta) = \vec{F} \cdot \vec{d}, \tag{122}$$

where $\vec{F}$ is a constant force that is applied to an object that moves in a straight line with a final displacement of $\vec{d}$. We now see that

$$\vec{d} = \langle 2, 4, 6\rangle m - \langle 0, 0, 1\rangle m = \langle 2, 4, 5\rangle m \tag{123}$$

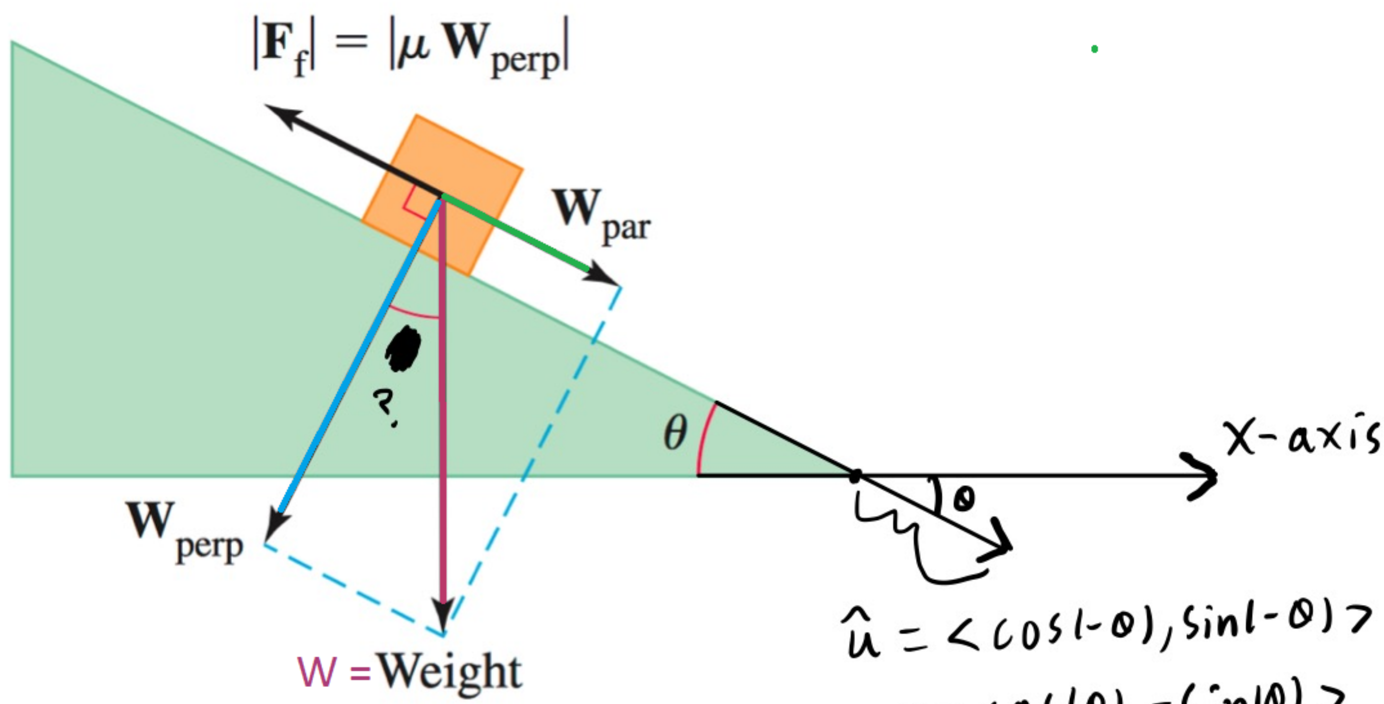
$$\rightarrow \text{Work} = \underbrace{\langle 2, 4, 1\rangle N}_{\vec{F}} \cdot \underbrace{\langle 2, 4, 5\rangle m}_{\vec{d}} = \boxed{25\text{J}}. \tag{124}$$

Problem 1.3: An object on an inclined plane does not slide provided the component of the object's weight parallel to the plane $|\mathbf{W}_{\text{par}}|$ is less than or equal to the magnitude of the opposing frictional force $|\mathbf{F}_f|$. The magnitude of the frictional force, in turn, is proportional to the component of the object's weight perpendicular to the plane $|\mathbf{W}_{\text{perp}}|$. The constant of proportionality is the coefficient of static friction $\mu > 0$. Suppose a 100lb block rests on a plane that is tilted at an angle of $\theta = 30°$ to the horizontal. What is the smallest possible value of μ?



We will present 2 solutions to this problem. The first solution is a direct approach but is computationally intensive. The second solution requires a little more ingenuity but is shorter. For the sake of generality, in both solutions we will solve the problem for a general angle θ and weight w and only plug in $\theta = 30°$ and $w = 100$ at the very end.

Solution 1: We see that $\mathbf{W} = \mathbf{W}_{\text{par}} + \mathbf{W}_{\text{perp}}$ is an decomposition of the force of gravity $\mathbf{W}$ into a sum of two orthogonal components. Since we know that $\mathbf{W} = \langle 0, -w\rangle$lb we only need to find $\mathbf{W}_{\text{par}}$ and it will then be easy to obtain $\mathbf{W}_{\text{perp}}$ through subtraction. To find $\mathbf{W}_{\text{par}}$ we calculate the orthogonal projection of $\mathbf{W}$ onto $\hat{u}$, the direction of the ramp as shown in the diagram below.



We now see that

$$(125) \qquad \mathbf{W}_{\text{par}} = \text{Proj}_{\hat{u}}\mathbf{W} = \frac{\mathbf{W}\cdot\hat{u}}{|\hat{u}|^2}\hat{u} = (\mathbf{W}\cdot\hat{u})\hat{u}$$

$$(126) \qquad = (\langle 0, -w\rangle \cdot \langle \cos(\theta), -\sin(\theta)\rangle)\langle \cos(\theta), -\sin(\theta)\rangle$$

$$(127) \qquad = \langle w\sin(\theta)\cos(\theta), -w\sin(\theta)^2\rangle$$

$$(128) \qquad \mathbf{W}_{\text{perp}} = \mathbf{W} - \mathbf{W}_{\text{par}} = \langle 0, -w\rangle - \langle w\sin(\theta)\cos(\theta), -w\sin(\theta)^2\rangle$$

$$(129) \qquad = \langle -w\sin(\theta)\cos(\theta), w(-1+\sin^2\theta)\rangle = \langle -w\sin(\theta)\cos(\theta), -w\cos^2(\theta)\rangle$$

We now recall that we are searching for μ for which

$$(130)\qquad |\mathbf{W}_{\text{par}}| = |\mathbf{F}_{\text{f}}| = |\mu \mathbf{W}_{\text{perp}}| = \mu |\mathbf{W}_{\text{perp}}| \to \mu = \frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|}$$

To this end, we see that[6]

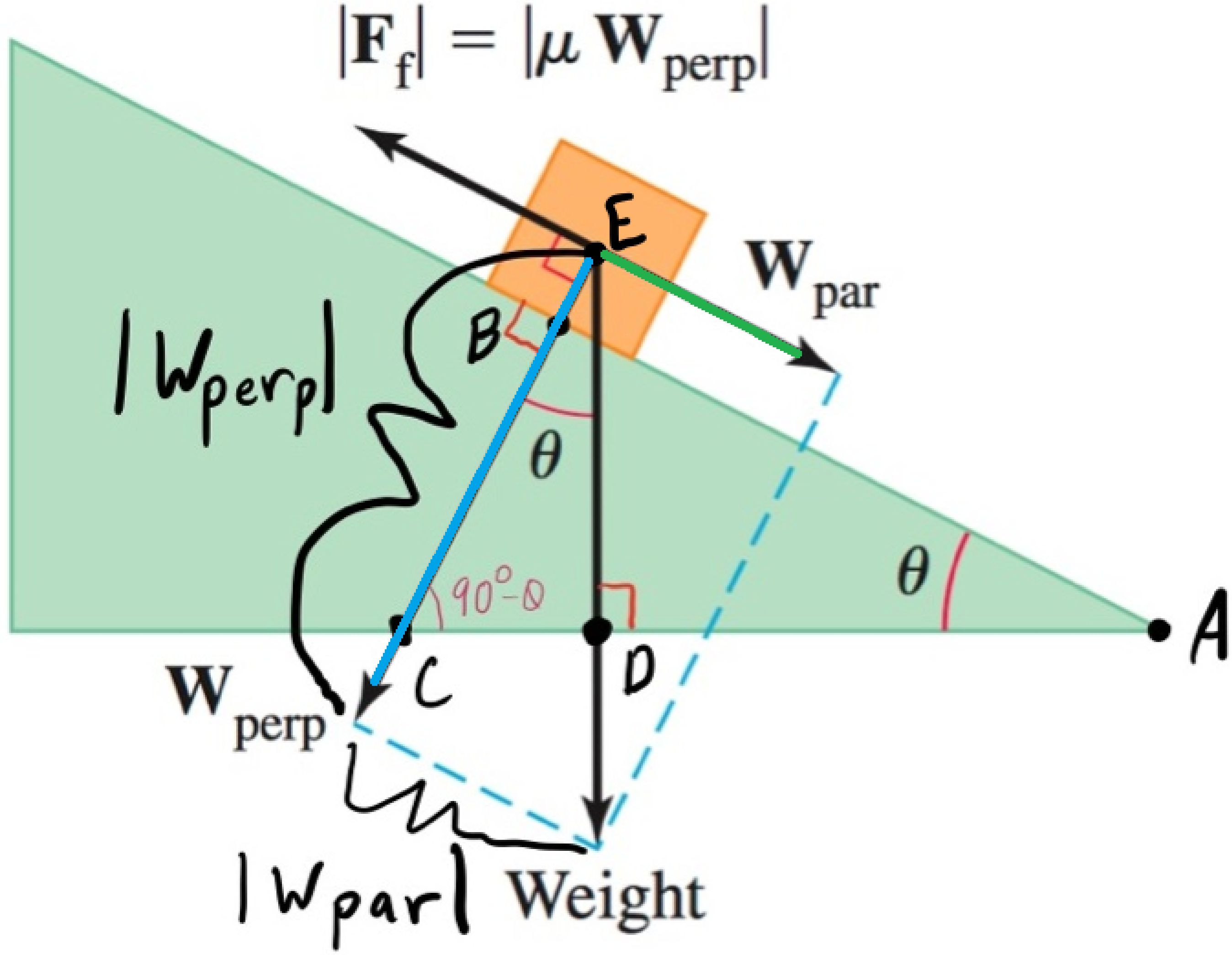
$$(131)\quad |\mathbf{W}_{\text{par}}| = \sqrt{(w\sin(\theta)\cos(\theta))^2 + (-w\sin(\theta)^2)^2} = w\sin(\theta)\sqrt{\cos^2(\theta)+\sin^2(\theta)} = w\sin(\theta), \text{ and}$$

$$(132)\quad |\mathbf{W}_{\text{perp}}| = \sqrt{(-w\sin(\theta)\cos(\theta))^2 + (-w\cos^2(\theta))^2} = w\cos(\theta)\sqrt{\sin^2(\theta)+\cos^2(\theta)} = w\cos(\theta), \text{ hence}$$

$$(133)\qquad \mu = \frac{|\mathbf{W}_{\text{par}}|}{|\mathbf{W}_{\text{perp}}|} = \frac{w\sin(\theta)}{w\cos(\theta)} = \boxed{\tan(\theta)} = \tan(30^\circ) = \boxed{\frac{1}{\sqrt{3}}}.$$

Solution 2: First, let us verify that the two angles labeled with θ in the given diagram are indeed the same angle. We begin by labeling points on the original diagram as shown in the new diagram below in order to obtain the subsequent calculations.

[6] Recall that $\sin(\theta), \cos(\theta) \geq 0$ when $0 \leq \theta \leq 90^\circ$, so we don't need to write $|\sin(\theta)|$ or $|\cos(\theta)|$ in this case.



(134) $$\angle ACB = 90^\circ - \angle BAC = 90^\circ - \theta \text{ and}$$

(135) $$\angle CED = 90^\circ - \angle DCE = 90^\circ - \angle ACB = 90^\circ - (90^\circ - \theta) = \theta,$$

so the given diagram was indeed correctly labeled. We now recall that we are searching for μ for which

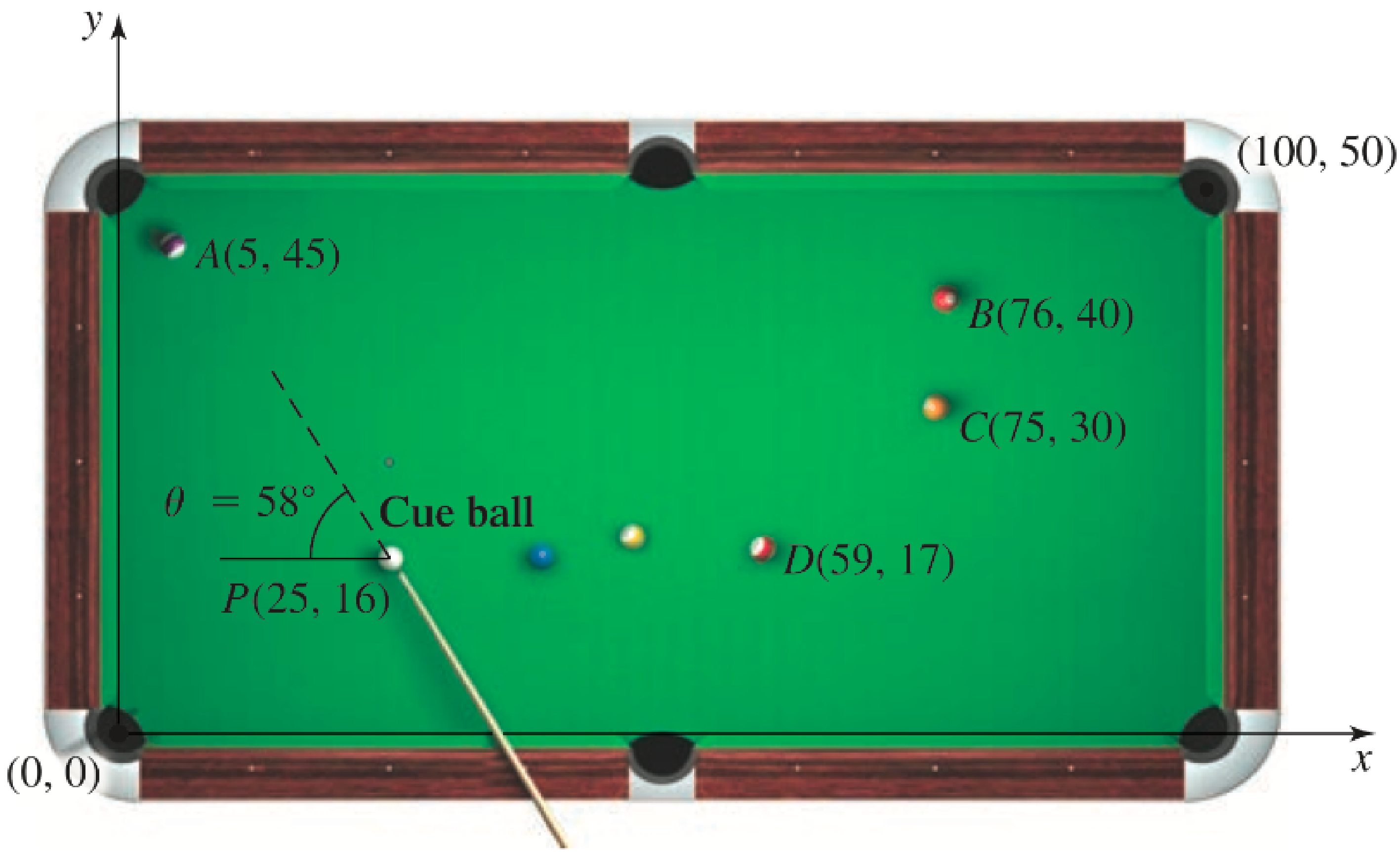
(136) $$|\mathbf{W}_{par}| = |\mathbf{F}_f| = |\mu \mathbf{W}_{perp}| = \mu|\mathbf{W}_{perp}| \to \mu = \frac{|\mathbf{W}_{par}|}{|\mathbf{W}_{perp}|}$$

After taking a look at our labeled diagram we see that

(137) $$\frac{|\mathbf{W}_{par}|}{|\mathbf{W}_{perp}|} = \boxed{\tan(\theta)} = \tan(30^\circ) = \boxed{\frac{1}{\sqrt{3}}}.$$

Problem 1.4: A cue ball in a billiards video game lies at P(25, 16). We assume that each ball has a diameter of 2.25 screen units, and pool balls are represented by the point at their center.

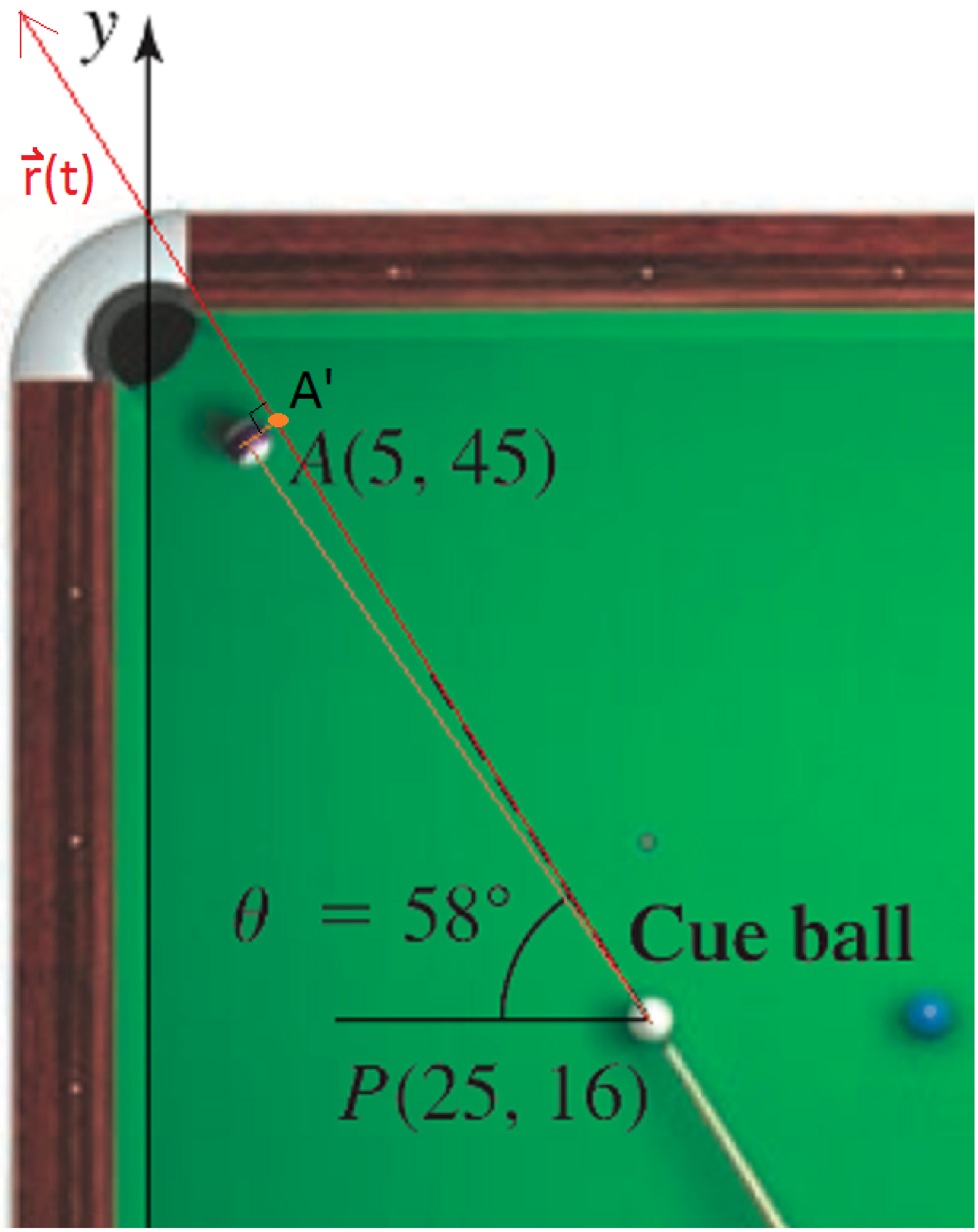
a. The cue ball is aimed at an angle of 58° above the negative x-axis toward a target ball at A(5, 45). Do the balls collide?
b. The cue ball is aimed at the point (50, 25) in an attempt to hit a target ball at B(76, 40). Do the balls collide?
c. The cue ball is aimed at an angle θ above the x-axis in the general direction of a target ball at C(75, 30). What range of angles (for $0 \leq \theta \leq \frac{\pi}{2}$) will result in a collision? Express your answer in degrees.



Solution to a: Since the diameter of each ball is 2.25 units, the balls collide if at some point in time their centers are at most 2.25 units away from each other. Therefore we can solve this problem by calculating the distance between the point A and the straight line that is the trajectory of the cue ball. To this end we observe that the cue ball is aimed at an angle of $180° - 58° = 122°$ above the positive x-axis, so we can parametrize the trajectory of the cue ball (even though we really only need the direction of the trajectory) by

$$(138)\quad \vec{r}(t) = \langle 25, 16\rangle + t\underbrace{\langle \cos(122^\circ), \sin(122^\circ)\rangle}_{\hat{u},\text{ the direction of the trajectory}} = \langle 25 + t\cos(122^\circ), 16 + t\sin(122^\circ)\rangle.$$

Now let A' be the point on $\vec{r}(t)$ that is closest to A, which happens to be the orthogonal projection of A onto $\vec{r}(t)$ as shown in the diagram below.



We have now reduced to problem down to whether or not $|\overrightarrow{A'A}|$ is larger than 2.25 or not. Since $\overrightarrow{A'A} = \overrightarrow{PA} - \overrightarrow{PA'}$, we first observe that

$$\overrightarrow{PA} = \underbrace{\langle 5, 45\rangle}_{\text{A}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle -20, 29\rangle, \tag{139}$$

and we will now proceed to find $\overrightarrow{PA'}$. We see that $\overrightarrow{PA'}$ is the orthogonal projection of $\overrightarrow{PA}$ onto $\vec{r}(t)$, but $\hat{u}$ points in the same direction as $\vec{r}(t)$, so $\overrightarrow{PA'}$ is also the orthogonal projection of $\overrightarrow{PA}$ onto $\hat{u}$, hence

$$\overrightarrow{PA'} = (\overrightarrow{PA} \cdot \hat{u})\hat{u} \tag{140}$$

$$= \Big(\langle -20, 29\rangle \cdot \langle \cos(122^\circ), \sin(122^\circ)\rangle\Big)\langle \cos(122^\circ), \sin(122^\circ)\rangle \tag{141}$$

$$\approx \langle -18.65, 29.84\rangle. \tag{142}$$

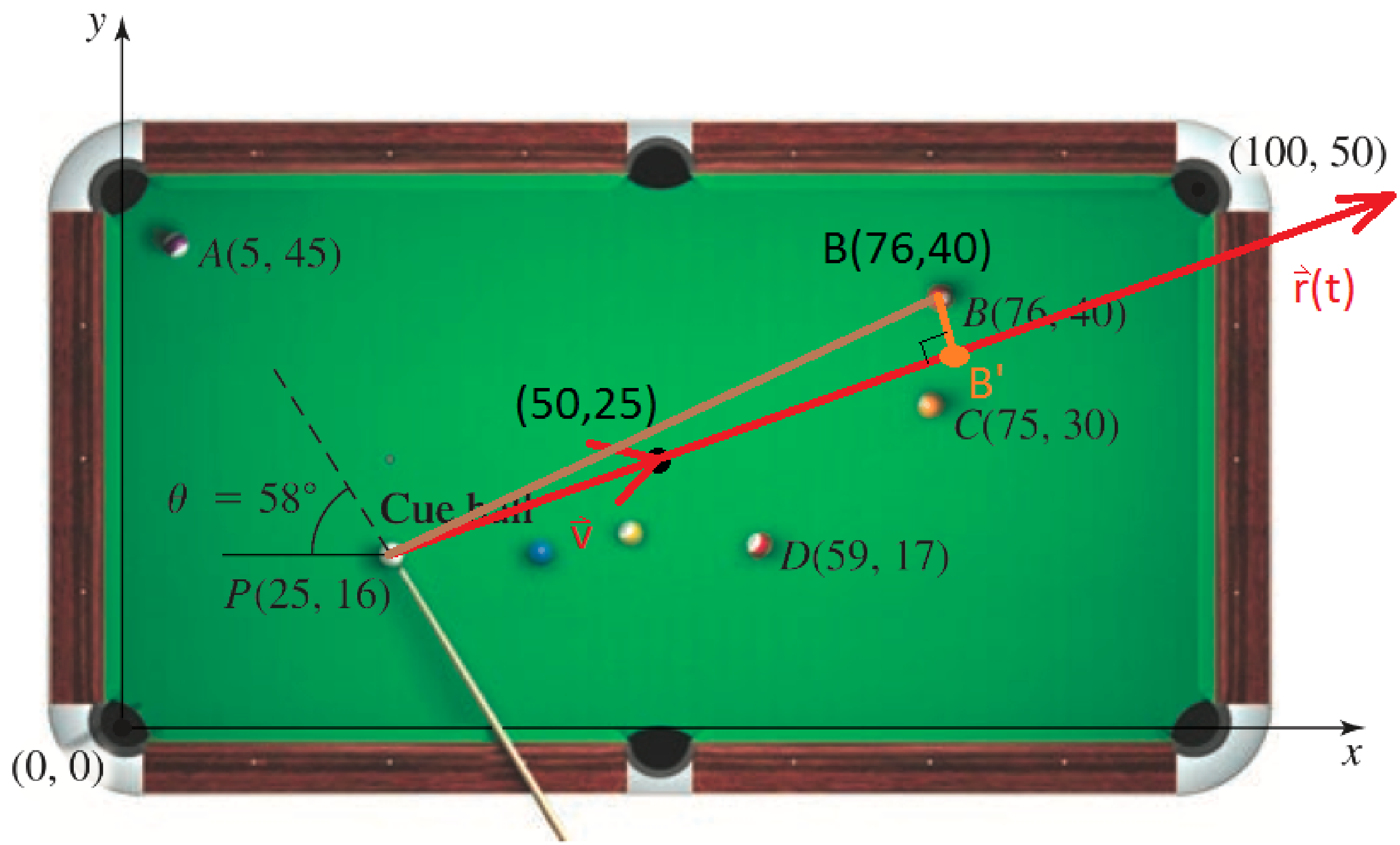
Putting everything together we see that

$$|\overrightarrow{A'A}| = |\overrightarrow{PA} - \overrightarrow{PA'}| \approx |\langle -20, 29\rangle - \langle -18.65, 29.84\rangle| \tag{143}$$

$$= |\langle -1.35, -0.84\rangle| = 1.59 < 2.25, \tag{144}$$

so the balls do collide.

Solution to b: We use the same strategy that we used in part a. The only difference is that we will use slightly different computations to obtain a parametrization of (or more importantly, the direction of) the path of the cue ball since we were given a point on its trajectory rather than the angle that the trajectory makes with the x-axis.



We see that

$$\vec{r}(t) = \langle 25, 16\rangle + t \underbrace{(\langle 50, 25\rangle - \langle 25, 16\rangle)}_{\text{Points in the direction of } \vec{r}(t)} \tag{145}$$

$$= \langle 25, 16\rangle + t \underbrace{\langle 25, 9\rangle}_{\vec{v}} = \langle 25 + 25t, 16 + 9t\rangle \tag{146}$$

Since $\overrightarrow{BB'} = \overrightarrow{PB} - \overrightarrow{PB'}$, we first observe that

$$\overrightarrow{PB} = \underbrace{\langle 76, 40\rangle}_{\text{B}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle 51, 24\rangle, \tag{147}$$

and we will now proceed to find $\overrightarrow{PB'}$. Since $\overrightarrow{PB'}$ is the orthogonal projection of $\overrightarrow{PB}$ onto $\vec{r}(t)$ we see as in part a that $\overrightarrow{PB'}$ is also the orthogonal projection of $\overrightarrow{PB}$ onto $\vec{v}$, so

$$\overrightarrow{PB'} = \frac{\overrightarrow{PB} \cdot \vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\langle 51, 24\rangle \cdot \langle 25, 9\rangle}{|\langle 25, 9\rangle|^2}\langle 25, 9\rangle \tag{148}$$

$$= \langle \frac{37275}{706}, \frac{13419}{706} \rangle \approx \langle 52.80, 19.01 \rangle \tag{149}$$

Putting everything together we see that

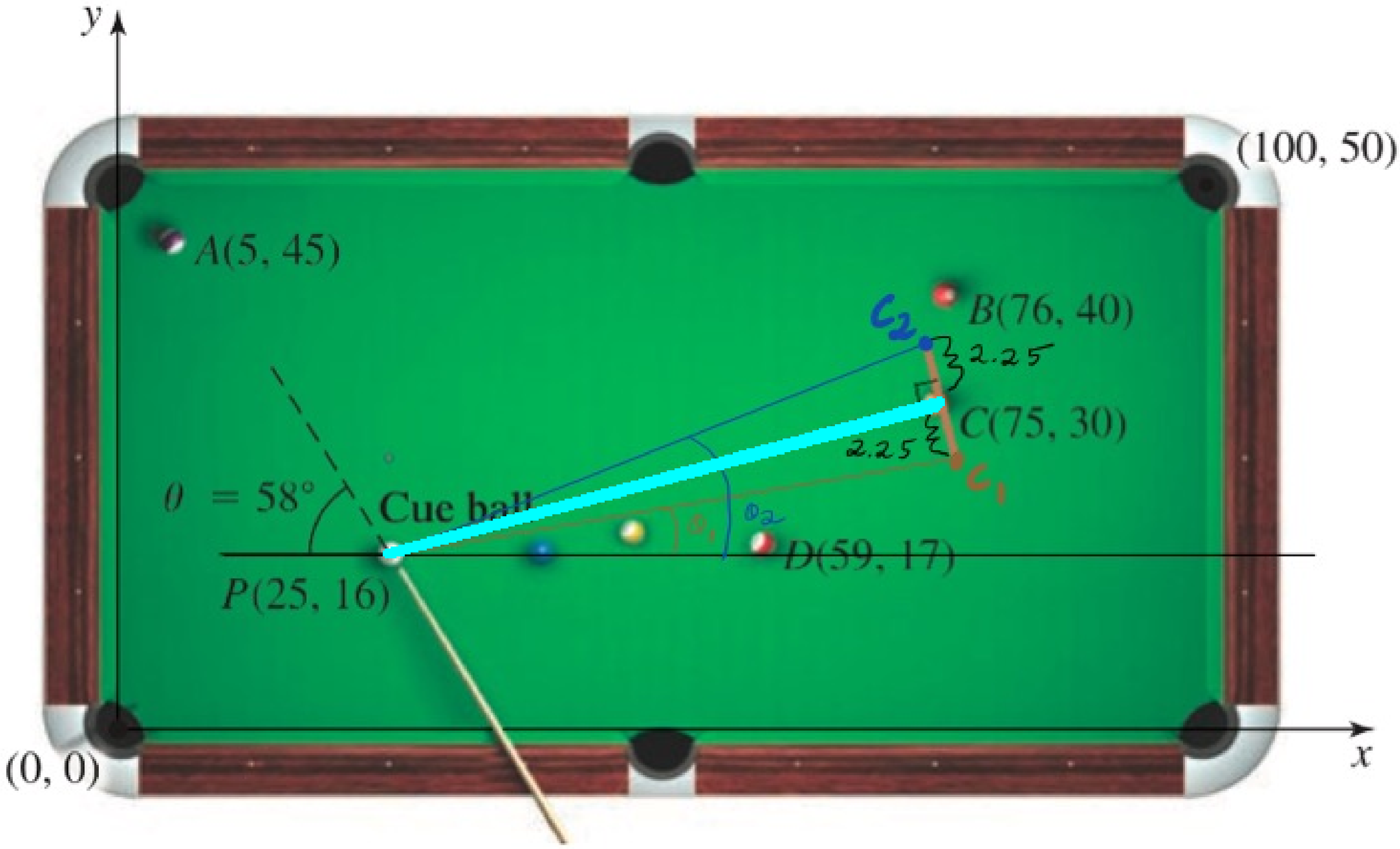
$$|\overrightarrow{BB'}| = |\overrightarrow{PB} - \overrightarrow{PB'}| \approx |\langle 51, 24 \rangle - \langle 52.80, 19.01 \rangle| \tag{150}$$

$$|\langle -1.80, 4.99 \rangle| \approx 5.30 > 2.25, \tag{151}$$

so the balls do not collide.

Remark: While we did not use the parametrizations $\vec{r}(t)$ in either of parts a or b to calculate the relevant orthogonal projections, it is worth noting that an alternative solution to these problems is to use single variable calculus to find the minimum value of the function $d_A(t) = d((5, 45), \vec{r}(t))$ for part a and the function $d_B(t) = d((76, 40), \vec{r}(t))$ for part b.

Solution to c: Let the points C_1 and C_2 be such that $|\overrightarrow{CC_1}| = |\overrightarrow{CC_2}| = 2.25$ and each of $\overrightarrow{CC_1}$ and $\overrightarrow{CC_2}$ are orthogonal to $\overrightarrow{PC}$. $\overrightarrow{PC_1}$ and $\overrightarrow{PC_2}$ represent the trajectories in which the cue ball just barely touches the ball at C, so we want to determine the angles θ_1 and θ_2 as shown in the diagram below.



To this end we begin by observing that

(152) $$\overrightarrow{PC} = \underbrace{\langle 75, 30\rangle}_{\text{C}} - \underbrace{\langle 25, 16\rangle}_{\text{P}} = \langle 50, 14\rangle.$$

Recalling that for a given vector $\vec{w} := \langle x, y\rangle$ the vectors $\langle -y, x\rangle$ and $\langle y, -x\rangle$ are orthogonal to $\vec{w}$, we see that

(153) $$\overrightarrow{PC_1} = \overrightarrow{PC} + 2.25 \cdot \frac{\langle 14, -50\rangle}{|\langle 14, -50\rangle|} \approx \langle 50.61, 11.83\rangle, \text{ and}$$

(154) $$\overrightarrow{PC_2} = \overrightarrow{PC} + 2.25 \cdot \frac{\langle -14, 50\rangle}{|\langle -14, 50\rangle|} \approx \langle 49.39, 16.17\rangle.$$

We now see that

(155) $$\theta_1 = \tan^{-1}\left(\frac{11.83}{50.61}\right) \approx 13.16°, \text{ and} \theta_2 = \tan^{-1}\left(\frac{16.17}{49.39}\right) \approx 18.13°,$$

so the balls will collide if $13.16° \leq \theta \leq 18.13°$

Remark: An alternative approach to solving this problem is to first find a parametrization $\vec{r_\theta}(t)$ of the trajectory of the cue ball when it makes an angle of $0 \leq \theta \leq \frac{\pi}{2}$ with the positive x-axis using the techniques from part a. Then for each such θ we use single variable calculus find $m(\theta)$, the minimum value of $d_{C,\theta}(t) = d((75, 30), \vec{r_\theta}(t))$ as t varies (and θ is still fixed). To finish the problem we find the values of θ for which $m(\theta) = 2.25$.

Problem 1.5: Determine whether the lines $\vec{r}(t) = \langle 1,3,2\rangle + t\langle 6,-7,1\rangle$ and $R(s) = \langle 10,6,14\rangle + s\langle 8,1,4\rangle$ are parallel or skew, and find their intersection(s) if any exist.

Solution: Let us first determine whether or not the lines paramterized by $\vec{r}(t)$ and $\vec{R}(s)$ are parallel since that requires the least computations. We see that the line paramterized by $\vec{r}(t)$ has the same direction as the vector $\langle 6,-7,1\rangle$ and the line parameterized by $\vec{R}(s)$ has the same direction as the vector $\langle 8,1,4\rangle$. It is clear that there is no constant c for which

$$\langle 6,-7,1\rangle = c\langle 8,1,4\rangle = \langle 8c, c, 4c\rangle \tag{156}$$

since we cannot simulataneously have $c = -7$ and $4c = 1$, so the lines in question are not parallel. Now let us search for the intersection(s) of the lines in questions while recalling that the lines will be skew if there are no intersections (since we have already shown that they are not parallel). To do this, we want to find all $t, s \in \mathbb{R}$ for which $\vec{r}(t) = \vec{R}(s)$, which results in the following computations:

$$\underbrace{\langle 1,3,2\rangle + t\langle 6,-7,1\rangle}_{\vec{r}(t)} = \underbrace{\langle 10,6,14\rangle + s\langle 8,1,4\rangle}_{\vec{R}(s)} \tag{157}$$

$$\Leftrightarrow \langle 1+6t, 3-7t, 2+t\rangle = \langle 10+8s, 6+s, 14+4s\rangle \tag{158}$$

$$\Leftrightarrow \begin{aligned} 1+6t &= 10+8s \\ 3-7t &= 6+s \\ 2+t &= 14+4s \end{aligned} \tag{159}$$

$$\to s = -3-7t \tag{160}$$

$$\to 2+t = 14+4s = 14+4(-3-7t) = 2-28t \tag{161}$$

$$\to t = 0 \to s = -3. \tag{162}$$

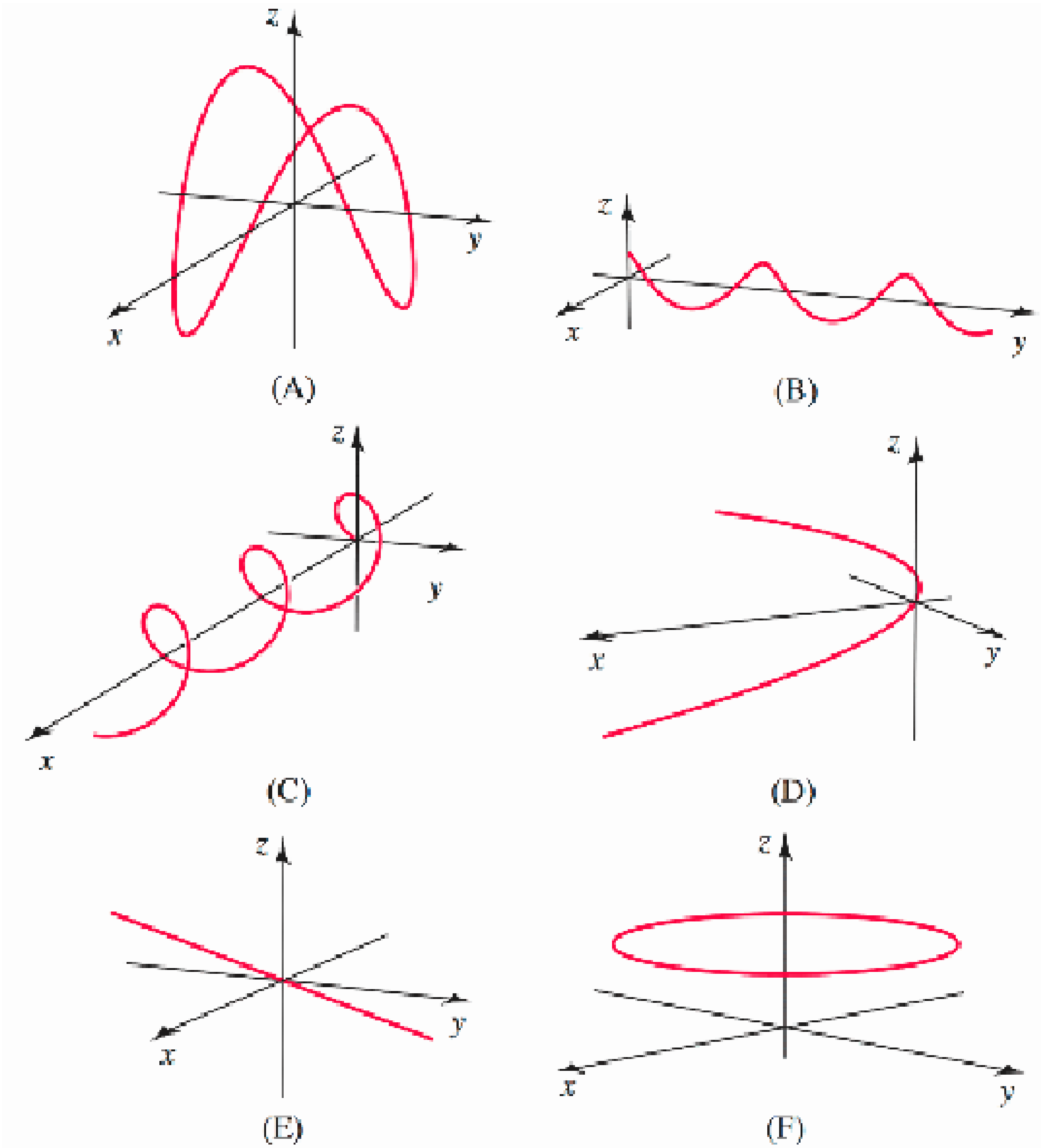
However, since

$$1 + 6 \cdot 0 \neq 10 + 8 \cdot (-3), \tag{163}$$

we see that there are no $s, t \in \mathbb{R}$ for which $\vec{r}(t) = \vec{R}(s)$, so the lines in question are skew.

Problem 1.7 Match function a-f with the appropriate graph A-F.

a. $\vec{r}(t) = \langle t, -t, t \rangle$.
b. $\vec{r}(t) = \langle t^2, t, t \rangle$.
c. $\vec{r}(t) = \langle 4\cos(t), 4\sin(t), 2 \rangle$.
d. $\vec{r}(t) = \langle 2t, \sin(t), \cos(t) \rangle$.
e. $\vec{r}(t) = \langle \sin(t), \cos(t), \sin(2t) \rangle$.
f. $\vec{r}(t) = \langle \sin(t), 2t, \cos(t) \rangle$.



Solution:

a $\leftrightarrow$ E It is clear that $\vec{r}(t)$ is the paramterization of a straight line and (E) is the only graph of a straight line in the available options.

b $\leftrightarrow$ D We see that $\vec{r}(t)$ is a parabola if we make a plane with the line $y = z$ taking the place of (what is normally) the x-axis and the line $y = z = 0$ taking the place of (what is normally) the y-axis and (D) is the only graph of a parabola.

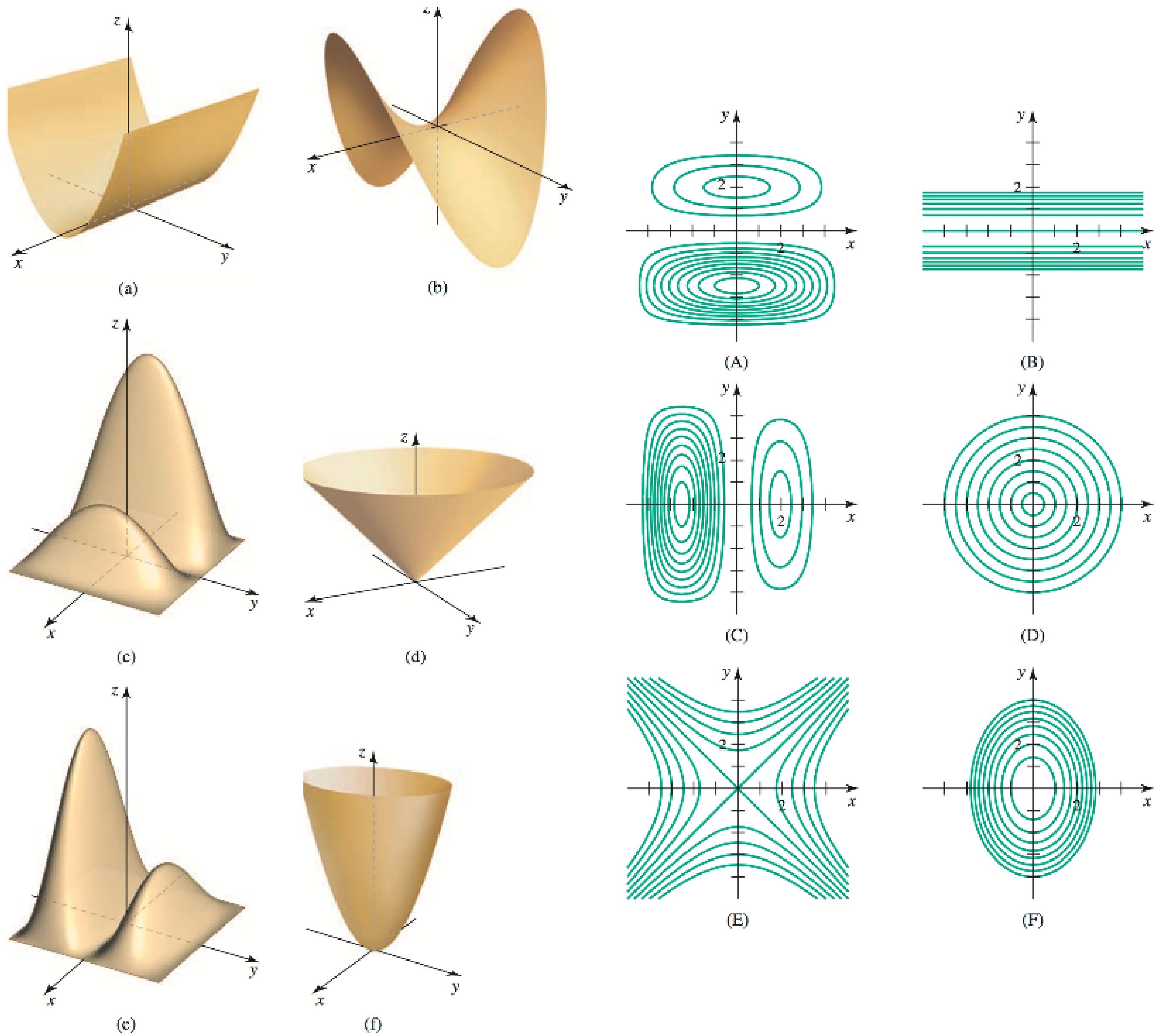
c $\leftrightarrow$ F We see that the z-coordinate of $\vec{r}(t)$ is constant so the graph of $\vec{r}(t)$ lies in a horizontal plane and (F) is the only such graph.

d $\leftrightarrow$ C We see that when the x-coordinate of $\vec{r}(t)$ is ignored the result is a parameterization of the unit circle in the yz-plane, so if the graph of $\vec{r}(t)$ is "smushed down to the yz-plane" then the result will be the unit circle. Furthermore, it is clear that the x-coordinate of $\vec{r}(t)$ is unbounded and (C) is the only graph that satisfies the previous two properties.

e $\leftrightarrow$ A We see that all three components of $\vec{r}(t)$ are bounded and that none of the components are constant and (A) is the only graph that satisfies these properties.

f $\leftrightarrow$ B We see that when the y-coordinate of $\vec{r}(t)$ is ignored the result is a parameterization of the unit circle in the xz-plane, so if the graph of $\vec{r}(t)$ is "smushed down to the xz-plane" then the result will be the unit circle. Furthermore, it is clear that the y-coordinate of $\vec{r}(t)$ is unbounded and (B) is the only graph that satisfies the previous two properties.

Problem 1.8: Match surfaces a-f in the figure below with level curves A-F.



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a⇔(B): We see that the figure in a is similar to the cylinder generated by the curve $z = y^2$ in the yz-plane with the line $z = y = 0$, so the level sets will consist of lines that are parallel to the x-axis, which allows us to match a with (B).

b⇔(E): Since b is a hyperbolic paraboloid we know that its level sets are lines (for the level set of $z = 0$) and hyperbolas (for level sets of $z = z_0 \neq 0$), which allows us to match b with (E).

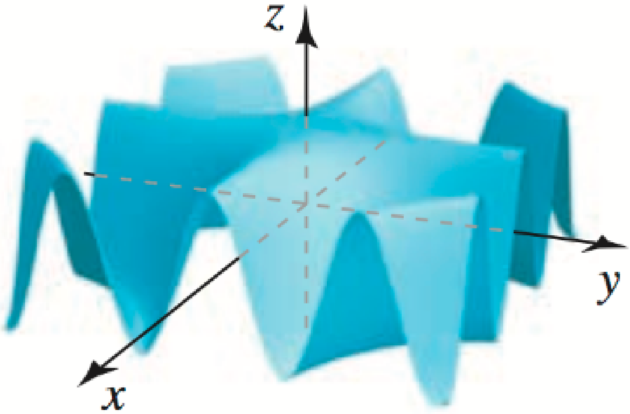
c⇔(C): The level sets of c are easily seen to be one or two (depending on the height of the level set) ellipse-like shapes whose major axes (by analogy, not literally) are parallel to the y-axis, which allows us to match c with (C).

d⇔(D): Since d is a cone we know that all of its level sets are circles, which allows us to match d with (D).

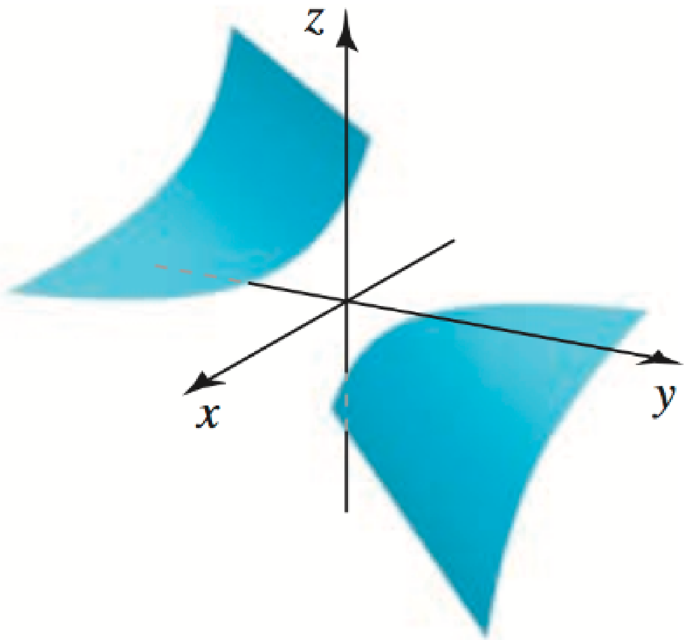
e⇔(A): The level sets of e are easily seen to be one or two (depending on the height of the level set) ellipse-like shapes whose major axes (by analogy, not literally) are parallel to the x-axis, which allows us to match e with (A).

f⇔(F): Since f is an elliptic paraboloid we know that all of its level sets are ellipses, which allws us to match f with (A). **Problem 1.9:** Match functions a-d with surfaces A-D in the figure below.

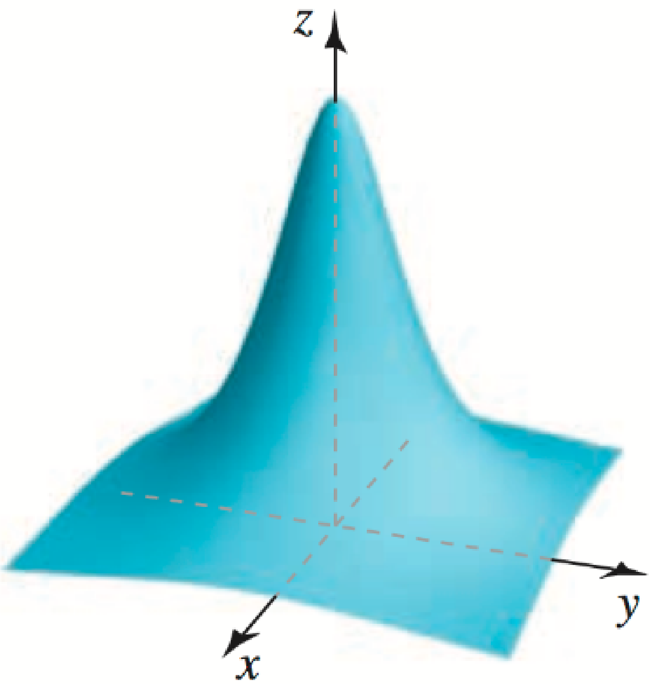
a. $f(x, y) = \cos(xy)$
b. $g(x, y) = \ln(x^2 + y^2)$
c. $h(x, y) = \frac{1}{x-y}$
d. $p(x, y) = \frac{1}{1+x^2+y^2}$



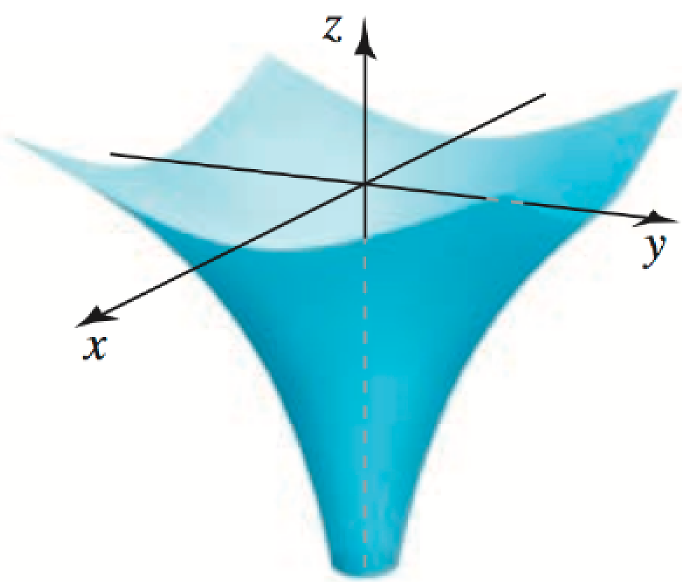
(A)



(B)



(C)



(D)

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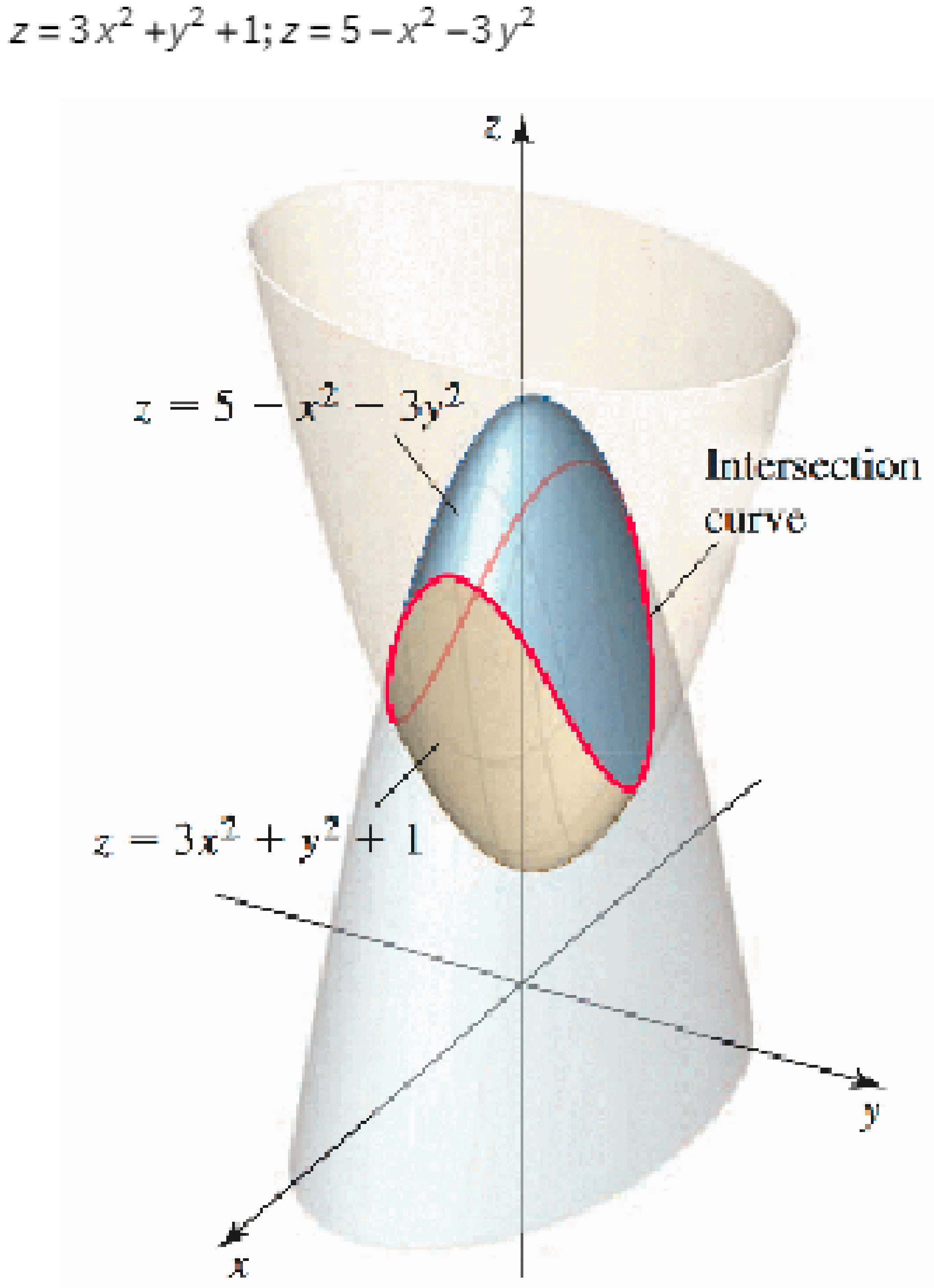
a⇔(A): We see that $1 = \cos(0 \cdot y) = \cos(x \cdot 0)$ for all $x, y \in \mathbb{R}$, so f takes on the value 1 on the x-axis as well as the y-axis, which is enough to match a with (A). Alternatively, we may examine the behavior of f along some other line $y = mx$, i.e., examine $f(x, mx) = \cos(mx^2)$. One of the easiest such lines to examine is $y = x$, and we see that $f(x,x) = \cos(x^2)$ oscillates indefinitely, which is again enough information to yield the desired result.

b⇔(D): We begin by examining the level sets of $g(x,y)$. We see that $\ln(x^2 + y^2) = c \Leftrightarrow x^2 + y^2 = e^c$, so the level sets of $g(x,y)$ are circles centered at the origin. We also observe that $g(0,0)$ is undefined and that $g(x,y)$ increases as $x^2 + y^2$ increases, so we deduce that b is matched with (D).

c⇔(B): We begin by examining the level sets of $h(x,y)$. We see that $\frac{1}{x-y} = c \Leftrightarrow x - y = \frac{1}{c}$, so the level sets of $h(x,y)$ are lines parallel to the line $x = y$, i.e., lines that make an angle of $45°$ with the positive x-axis. Furthermore, we see that if $x - y = c_1$ with $c_1 > 0$ then $h(x,y) > 0$, if $x - y = 0$ then $h(x,y)$ is undefined, and if $x - y = c_2$ with $c_2 < 0$ then $h(x,y) < 0$, which is enough information to match c with (B).

d⇔(C): We begin by examining the level sets of $p(x,y)$. We see that $\frac{1}{1+x^2+y^2} = c \Leftrightarrow x^2+y^2 = \frac{1}{c} - 1$, so the level sets of $p(x,y)$ are circles centered at the origin. We observe that $p(x,y)$ is defined for all $x, y \in \mathbb{R}$ and that $p(x,y)$ decreases as $x^2 + y^2$ increases, so we deduce that d is matched with (C).

Problem 1.10: Find a function $\vec{r}(t)$ that describes the curve $\mathcal{C}$ which is the intersection of the surfaces $z = 3x^2 + y^2 + 1$ and $z = 5 - x^2 - 3y^2$. Note that there is not a unique answer to this question since any curve possess infinitely many distinct paramterizations.



Solution: Writing $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, we see that we only need to determine $x(t)$ and $y(t)$ since it will then be very easy to determine $z(t)$ using the defining equations of the surfaces in play. For the sake of readability we will write x, y, z instead of $x(t), y(t), z(t)$ for calculations in this problem. We see that

$$3x^2 + y^2 + 1 = z = 5 - x^2 - 3y^2 \to 4x^2 + 4y^2 = 4 \to x^2 + y^2 = 1. \tag{164}$$

It follows that $\langle x(t), y(t)\rangle$ traces out the unit circle, so we may set $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$. We now see that

$$z = 3x^2 + y^2 + 1 = 3\cos^2(t) + \sin^2(t) + 1 = 2\cos^2(t) + 2 = \cos(2t) + 3. \tag{165}$$

Putting everything together we see that we may take

$$\boxed{\vec{r}(t) = \langle \cos(t), \sin(t), \cos(2t) + 3\rangle, 0 \le t \le 2\pi}. \tag{166}$$

Problem 1.11: Suppose that $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector valued functions satisfying $\vec{u}(0) = \langle 0,1,1\rangle, \vec{u}\,'(0) = \langle 0,7,1\rangle, \vec{v}(0) = \langle 0,1,1\rangle$, and $\vec{v}\,'(0) = \langle 1,1,2\rangle$. Evaluate the following expressions.

a. $\frac{d}{dt}\Big(\vec{u}(t)\cdot\vec{v}(t)\Big)\Big|_{t=0}$

b. $\frac{d}{dt}\Big(\vec{u}(t)\times\vec{v}(t)\Big)\Big|_{t=0}$

c. $\frac{d}{dt}\Big(\cos(t)\vec{u}(t)\Big)\Big|_{t=0}$

d. $\frac{d}{dt}\Big(\vec{u}(\sin(t))\Big)\Big|_{t=0}$

Solution to a: Recalling that the product rule for dot products looks exactly the same as the ordinary product rule for scalar valued functions we see that

(167) $$\frac{d}{dt}\Big(\vec{u}(t)\cdot\vec{v}(t)\Big)\Big|_{t=0} = \Big(\frac{d}{dt}\vec{u}(t)\Big)\cdot\vec{v}(t)+\vec{u}(t)\cdot\Big(\frac{d}{dt}\vec{v}(t)\Big)\Big|_{t=0}$$

(168) $$= \vec{u}\,'(0)\cdot\vec{v}(0)+\vec{u}(0)\cdot\vec{v}\,'(0) = \langle 0,7,1\rangle\cdot\langle 0,1,1\rangle+\langle 0,1,1\rangle\cdot\langle 1,1,2\rangle$$

(169) $$= (0+7+1)+(0+1+2) = \boxed{11}.$$

Solution to b: Recalling that the product rule for cross products looks exactly the same as the ordinary product rule for scalar valued functions we see that

(170) $$\frac{d}{dt}\Big(\vec{u}(t)\times\vec{v}(t)\Big)\Big|_{t=0} = \Big(\frac{d}{dt}\vec{u}(t)\Big)\times\vec{v}(t)+\vec{u}(t)\times\Big(\frac{d}{dt}\vec{v}(t)\Big)\Big|_{t=0}$$

(171) $$= \vec{u}\,'(0)\times\vec{v}(0)+\vec{u}(0)\times\vec{v}\,'(0) = \langle 0,7,1\rangle\times\langle 0,1,1\rangle+\langle 0,1,1\rangle\times\langle 1,1,2\rangle$$

(172) $$= \langle 6,0,0\rangle+\langle 1,1,-1\rangle = \boxed{\langle 7,1,-1\rangle}.$$

Solution to c: Recalling that the product rule for a scalar valued function with a vector valued function looks exactly the same as the ordinary product rule for scalar valued functions we see that

(173) $$\frac{d}{dt}\Big(f(t)\vec{u}(t)\Big)\Big|_{t=0} = \Big(\frac{d}{dt}f(t)\Big)\vec{u}(t)+f(t)\Big(\frac{d}{dt}\vec{u}(t)\Big)\Big|_{t=0},\text{ so}$$

$$\frac{d}{dt}\Big(\cos(t)\vec{u}(t)\Big)\Big|_{t=0} = \Big(\frac{d}{dt}\cos(t)\Big)\vec{u}(t) + \cos(t)\Big(\frac{d}{dt}\vec{u}(t)\Big)\Big|_{t=0} \tag{174}$$

$$= -\sin(0)\vec{u}(0) + \cos(0)\vec{u}\,'(0) = \vec{u}\,'(0) = \boxed{\langle 0,7,1\rangle}. \tag{175}$$

Solution to d: Recalling that the chain rule for a vector valued function composed with a scalar valued function looks exactly the same as the ordinary chain rule for scalar valued functions we see that

$$\frac{d}{dt}(\vec{u}\circ f)(t)\Big|_{t=0} = \left(\left((\frac{d}{dt}\vec{u})\circ f\right)\cdot\frac{d}{dt}f\right)(t)\Big|_{t=0}, \text{ so} \tag{176}$$

$$\frac{d}{dt}\Big(\vec{u}(\sin(t))\Big)\Big|_{t=0} = \frac{d}{dt}(\vec{u}\circ\sin)(t)\Big|_{t=0} = \left(\left((\frac{d}{dt}\vec{u})\circ\sin\right)\cdot\frac{d}{dt}\sin\right)(t)\Big|_{t=0} \tag{177}$$

$$= \vec{u}\,'(\sin(0))\cdot\cos(0) = \vec{u}\,'(0) = \boxed{\langle 0,7,1\rangle}. \tag{178}$$

Problem 2.1: Let $\vec{r}(t) = \langle t, 2, \frac{2}{t} \rangle$ for $t > 1$. Find the unit tangent vector $\hat{T}(t)$ at all points of the curve $\vec{r}(t)$.

Solution: Recalling that $\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ we see that

$$\vec{r}\,'(t) = \langle 1, 0, -\frac{2}{t^2} \rangle \tag{179}$$

$$\Rightarrow |\vec{r}\,'(t)| = \sqrt{1^2 + 0^2 + (-\frac{2}{t^2})^2} = \sqrt{\frac{t^4+4}{t^4}} = \frac{\sqrt{t^4+4}}{t^2} \tag{180}$$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{t^2}{\sqrt{t^4+4}} \langle 1, 0, -\frac{2}{t^2} \rangle = \boxed{\langle \frac{t^2}{\sqrt{t^4+4}}, 0, \frac{-2}{\sqrt{t^4+4}} \rangle}. \tag{181}$$

Problem 1.12: Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.

(a) If the speed of an object is constant, then its velocity components are constant.
(b) The functions $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ and $\vec{R}(t) = \langle \cos(t^2), \sin(t^2) \rangle$ generate the same set of points for $t \geq 0$. (Bonus: What about for $t \geq \pi^2$?)
(c) A velocity vector (vector valued function) of variable magnitude cannot have constant direction.
(d) If the acceleration of an object is $\vec{a}(t) = \vec{0}$, for all $t \geq 0$, then the velocity of the object is constant.
(e) If you double the initial speed of a projectile, its range also double (assume no forces other than gravity).
(f) If you double the initial speed of a projectile, its time of flight also doubles (assume no forces other than gravity).
(g) A trajectory with $\vec{v}(t) = \vec{a}(t) \neq \vec{0}$, for all t, is possible.

Solution to (a): False. An object with a velocity of $\vec{v}(t) = \langle \cos(t), \sin(t) \rangle$ has constant speed since $|\vec{v}(t)| = 1$, but it is clear that neither of the velocity components are constant.

Solution to (b): True. We see that $\vec{R}(t) = \vec{r}(t^2)$, and since $t \geq 0$, we also see that $\vec{r}(t) = \vec{R}(\sqrt{t})$. Since the map $t \mapsto t^2$ is a bijection of $[0, \infty)$ to itself, (as is the map $t \mapsto \sqrt{t}$), we see that $\vec{r}(t)$ and $\vec{R}(t) = \vec{r}(t^2)$ generate the same set of points for $t \geq 0$. Note that the previous reasoning was very general and didn't make use of what $\vec{r}(t)$ and $\vec{R}(t)$ were, but only the fact that they are related by the equation $\vec{R}(t) = \vec{r}(t^2)$. For the bonus, the answer is still True. This time we just observe62 that $\vec{r}(t)$ and $\vec{R}(t)$ both trace out the unit circle, even if we only use $t \geq \pi^2$ instead of $t \geq 0$. In fact, we see that even for $t \geq \pi^2$, both $\vec{r}(t)$ and $\vec{R}(t)$ generate every point on the unit circle and infinite number of times. The reason that we had to change our method of proof for the bonus is that the map $t \mapsto t^2$ send $[\pi^2, \infty)$ onto $[\pi^4, \infty)$ and not $[\pi^2, \infty)$.

Solution to (c): False. The velocity function of $\vec{v}(t) = \langle t^2, t^2, t^2 \rangle$ has constant direction since the direction is always that of $\langle 1, 1, 1 \rangle$, but the magnitude is given by $|\vec{v}(t)| = t^2\sqrt{3}$.

Solution to (d): True. The velocity $\vec{v}(t)$ is given by

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)du = \vec{v}_0 + \int_0^t \langle 0, 0 \rangle du = \vec{v}_0 + \langle 0, 0 \rangle = \vec{v}_0. \tag{182}$$

Solution to (e): False. Let us examine the most general situation so that we can also answer part (f). We may assume that our projectile starts at $(0, 0)$ by choosing an appropriate reference frame, so $\vec{r}_0 = \langle 0, 0 \rangle$. We let $\vec{v}_0 = \langle v_1, v_2 \rangle$ and note that the acceleration of our object is given by $\vec{a}(t) = \langle 0, -g \rangle$ since no forces other than gravity are acting on our system. We see that the velocity $\vec{v}(t)$ is given by

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)du = \vec{v}_0 + \int_0^t \langle 0, -g \rangle du = \vec{v}_0 + \langle 0, -gt \rangle = \langle v_1, v_2 - gt \rangle. \tag{183}$$

We now see that the trajectory $\vec{r}(t)$ is given by

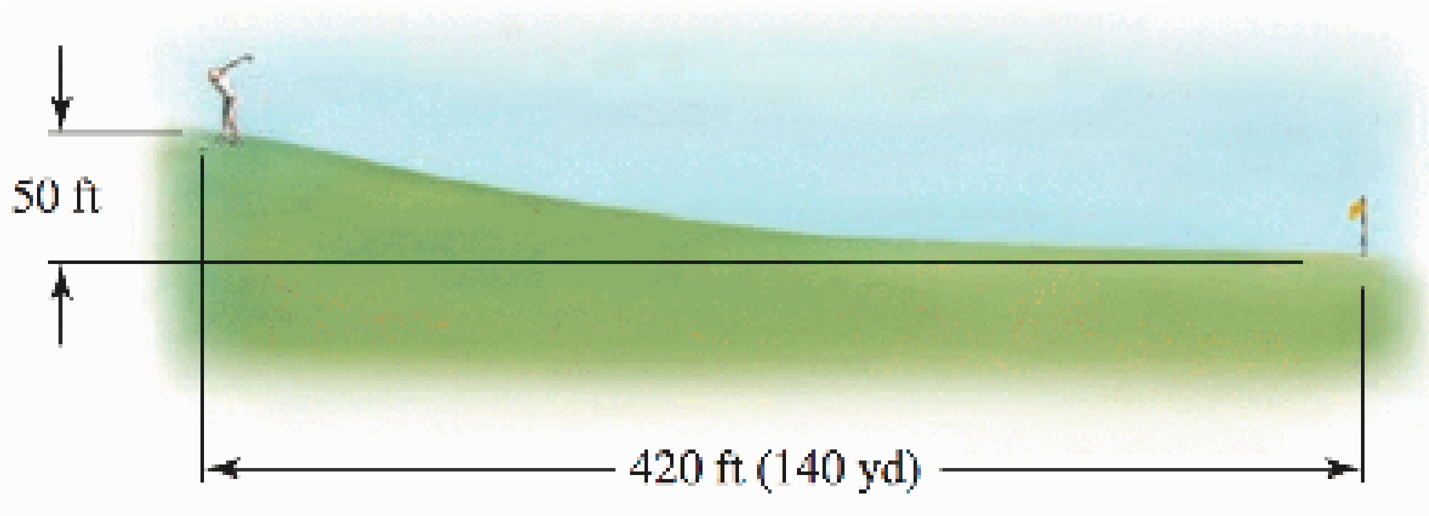
$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u)du = \int_0^t \langle v_1, v_2 - gu \rangle du = \langle v_1 t, v_2 t - \frac{g}{2}t^2 g \rangle. \tag{184}$$

The object stops moving once it hits the floor, which happens when $v_2 t - \frac{g}{2}t^2 = 0$, which happens when $t = \frac{2v_2}{g}$ since $t = 0$ corresponds to the fact that our object started on the floor. We now see that if $\vec{v}_0{}' = 2\vec{v}_0 = \langle 2v_1, 2v_2 \rangle$, then the new time of flight will be $t' = \frac{2v_2'}{g} = \frac{2 \cdot 2v_2}{g} = \frac{4v_2}{g}$, which is twice the previous time of flight, so part (f) is true. We now see that the original range is $v_1 t = \frac{2v_1 v_2}{g}$, while the new range is $v_1' t' = 2v_1 \frac{4v_2}{g} = \frac{8v_1 v_2}{g}$, so the range quadruples instead of doubling.

Solution to (f): True. See the explanation to part (e).

Solution to (g): True. The trajectory given by $\vec{r}(t) = \langle e^t, e^t \rangle$ satisfies $\vec{v}(t) = \vec{r}\,'(t) = \langle e^t, e^t \rangle$ and $\vec{a}(t) = \vec{v}\,'(t) = \langle e^t, e^t \rangle$, so we even have that $\vec{r}(t) = \vec{v}(t) = \vec{a}(t)$ for all t!

Problem 1.13: A golfer stands 420ft (140yd) horizontally from the hole and 50ft above the hole (see figure). Assuming the ball is hit with an initial speed of 120ft/s, at what angle(s) should it be hit to land in the hole? Assume the path of the ball lies in a plane. You may approximate earth's gravitational constant by 32ft/s^2.



Solution: We can model the situation in the xy-plane by placing the golfer at $(0,0)$ and observing that he is aiming the ball at $(420,-50)$. Let $\vec{v}_0$ be the initial velocity with which the golder hits the ball and note that $|\vec{v}_0| = 120$, so $\vec{v}_0 = \langle 120\cos(\theta), 120\sin(\theta)\rangle$, where θ is the angle that $\vec{v}_0$ makes with the positive x-axis. Since the only force acting on the golf ball after the initial shot is gravity, we see that the acceleration of the golf ball is given by $\vec{a}(t) \approx \langle 0,-32\rangle$. It follows that the velocity of the golf ball is given by

$$\vec{v}(t) = \vec{v}_0 + \int_0^t \vec{a}(u)du = \vec{v}_0 + \int_0^t \langle 0,-32\rangle du = \vec{v}_0 + \langle 0,-32t\rangle \tag{185}$$

$$= \langle 120\cos(\theta), 120\sin(\theta) - 32t\rangle. \tag{186}$$

Recalling that the ball starts at $(0,0)$ with the golfer, we let $\vec{r}(t)$ denote the trajectory of the ball and note that $\vec{r}_0 = \vec{r}(0) = (0,0)$, so $\vec{r}(t)$ is given by

$$\vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u)du = \int_0^t \langle 120\cos(\theta), 120\sin(\theta) - 32u\rangle du \tag{187}$$

$$\overset{*}{=} \langle 120t\cos(\theta), 120t\sin(\theta) - 16t^2\rangle. \tag{188}$$

To perform the calculation at equation $(*)$, we recall that θ is the initial angle at which the golf ball is hit, so θ does not change, so we treat θ as a constant (like 2 or π) when performing the integrations. Since we want the ball to pass through $(420, -50)$ at some point in time, we obtain the following system of equations that we will proceed to solve.

$$\vec{r}(t) = (420, -50) \Leftrightarrow \begin{array}{rcl} 120t\cos(\theta) & = & 420 \\ 120t\sin(\theta) - 16t^2 & = & -50 \end{array} \tag{189}$$

$$\Leftrightarrow \begin{array}{rl} \cos(\theta) = & \frac{7}{2t} \\ \sin(\theta) = & \frac{8t^2-25}{60t} \end{array} \tag{190}$$

$$\rightarrow 1 = \sin^2(\theta) + \cos^2(\theta) = \frac{49}{4t^2} + \frac{64t^4 - 400t^2 + 625}{3600t^2} \tag{191}$$

$$\rightarrow 3600t^2 = 49 \cdot 900 + 64t^4 - 400t^2 + 625 \tag{192}$$

$$\rightarrow 0 = 64t^4 - 4000t^2 + 44725 = 64(t^2)^2 - 4000(t^2) + 44725 \tag{193}$$

$$\rightarrow t^2 = \frac{4000 \pm \sqrt{4000^2 - 4 \cdot 64 \cdot 44725}}{2 \cdot 64} = \frac{125}{4} \pm \frac{15\sqrt{79}}{8} \tag{194}$$

$$\rightarrow t \in \{\pm\sqrt{\frac{125}{4} \pm \frac{15\sqrt{79}}{8}}\} \tag{195}$$

$$= \{\sqrt{\frac{125}{4} + \frac{15\sqrt{79}}{8}}, -\sqrt{\frac{125}{4} + \frac{15\sqrt{79}}{8}}, \sqrt{\frac{125}{4} - \frac{15\sqrt{79}}{8}}, -\sqrt{\frac{125}{4} - \frac{15\sqrt{79}}{8}}\}.$$

Due to the real world context of our problem, we only consider the positive values of t, of which there are only 2. Similarly, the real world context of our problem tells us that $0 \le \theta \le \frac{\pi}{2}$, so we may safely use the $\cos^{-1}$ and $\sin^{-1}$ to calculate θ without worrying about adjustments by π and such. We see that when $t = \sqrt{\frac{125}{4} + \frac{15\sqrt{79}}{8}}$ we have

$$\theta = \cos^{-1}(\frac{7}{6t}) \approx 1.401 \approx \boxed{80.3^\circ}, \tag{196}$$

and when $t = \sqrt{\frac{125}{4} - \frac{15\sqrt{79}}{8}}$ we have

$$\theta = \cos^{-1}(\frac{7}{6t}) \approx 1.260 \approx \boxed{72.2^\circ}. \tag{197}$$

Problem 2.2: Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.

(a) If an object moves on a trajectory with constant speed S over a time interval $a \leq t \leq b$, then the length of the trajectory is $S(b-a)$.

(b) The curves defined by

(198) $$\vec{r}(t) = \langle f(t), g(t)\rangle \text{ and } \vec{R}(t) = \langle g(t), f(t)\rangle$$

have the same length over the interval $[a,b]$.

(c) The curve $\vec{r}(t) = \langle f(t), g(t)\rangle$, for $0 \leq a \leq t \leq b$, and the curve $\vec{R}(t) = \langle f(t^2), g(t^2)\rangle$, for $\sqrt{a} \leq t \leq \sqrt{b}$, have the same length.

(d) The curve $\vec{r}(t) = \langle t, t^2, 3t^2\rangle$, for $1 \leq t \leq 4$, is parameterized by arclength.

Solution to (a): True. It is acceptable to say that this is intuitively obvious. For a more technical explanation, we observe that for a trajectory paramterized by $\vec{r}(t)$, $a \leq t \leq b$, we have $S = |\vec{v}(t)| = |\vec{r}\,'(t)|$. We now see that the length L of the trajectory is given by

(199) $$L = \int_a^b |\vec{r}\,'(t)|dt = \int_a^b S dt = St\Big|_a^b = S(b-a).$$

Solution to (b): True. We see that the graph of one of $\vec{r}(t)$ or $\vec{R}(t)$ can be obtained from the other by a reflection over the line $y = x$, and reflections preserve arclength, so the two curves have the same arclength.[7] We can also verify this algebraically by letting L_r and L_R denote the arclengths of $\vec{r}(t)$ and $\vec{R}(t)$ respectively and observing that

(200) $$L_r = \int_a^b |\vec{r}\,'(t)|dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt$$

(201) $$= \int_a^b \sqrt{(g'(t))^2 + (f'(t))^2}dt = \int_a^b |\vec{R}\,'(t)|dt = L_R.$$

[7] Technically it is not sufficient to only examine the graphs. $\vec{r}(t) = \langle\cos(t), \sin(t)\rangle, 0 \leq t \leq 2\pi$ and $\vec{R}(t) = \langle\cos(t), \sin(t)\rangle, 0 \leq t \leq 4\pi$ both have the unit circle as their graph, but $\vec{R}$ has twice as much arclength as $\vec{r}$ since it traverses the unit circle twice instead of once.

Solution to (c): True. Intuitively, $\vec{r}(t)$ and $\vec{R}(t) = \vec{r}(t^2)$ trace out the same curve when we use the given intervals, but at different speeds, so the total arclength of both curves is the same. We may also verify that the arclengths are the same through direct calculation. To this end, let L_r and L_R denote the arclengths of $\vec{r}(t)$ and $\vec{R}(t)$ respectively and observe that

$$L_R = \int_{\sqrt{a}}^{\sqrt{b}} |\vec{R}\,'(t)|dt = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{(2tf'(t^2))^2 + (2tg'(t^2))^2}dt \tag{202}$$

$$= \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{(f'(t^2))^2 + (g'(t^2))^2}(2tdt) \overset{u=t^2}{=} \int_a^b \sqrt{(f'(u))^2 + (g'(u))^2}du \tag{203}$$

$$= \int_a^b |\vec{r}\,'(t)|dt = L_r. \tag{204}$$

Solution to (d): False. We note that $\vec{r}\,'(t) = \langle 1, 2t, 6t \rangle$, so $|\vec{r}\,'(t)| = \sqrt{1 + 40t^2}$. The parameterization $\vec{r}(t)$ is a paramterization by arclength if and only if $|\vec{r}\,'(t)| = 1$ for every $1 \leq t \leq 4$, and this is clearly not the case.

Problem 2.3 Consider the curve $\mathcal{C}$ that is described by the parameterization $\vec{r}(t) = \langle t^m, t^m, t^{\frac{3}{2}m}\rangle$ where $0 \le a \le t \le b$ and $m \neq 0$.

(a) Find the arclength function $s(t)$. Note that your answer may include a, b, and m in it.

(b) Find the parameterization by arclength for $\mathcal{C}$ when $a = \sqrt{\frac{28}{9}}, b = 4$, and $m = 2$.

Solution to (a): Firstly, we observe that

(205) $$\vec{r}\,'(t) = \langle mt^{m-1}, mt^{m-1}, \frac{3}{2}mt^{\frac{3}{2}m-1}\rangle, \text{ so}$$

(206) $$\begin{aligned} |\vec{r}\,'(t)| &= \sqrt{(mt^{m-1})^2 + (mt^{m-1})^2 + (\tfrac{3}{2}mt^{\frac{3}{2}m-1})^2} \\ &= \sqrt{(2m^2t^{2m-2} + \tfrac{9}{4}m^2t^{3m-2}} \\ &= |m|t^{m-1}\sqrt{2 + \tfrac{9}{4}t^m}. \end{aligned}$$

It follows that for $a \le t \le b$ we have

(207) $$s(t) = \int_a^t |\vec{r}\,'(u)|du = \int_a^t |m|u^{m-1}\sqrt{2 + \frac{9}{4}u^m}du$$

(208) $$= \frac{|m|}{m}\int_a^t \sqrt{2 + \frac{9}{4}u^m}(mu^{m-1}du) \overset{v=u^m}{=} \frac{|m|}{m}\int_{u=a}^t \sqrt{2 + \frac{9}{4}v}dv$$

(209) $$= \frac{|m|}{m}\cdot\frac{8}{27}(2 + \frac{9}{4}v)^{\frac{3}{2}}\Big|_{u=a}^t = \frac{8|m|}{27m}(2 + \frac{9}{4}u^m)^{\frac{3}{2}}\Big|_a^t$$

(210) $$\boxed{\frac{8|m|}{27m}\left((2 + \frac{9}{4}t^m)^{\frac{3}{2}} - (2 + \frac{9}{4}a^m)^{\frac{3}{2}}\right)}.$$

Solution to (b): We begin by plugging $a = \sqrt{\frac{28}{9}}, b = 4$, and $m = 2$ into our answer from part (a) to see that

(211) $$s(t) = \frac{8}{27}(2 + \frac{9}{4}t^2)^{\frac{3}{2}} - 8.$$

To obtain the parameterization by arclength, we have to switch from finding arclength as a function of time to finding time as a function of arclength. To this end we see that

$$\begin{aligned} s = \tfrac{8}{27}(2+\tfrac{9}{4}t^2)^{\frac{3}{2}} - 8 &\Leftrightarrow \\ s + 8 = \tfrac{8}{27}(2+\tfrac{9}{4}t^2)^{\frac{3}{2}} &\Leftrightarrow \\ \tfrac{27}{8}s + 27 = (2+\tfrac{9}{4}t^2)^{\frac{3}{2}} &\Leftrightarrow \\ (\tfrac{27}{8}s + 27)^{\frac{2}{3}} = 2+\tfrac{9}{4}t^2 &\Leftrightarrow \\ (\tfrac{27}{8}s + 27)^{\frac{2}{3}} - 2 = \tfrac{9}{4}t^2 &\Leftrightarrow \\ \tfrac{4}{9}(\tfrac{27}{8}s + 27)^{\frac{2}{3}} - \tfrac{8}{9} = t^2 &\Leftrightarrow \\ \sqrt{(s+8)^{\frac{2}{3}} - \tfrac{8}{9}} = t. & \end{aligned} \tag{212}$$

It follows that the parameterization by arclength $\vec{r}(s)$, $0 \leq s \leq s(b)$, is given by

$$\vec{r}(s) = \vec{r}(t(s)) = \langle t(s)^m, t(s)^m, t(s)^{\frac{3}{2}m} \rangle \tag{213}$$

$$= \boxed{\left\langle ((s+8)^{\frac{2}{3}} - \frac{8}{9})^{\frac{m}{2}}, ((s+8)^{\frac{2}{3}} - \frac{8}{9})^{\frac{m}{2}}, ((s+8)^{\frac{2}{3}} - \frac{8}{9})^{\frac{3m}{4}}, \right\rangle} \tag{214}$$

Problem 2.4: Determine whether the following statements are true or false. If a statement is true, then explain why. If a statement is false, then provide a counterexample.

(a) The position, unit tangent, and principal unit normal vectors ($\vec{r}, \hat{T}$, and $\hat{N}$) at a point lie in the same plane.
(b) The vectors $\hat{T}$ and $\hat{N}$ at a point depend on the orientation of a curve.
(c) The curvature at a point depends on the orientation of a curve.
(d) An object with unit speed ($|\vec{v}| = 1$) on a circle of radius R has an acceleration of $\vec{a} = \frac{1}{R}\hat{N}$.
(e) If the speedometer of a car reads a constant 60 mi/hr, the car is not accelerating.
(f) A curve in the xy-plane that is concave up at all points has positive torsion.
(g) A curve with large curvature also has large torsion.

Solution to (a): False. To see a concrete counterexample, we simply consider $\vec{r}(t) = \langle \cos(t), \sin(t), 1\rangle$. Since $\hat{T}(t) = \vec{r}\,'(t) = \langle -\sin(t), \cos(t), 0\rangle$, we see that $\hat{N} = \langle -\cos(t), -\sin(t), 0\rangle$. Since $\hat{T}(t)$ and $\hat{N}(t)$ are always in the xy-plane, but $\vec{r}(t)$ is never in the xy-plane, we see that we have indeed produced a counterexample.

In fact, we can generalize the idea behind the previous counterexample. If $\vec{r}_2(t) = \vec{r}_1(t) + \vec{v}_0$ for some constant vector $\vec{v}_0$, then $\hat{T}_1 = \hat{T}_2$ and $\hat{N}_1 = \hat{N}_2$, so if $\vec{r}_1(t_0), \hat{T}_1(t_0), \hat{N}_1(t_0)$ are coplanar for some t_0, then we can take $\vec{v}_0$ to be any vector that is outside of the plane P of $\hat{T}_1(t_0)$ and $\hat{N}_1(t_0)$ so that $\vec{r}_2(t_0) = \vec{r}_1(t_0) + \vec{v}_0$ will be outside of P. Since P is still the plane of $\hat{T}_2(t_0)$ and $\hat{N}_2(t_0)$, we see that $\vec{r}_2(t_0), \hat{T}_2(t_0)$, and $\hat{N}_2(t_0)$ are not coplanar. What we have essentially done is use the fact that $\hat{T}$ and $\hat{N}$ don't change as a result of a translation, so we can use a translation to ensure that $\vec{r}(t_0), \hat{T}(t_0)$, and $\hat{N}(t_0)$ are not coplanar at a given point t_0 regardless of the initial curve $\vec{r}(t)$.

Solution to (b): True. Consider the curves $y = x^2$ and $x = y^2$. The unit tangent vector to $y = x^2$ at the point $(0, 0)$ is either $(1, 0)$ or $(-1, 0)$ depending on the parameterization that is used. Similarly, the unit tange vector to $x = y^2$ at $(0, 0)$ is either $(0, 1)$ or $(0, -1)$ depending on the parameterization that is used. We see that the cure $x = y^2$ is the curve $y = x^2$ rotated 90° clockwise, and that this will result in the unit tangent vector at $(0,0)$ being rotated by $90°$ as well (if the correct parameterization of $x = y^2$ and $y = x^2$ are used).

Solution to (c): False. The curvature measures the rate at which the unit tangent vector $\hat{T}$ changes direction. If you alter a curves orientation via translations and rotations, then the translations do not affect the unit tangent vectors, but the rotations will rotate all of the unit tangent vectors as well. Since the same rotation is being applied to all of the unit tangent vectors, the rate at which their directions change will remain the same.

Solution to (d): True. We present two verifications of this claim. For the first verification, we use the tangential and normal components of acceleration. In particular, we recall that

(215) $$\vec{a}(t) = \frac{d^2s}{dt^2}\hat{T} + \kappa\left(\frac{ds}{dt}\right)^2\hat{N}.$$

Since $\frac{ds}{dt} = |\vec{v}(t)| = 1$, we see that $\frac{d^2s}{dt^2} = 0$. Since a circle of radius R has curvature $\kappa = \frac{1}{R}$, we see that $\vec{a}(t) = \kappa\hat{N} = \frac{1}{R}\hat{N}(t)$.

Our next verification will be a direct calculation of $\vec{a}(t)$. Since the acceleration does not depend on where the circle is cetered, we may assume that we are working with a circle centered at $(0,0)$ in the xy-plane. Since $|\vec{v}(t)| = 1$, we are working with a paramaterization by arclength of our circle of radius R, which is given by

(216) $$\vec{r}(t) = \langle R\cos(\frac{t}{R}), R\sin(\frac{t}{R})\rangle, 0 \leq t \leq 2\pi R.$$

We see that

(217) $$\hat{T}(t) = \vec{v}(t) = \langle -\sin(\frac{t}{R}), \cos(\frac{t}{R})\rangle \text{ and } \vec{a}(t) = \langle -\frac{1}{R}\cos(\frac{t}{R}), -\frac{1}{R}\sin(\frac{t}{R})\rangle.$$

Since $|\hat{T}'(t)| = |\vec{a}(t)| = \frac{1}{R}$, we see that

(218) $$\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} = \langle -\cos(\frac{t}{R}), -\sin(\frac{t}{R})\rangle,$$

so we do indeed have that $\vec{a}(t) = \frac{1}{R}\hat{N}(t)$.

Solution to (e): False. While the speed of the car is not changing, the car could still be changing its direction of motion, which would mean that the car is accelerating. This is easily seen to be the case if the cars trajectory is modeled by $\vec{r}(t) = \langle 60\cos(t), 60\sin(t)\rangle$ since $\vec{v}(t) = \langle -60\sin(t), 60\cos(t)\rangle$ satisfies $|\vec{v}(t)| = 60$ and $\vec{a}(t) = \langle -60\cos(t), -60\sin(t)\rangle \neq \vec{0}$.

Solution to (f): False. The torsion at a point $\vec{r}(t_0)$ measures the rate at which the curve $\vec{r}(t)$ twists out of the plane determined by $\hat{T}(t_0)$ and $\hat{N}(t_0)$. If the curve $\vec{r}(t)$ is contained in the xy-plane (regardless of whether it is concave up or even convex) then $\hat{T}(t_0)$ and $\hat{N}(t_0)$ will be contained in the xy-plane as well for every t_0, so the torision will always be 0 since the curve does not twist out of the xy-plane at all.

Solution to (g): False. We recall that a circle in the xy-plane (or any other plane) of radius r has a curvature of $\kappa = \frac{1}{r}$. We already saw in part (f) that any such circle has 0 torsion at all points, regardless of the radius r. As r gets closer to 0, κ grows larger without bound, but the torsion is always 0.

Problem 2.5: Compute the unit binormal vector $\hat{B}$ and torsion τ of the curve parameterized by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), -t\rangle, t \in \mathbb{R}(-\infty < t < \infty)$.

Solution: Since $\hat{B} = \hat{T} \times \hat{N}$ and $\tau = -\frac{d\hat{B}}{ds} \cdot \hat{N} = -\frac{1}{|\vec{v}(t)|}\frac{d\hat{B}}{dt} \cdot \hat{N}$, we see that we should begin by calculating $\hat{T}(t)$ and $\hat{N}(t)$. To this end, we see that

$$\vec{r}\,'(t) = \langle -2\sin(t), 2\cos(t), -1\rangle \tag{219}$$

$$\to |\vec{r}\,'(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2 + (-1)^2} = \sqrt{5}. \tag{220}$$

$$\to \hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\frac{2}{\sqrt{5}}\sin(t), \frac{2}{\sqrt{5}}\cos(t), -\frac{1}{\sqrt{5}}\rangle. \tag{221}$$

Recalling that $\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|}$, we see that

$$\hat{T}'(t) = \langle -\frac{2}{\sqrt{5}}\cos(t), -\frac{2}{\sqrt{5}}\sin(t), 0\rangle \tag{222}$$

$$\to |\hat{T}'(t)| = \sqrt{(-\frac{2}{\sqrt{5}}\cos(t))^2 + (-\frac{2}{\sqrt{5}}\sin(t))^2 + 0^2} = \frac{2}{\sqrt{5}} \tag{223}$$

$$\to \hat{N}(t) = \frac{\langle -\frac{2}{\sqrt{5}}\cos(t), -\frac{2}{\sqrt{5}}\sin(t), 0\rangle}{\frac{2}{\sqrt{5}}} = \langle -\cos(t), -\sin(t), 0\rangle. \tag{224}$$

$$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{2}{\sqrt{5}}\sin(t) & \frac{2}{\sqrt{5}}\cos(t) & -\frac{1}{\sqrt{5}} \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix} \tag{225}$$

$$= \hat{i}(\frac{2}{\sqrt{5}}\cos(t)\cdot 0-(-\sin(t))\cdot(-\frac{1}{\sqrt{5}})) \tag{226}$$
$$-\hat{j}((-\frac{2}{\sqrt{5}}\sin(t))\cdot 0-(-\cos(t))\cdot(-\frac{1}{\sqrt{5}}))$$
$$+\hat{k}((-\frac{2}{\sqrt{5}}\sin(t))\cdot(-\sin(t))-(-\cos(t))\cdot\frac{2}{\sqrt{5}}\cos(t))$$

. . . .

$$= -\frac{1}{\sqrt{5}}\sin(t)\hat{i}+\frac{1}{\sqrt{5}}\cos(t)\hat{j}+\frac{2}{\sqrt{5}}(\sin^2(t)+\cos^2(t))\hat{k} \tag{227}$$

. . . .

$$= \boxed{\langle -\frac{1}{\sqrt{5}}\sin(t), \frac{1}{\sqrt{5}}\cos(t), \frac{2}{\sqrt{5}}\rangle}. \tag{228}$$

Recalling that $|\vec{v}(t)| = |\vec{r}\,'(t)|$ we see that

$$\tau = \tau(t) = -\frac{1}{|\vec{v}(t)|}\frac{d\hat{B}}{dt}\cdot\hat{N} \tag{229}$$

$$= -\frac{1}{\sqrt{5}}\langle -\frac{1}{\sqrt{5}}\cos(t), -\frac{1}{\sqrt{5}}\sin(t), 0\rangle\cdot\langle -\cos(t), -\sin(t), 0\rangle = \boxed{-\frac{1}{5}} \tag{230}$$

Problem 2.7: Verify that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y} = 1. \tag{231}$$

Solution: We begin by reviewing one of the sum to product trigonometric identities. Observe that

$$\boxed{\sin(x)+\sin(y)} = \sin(\frac{x+y}{2}+\frac{x-y}{2}) + \sin(\frac{x+y}{2}-\frac{x-y}{2}) \tag{232}$$

$$= \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) + \sin(\frac{x+y}{2})\cos(-\frac{x-y}{2}) + \sin(-\frac{x-y}{2})\cos(\frac{x+y}{2}) \tag{233}$$

$$= \sin(\frac{x+y}{2})\cos(\frac{x-y}{2}) + \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) + \sin(\frac{x+y}{2})\cos(+\frac{x-y}{2}) - \sin(\frac{x-y}{2})\cos(\frac{x+y}{2}) \tag{234}$$

$$= \boxed{2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})}. \tag{235}$$

Recalling that

$$\lim_{x\to0} \frac{\sin(x)}{x} = 1, \tag{236}$$

we let $z = \frac{x+y}{2}$ and see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x)+\sin(y)}{x+y} = \lim_{(x,y)\to(0,0)} \frac{2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})}{x+y} \tag{237}$$

$$\left(\lim_{(x,y)\to(0,0)} \frac{\sin(\frac{x+y}{2})}{\frac{x+y}{2}}\right)\left(\lim_{(x,y)\to(0,0)} \cos(\frac{x-y}{2})\right) \tag{238}$$

. . . .

$$\left(\lim_{z\to 0} \frac{\sin(z)}{z}\right)\left(\lim_{(x,y)\to(0,0)} \cos(\frac{x-y}{2})\right) = (1)(\cos(\frac{0-0}{2})) = 1. \tag{239}$$

Problem 2.8: Consider the function

$$f(x,y) = \frac{xy^2}{x^2+y^4}. \tag{240}$$

(a) Show that if L is a line that passes through the origin, then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = 0. \tag{241}$$

(b) Show that

$$\lim_{(x,y)\to(0,0)} f(x,y) \tag{242}$$

does not exist.

Solution to (a): Firstly, we see that if L is a line of the form $y = mx$ for some $m \in \mathbb{R}$, then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{x\to0} f(x,mx) = \lim_{x\to0}\frac{x(mx)^2}{x^2+(mx)^4} \tag{243}$$

$$= \lim_{x\to0}\frac{m^2x^3}{x^2+m^4x^4} = \lim_{x\to0}\frac{m^2x}{1+m^4x^2} = 0. \tag{244}$$

The only line L left to consider is the line through the origin with infinite slope, which is just the line $x = 0$. In this case we see that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{y\to0} f(0,y) = \lim_{y\to0}\frac{0\cdot y^2}{0^2+y^4} = 0. \tag{245}$$

Solution to (b): In order to show that the limit in equation (242) does not exist we need to use the 2 path test. Based on part (a), we see that our second path needs cannot be a line. Thankfully, we only need to find a path P that results in any nonzero value when the limit is taken along P. If we try the parabolic path $y = x^2$, then we again get a value of 0 for the limit, but if we try the path $x = y^2$ then we get a value of $\frac{1}{2}$! In fact, we see that for $m \in \mathbb{R}$ and the path P_m given by $x = my^2$ we have

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f(x,y) = \lim_{y\to 0} f(my^2, y) = \lim_{y\to 0} \frac{(my^2)y^2}{(my^2)^2 + y^4} \tag{246}$$

$$= \lim_{y\to 0} \frac{my^4}{m^2y^4 + y^4} = \lim_{y\to 0} \frac{m}{m^2+1}. \tag{247}$$

Since the range of the function $g(m) = \frac{m}{m^2+1}$ is $[-1, 1]$, we see that the limit can take on any value between -1 and 1 if the correct path is chosen. While we only need 2 paths that result in different values to apply the 2 path test, it is amusing to see that we have found infinitely many paths that result in infinitely many different values.

Problem 2.9: Consider the function $f(x,y) = \sqrt{|xy|}$.

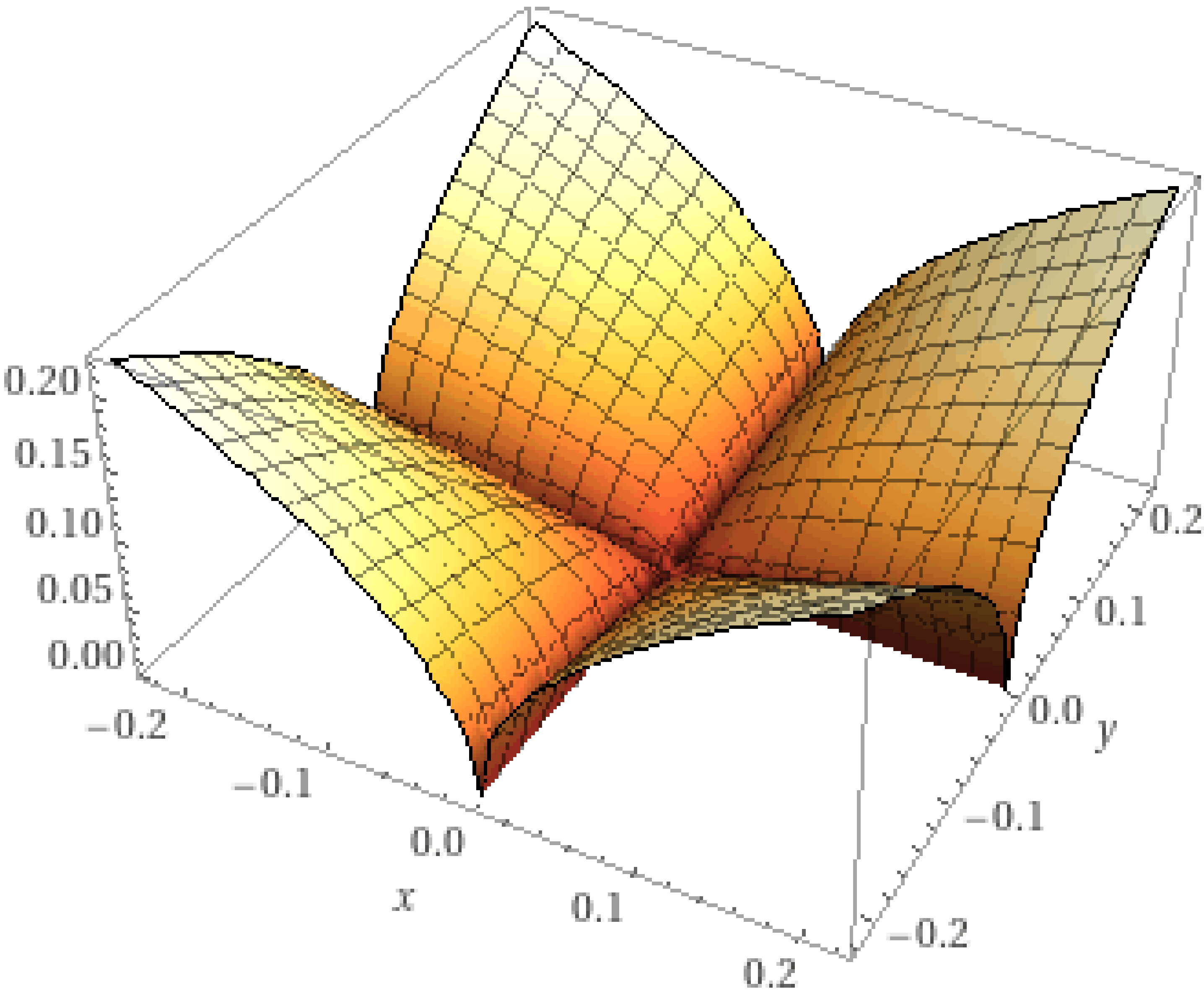


FIGURE 10. A graph of $z = \sqrt{|xy|}$.

(a) Is f continuous at $(0,0)$?
(b) Show that $f_x(0,0)$ and $f_y(0,0)$ exist by calculating their values.
(c) Determine whether f_x and f_y are continuous at $(0,0)$.
(d) Is f differentiable at $(0,0)$?

Solution to (a): Yes. We will show that $f(x,y)$ is continuous on all of $\mathbb{R}^2$. The function $f_1(x,y) = xy$ is a continuous function since it is a polynomial function. The function $f_2(x) = |x|$ is also a continuous function, and the composition of continuous functions is again continuous, so we see that $f_3(x,y) := f_2(f_1(x,y)) = |xy|$ is a continuous function. Since $|xy|$ only takes on nonnegative values and the function $f_4(x) = \sqrt{x}$ is continuous on the domain $[0, \infty)$, we see that $f(x, y) = f_4(f_3(x, y))$ is indeed a continuous function.

Solution to (b): We see that

$$f_y(0,0) = \lim_{y\to 0}\frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0}\frac{\sqrt{|0\cdot y|} - \sqrt{|0\cdot 0|}}{y} = 0, \text{ and} \tag{248}$$

$$f_x(0,0) = \lim_{x\to 0}\frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0}\frac{\sqrt{|x\cdot 0|} - \sqrt{|0\cdot 0|}}{x} = 0. \tag{249}$$

Solution to (c): We will show that neither of f_x and f_y are continuous at $(0, 0)$. We note that for all $x, y > 0$ we have $f(x, y) = \sqrt{|xy|} = \sqrt{xy}$. It follows that for $x, y > 0$ we have

$$f_x(x,y) = \frac{\partial}{\partial x}((xy)^{\frac{1}{2}}) = \frac{1}{2}(xy)^{-\frac{1}{2}}\cdot y = \frac{1}{2}\sqrt{\frac{y}{x}}, \text{ and} \tag{250}$$

$$f_y(x,y) = \frac{\partial}{\partial y}((xy)^{\frac{1}{2}}) = \frac{1}{2}(xy)^{-\frac{1}{2}}\cdot x = \frac{1}{2}\sqrt{\frac{x}{y}}. \tag{251}$$

We now use the 2 path test to show that neither function is continuous. Let us consider the path P_m given by $y = mx$ with $m, x > 0$ so that the path lies in the first quadrant. We see that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f_x(x,y) = \lim_{x\to 0^+} f_x(x,mx) = \lim_{x\to 0^+}\frac{1}{2}\sqrt{\frac{mx}{x}} = \frac{m}{2}, \text{ and} \tag{252}$$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in P_m}} f_y(x,y) = \lim_{x\to 0^+} f_y(x,mx) = \lim_{x\to 0^+}\frac{1}{2}\sqrt{\frac{x}{mx}} = \frac{1}{2m}. \tag{253}$$

We see that the paths P_1 and P_2 result in the values of $\frac{1}{2}$ and 1 respectively for the value of $f_x(x, y)$ as (x, y) approaches $(0, 0)$, so f_x is not continuous at $(0, 0)$. Similarly, we see that the paths P_1 and P_2 result in the values of $\frac{1}{2}$ and $\frac{1}{4}$ respectively for the value of $f_y(x, y)$ as (x, y) approaches $(0, 0)$, so f_y is not continuous at $(0, 0)$.

Solution to (d): $\boxed{\text{No.}}$ We begin by examining the directional derivative in the direction of the vector $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ at $(0, 0)$. We see that

$$D_{\hat{u}} f(0,0) = \lim_{t \to 0} \frac{f\left((0,0) + t\hat{u}\right) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}) - 0}{t} \tag{254}$$

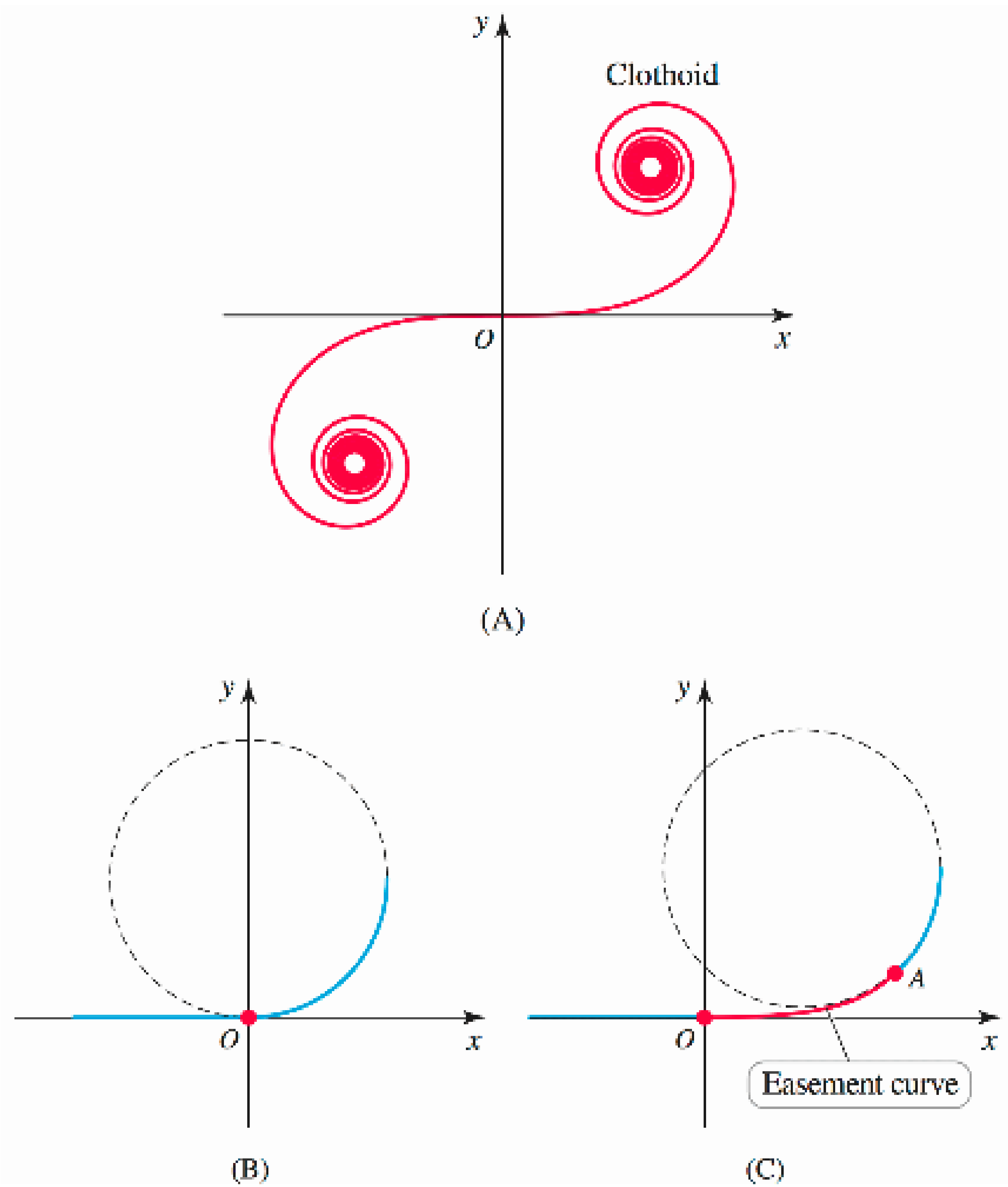
$$= \lim_{t \to 0} \frac{\sqrt{|\frac{t^2}{2}|}}{t} = \lim_{t \to 0} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}. \tag{255}$$

If f was differentiable at $(0, 0)$, then we would have

$$D_{\hat{u}} f(0,0) = \nabla f(0,0) \cdot \hat{u} = \langle f_x(0,0), f_y(0,0) \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = 0. \tag{256}$$

Since this is not the case, we see that f is not differentiable at $(0, 0)$.

Problem 2.6: The function $\vec{r}(t) = \langle \int_0^t \cos(\frac{1}{2}u^2)du, \int_0^t \sin(\frac{1}{2}u^2)du\rangle, t \in \mathbb{R}$ whose graph is called a **clothoid** or **Euler Spiral**, has applications in the design of railroad tracks, rollercoasters, and highways.



(a) A car moves from left to right on a straight highway, approaching a curve at the origin (Figure B). Sudden changes in curvature at the start of the curve may cause the driver to jerk the steering wheel. Suppose the curve starting at the origin is a segment of a circle of radius a. Explain why there is a sudden change in the curvature of the road at the origin.

(b) A better approach is to use a segment of a clothoid as an easement curve, in between the straight highway and a circle, to avoid sudden changes in curvature (Figure C). Assume the easement curve corresponds to the clothoid $\vec{r}(t)$, for $0 \leq t \leq 1.2$. Find the curvature of the easement curve as a function of t and explain why this curve eliminates the sudden change in curvature at the origin.

(c) Find the radius of a circle connected to the easement curve at point A (that corresponds to $t = 1.2$ on the curve $\vec{r}(t)$) so that the curvature of the circle matches the curvature of the easement curve at point A.

Solution to (a): We recall that straight lines have a curvature of 0 at every point and circles of radius a has a curvature of $\frac{1}{a}$ at every point.[8] Since $\frac{1}{a} \neq 0$ we see that there is a change in curvature when the line segment converts into a circular arc. Since the curvature κ is given by $\kappa = |\frac{d\hat{T}}{ds}|$, we recall that curvature tells us how quickly our path is changing direction. The driver on the curve in this part will notice that change in curvature as a jerk in their driving since the direction in which they are driving will change sharply instead of smoothly.

Solution to (b): Since $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \frac{1}{|\vec{v}(t)|}\left|\frac{d\hat{T}}{dt}\right|$, we begin by calculating $\hat{T}(t)$. To this end, we see that

$$\vec{r}\,'(t) = \langle\frac{d}{dt}\int_0^t \cos(\frac{1}{2}u^2)du, \frac{d}{dt}\int_0^t \sin(\frac{1}{2}u^2)du\rangle = \langle\cos(\frac{1}{2}t^2), \sin(\frac{1}{2}t^2)\rangle. \tag{257}$$

$$|\vec{v}(t)| = |\vec{r}\,'(t)| = \sqrt{\cos^2(\frac{1}{2}t^2) + \sin^2(\frac{1}{2}t^2)} = 1. \tag{258}$$

Conveniently, we see that the parameterization for the clothoid that we were given happened to be a parameterization by arclength, so we may interchange s and t in this situation. We now see that

[8] We see that this is one of many instances in math in which it is useful to imagine that a line is just a circle of infinite radius.

(259) $$\kappa(t) = \kappa(s) = \left|\frac{\hat{T}(s)}{ds}\right| = |\langle -t\sin(\frac{1}{2}t^2), t\cos(\frac{1}{2}t^2)\rangle|$$

(260) $$= \sqrt{(-t\sin(\frac{1}{2}t^2))^2 + (t\cos(\frac{1}{2}t^2)^2} = \sqrt{t^2(\cos^2(\frac{1}{2}t^2) + \sin^2(\frac{1}{2}t^2))} = \boxed{t}.$$

Since $\kappa(0) = 0$, we see that the clothoid and the linesegment have the same curvature at the point of transition, so the driver will not notice any jerking when switching from one part of the highway to the next.

Solution to (c): We were reminded in part (a) that a circle of radius r has a curvature of $\kappa = \frac{1}{r}$. It is clear from the preceding formula that $r = \frac{1}{\kappa}$, so the radius of a circle can also be determined using only its curvature. We see that $\kappa(1.2) = 1.2 = \frac{6}{5}$, so we want to use a circle with radius $r = \frac{1}{\frac{6}{5}} = \boxed{\frac{5}{6}}$ in order for the curvature of the circle to align with the curvature of the clothoid at $t = 1.2$ to ensure another smooth transition.

Problem 1.6: Find an equation of the plane P through the points $\mathrm{R}(5,3,7)$, $\mathrm{S}(0,1,0)$, and $\mathrm{T}(1,2,1)$.

Solution: It will be relatively easy to find the equation of the plane P if we first find a vector $\vec{n}$ that is normal to P. To find such a $\vec{n}$ it suffices to take the cross product of any two nonparallel vectors lying in P. To this end, we see that

$$\overrightarrow{\mathrm{SR}} = \langle 5,3,7\rangle - \langle 0,1,0\rangle = \langle 5,2,7\rangle, \text{ and} \tag{261}$$

$$\overrightarrow{\mathrm{ST}} = \langle 1,2,1\rangle - \langle 0,1,0\rangle = \langle 1,1,1\rangle \tag{262}$$

are two nonparallel vectors lying in P. We now take

$$\vec{n} = \overrightarrow{\mathrm{SR}} \times \overrightarrow{\mathrm{ST}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 2 & 7 \\ 1 & 1 & 1 \end{vmatrix} \tag{263}$$

$$\hat{i}(2\cdot 1 - 1\cdot 7) - \hat{j}(5\cdot 1 - 1\cdot 7) + \hat{k}(5\cdot 1 - 1\cdot 2) \tag{264}$$

$$= -5\hat{i} + 2\hat{j} + 3\hat{k} = \langle -5,2,3\rangle. \tag{265}$$

To derive the equation of the plane P we recall that $\vec{n}$ is perpendicular to any vector that lies in P. It follows that if (x,y,z) is an arbitrary point in P, then since $(0,1,0)$ is also a point in P the vector

$$\vec{v} := \langle x,y,z\rangle - \langle 0,1,0\rangle = \langle x, y-1, z\rangle \tag{266}$$

is a vector contained in P, so we have

$$0 = \vec{n}\cdot\vec{v} = \langle -5,2,3\rangle \cdot \langle x,y-1,z\rangle = -5x + 2y - 2 + 3z \tag{267}$$

$$\to \boxed{2 = -5x + 2y + 3z}. \tag{268}$$

Remark: We can easily check our answer by verifying that R, S, and T all satisfy equation (268). Furthermore, we see that if we replace $\vec{n}$ by $c\vec{n}$ for any nonzero constant c, then $c\vec{n}$ will still be normal to P, which will result in seemingly different equations for P such as

$$-2 = 5x - 2y - 3z \tag{269}$$

when $c = -1$. In particular the order of the cross product in equation (263) and the order of the subtraction in equations (261) and (262) don't matter since the end result will at worst alter $\vec{n}$ by a negative sign.

3. Solutions

Problem 1.14: The electric field due to a point charge of strength Q at the origin has a potential function $V(x,y,z) = kQ/r$, where $r^2 = x^2+y^2+z^2$ is the square of the distance between a variable point $P(x,y,z)$ at the charge, and $k > 0$ is a physical constant. The electric field is given by $\mathbf{E}(x,y,z) = -\nabla V(x,y,z)$.

(a) Show that

$$\mathbf{E}(x,y,z) = kQ\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle. \tag{270}$$

(b) Show that $|\mathbf{E}| = kQ/r^2$. Explain why this relationship is called the inverse square law.

Solution to (a): We note that since r represents a distance, r is a nonnegative number, so

$$r = (x^2+y^2+z^2)^{\frac{1}{2}} \text{ (and not } -(x^2+y^2+z^2)^{\frac{1}{2}}). \tag{271}$$

It follows that

$$V(x,y,z) = kQ(x^2+y^2+z^2)^{-\frac{1}{2}} \rightarrow \tag{272}$$

$$\begin{aligned} V_x(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial x}(x^2+y^2+z^2) \\ &= -kQx(x^2+y^2+z^2)^{-\frac{3}{2}} & = -kQxr^{-3} \\ V_y(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial y}(x^2+y^2+z^2) \\ &= -kQy(x^2+y^2+z^2)^{-\frac{3}{2}} & = -kQyr^{-3} \\ V_z(x,y,z) &= -\tfrac{1}{2}(kQ(x^2+y^2+z^2)^{-\frac{3}{2}})\tfrac{\partial}{\partial z}(x^2+y^2+z^2) \\ &= -kQz(x^2+y^2+z^2)^{-\frac{3}{2}} & = -kQzr^{-3} \end{aligned} \tag{273}$$

It is now clear that

$$\mathbf{E}(x,y,z) = -\nabla V(x,y,z) = -\langle V_x, V_y, V_z\rangle \tag{274}$$

(275) $$= -\langle -kQxr^{-3}, -kQyr^{-3}, -kQzr^{-3}\rangle = kQ\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle.$$

Solution to (b): We see that

(276)
$$|\mathbf{E}| = |kQ\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle| = kQ|\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle| = kQ\left((\frac{x}{r^3})^2 + (\frac{y}{r^3})^2 + (\frac{z}{r^3})^2\right)^{\frac{1}{2}}$$

(277) $$= kQ\left(\frac{x^2+y^2+z^2}{r^6}\right)^{\frac{1}{2}} = kQ\left(\frac{r^2}{r^6}\right)^{\frac{1}{2}} = kQ\left(\frac{1}{r^4}\right)^{\frac{1}{2}} = \frac{kQ}{r^2}.$$

The fact that $|\mathbf{E}| = \frac{kQ}{r^2}$ is known as the inverse square law because the magnitude of the electric field $\mathbf{E}$ is proportional to the inverse of the square (or the square of the inverse) of the distance r.

Problem 1.15: Consider the function $F(x,y,z) = e^{xyz}$.

(a) Write F as a composite function $f \circ g$, where f is a function of one variable and g is a function of three variables.

(b) Calculate $\nabla F(x,y,z)$ as well as $\nabla g(x,y,z)$. Find a relationship between $\nabla F(x,y,z)$ and $\nabla g(x,y,z)$.

Solution to (a): Letting $f(t) = e^t$ and $g(x,y,z) = xyz$, we see that $F(x,y,z) = f(g(x,y,z))$, so $F = f \circ g$.

Solution to (b): We see that

$$\begin{aligned} F_x(x,y,z) &= e^{xyz}\frac{\partial}{\partial x}(xyz) = yze^{xyz} \\ F_y(x,y,z) &= e^{xyz}\frac{\partial}{\partial y}(xyz) = xze^{xyz} \\ F_z(x,y,z) &= e^{xyz}\frac{\partial}{\partial z}(xyz) = xye^{xyz} \end{aligned} \tag{278}$$

$$\to \nabla F(x,y,z) = \langle F_x(x,y,z), F_y(x,y,z), F_z(x,y,a)\rangle \tag{279}$$

$$= yze^{xyz}, xze^{xyz}, xye^{xyz}\rangle = e^{xyz}\langle yz, xz, xy\rangle, \text{ and} \tag{280}$$

$$\begin{aligned} g_x(x,y,z) &= yz \\ g_y(x,y,z) &= xz \\ g_z(x,y,z) &= xy \end{aligned} \tag{281}$$

$$\to \nabla g(x,y,z) = \langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z)\rangle = \langle yz, xz, xy\rangle. \tag{282}$$

We now see that

$$\nabla F(x,y,z) = f(g(x,y,z))\nabla g(x,y,z) = F(x,y,z)\nabla g(x,y,z). \tag{283}$$

However, this is purely a coincidence. We will see later on that if if F, f, and g are functions for which $F = f \circ g$, then

$$\boxed{\nabla F(x,y,z) = f'(g(x,y,z))\nabla g(x,y,z)}. \tag{284}$$

Problem 1.16: Consider the function $f(x,y) = \ln(1+4x^2+3y^2)$ and the point $P = (\frac{3}{4}, -\sqrt{3})$.

(a) Find the gradient field $\nabla f(x,y)$ of $f(x,y)$ and then evaluate it at P.
(b) Find the angles θ (with respect to the x-axis) associated with the directions of maximum increase, maximum decrease, and zero change.
(c) Write the directional derivative at P as a function of θ; call this function $g(\theta)$.
(d) Find the value of θ that maximizes $g(\theta)$ and find the maximum value.
(e) Verify that the value of θ that maximizes g corresponds to the direction of the gradient vector at P. Verify that the maximum value of g equals the magnitude of the gradient vector at P.

Solution to (a): We see that

$$\begin{aligned} f_x(x,y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial x}(1+4x^2+3y^2) = \tfrac{8x}{1+4x^2+3y^2} \\ f_y(x,y) &= \tfrac{1}{1+4x^2+3y^2}\tfrac{\partial}{\partial y}(1+4x^2+3y^2) = \tfrac{6y}{1+4x^2+3y^2} \end{aligned} \tag{285}$$

$$\to \nabla f(x,y) = \langle \frac{8x}{1+4x^2+3y^2}, \frac{6y}{1+4x^2+3y^2} \rangle. \tag{286}$$

$$\nabla f(\frac{3}{4}, -\sqrt{3}) = \langle \frac{6}{1+\frac{9}{4}+9}, \frac{-6\sqrt{3}}{1+\frac{9}{4}+9} \rangle = \boxed{\langle \frac{24}{49}, \frac{-24\sqrt{3}}{49} \rangle}. \tag{287}$$

Solution to (b): We recall that $\nabla f(P)$ points in the direction of maximum increase from P. Since $\nabla f(P)$ is in the fourth quadrant, we see that

$$\theta_{\max} = \tan^{-1}(\frac{\frac{-24\sqrt{3}}{49}}{\frac{24}{49}}) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}. \tag{288}$$

is the angle associated with the direction of maximum increase. Since $-\nabla f(P)$ points in the direction of maximum decrease from P, we see that $\theta_{\min} = \theta_{\max} + \pi = \frac{2\pi}{3}$ is the angle associated with the direction of maximum decrease. Since the directions of no change are orthogonal to $\nabla f(P)$ (and to $-\nabla f(P)$), we see that $\theta_1 = \theta_{\max} + \frac{\pi}{2} = \frac{5\pi}{6}$ and $\theta_2 = \theta_{\max} - \frac{\pi}{2} = -\frac{\pi}{6}$ are the angles associated to the directions of zero change.

Solution to (c): We recall that $\vec{u}(\theta) = \langle \cos(\theta), \sin(\theta) \rangle$ is the unit vector associated with the angle θ. We also recall that for any unit vector $\vec{u}$, we have that

$$d_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}, \text{ so} \tag{289}$$

$$g(\theta) = d_{\vec{u}(\theta)} f(P) = \nabla f(P) \cdot \vec{u}(\theta) = \langle \frac{24}{49}, \frac{-24\sqrt{3}}{49} \rangle \cdot \langle \cos(\theta), \sin(\theta) \rangle \tag{290}$$

$$= \boxed{\frac{24}{49}\cos(\theta) - \frac{24\sqrt{3}}{49}\sin(\theta)}. \tag{291}$$

Solution to (d): We see that

$$g'(\theta) = -\frac{24}{49}\sin(\theta) - \frac{24\sqrt{3}}{49}\cos(\theta) \rightarrow \tag{292}$$

$$g'(\theta) = 0 \Leftrightarrow -\frac{24}{49}\sin(\theta) = \frac{24\sqrt{3}}{49}\cos(\theta) \Leftrightarrow \tan(\theta) = -\sqrt{3} \Leftrightarrow \tag{293}$$

$$\theta = -\frac{\pi}{3}, \frac{2\pi}{3} \tag{294}$$

We see that

$$g''(\theta) = -\frac{24}{49}\cos(\theta) + \frac{24\sqrt{3}}{49}\sin(\theta) \tag{295}$$

$$\rightarrow g''(-\frac{\pi}{3}) = -\frac{24}{49}\cos(-\frac{\pi}{3}) + \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = -\frac{48}{89} < 0. \tag{296}$$

The second derivative test shows us that $g(\theta)$ has a local maximum at $\theta = -\frac{\pi}{3}$.

$$g(-\frac{\pi}{3}) = \frac{24}{49}\cos(-\frac{\pi}{3}) - \frac{24\sqrt{3}}{49}\sin(-\frac{\pi}{3}) = \frac{48}{49}. \tag{297}$$

we see that g attains its maximum value of $\frac{48}{49}$ on $[0, 2\pi]$ at $\theta = -\frac{\pi}{3}$.

Solution to (e): From parts b and d we have already seen that the value of θ that maximizes $g(\theta)$ is the same as the angle θ associated with the direction of maximum increase. To finish, we just note that

$$|\nabla f(\frac{3}{4}, -\sqrt{3})| = |\langle\frac{24}{49}, \frac{-24\sqrt{3}}{49}\rangle| = \frac{24}{49}|\langle 1, -\sqrt{3}\rangle| \tag{298}$$

$$= \frac{24}{49}\sqrt{1^2 + (-\sqrt{3})^2} = \frac{48}{49}. \tag{299}$$

Problem 1.17: Find the gradient field $\vec{F} = \nabla\varphi$ for the potential function

$$\varphi(x,y) = \sqrt{x^2+y^2}, \quad \text{for } x^2+y^2 \leq 9, (x,y) \neq (0,0). \tag{300}$$

Sketch two level curves of φ and two vectors of $\vec{F}$ of your choice.

Solution: Firstly, we see that

$$\vec{F} = \nabla\varphi = \langle \varphi_x, \varphi_y \rangle = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle. \tag{301}$$

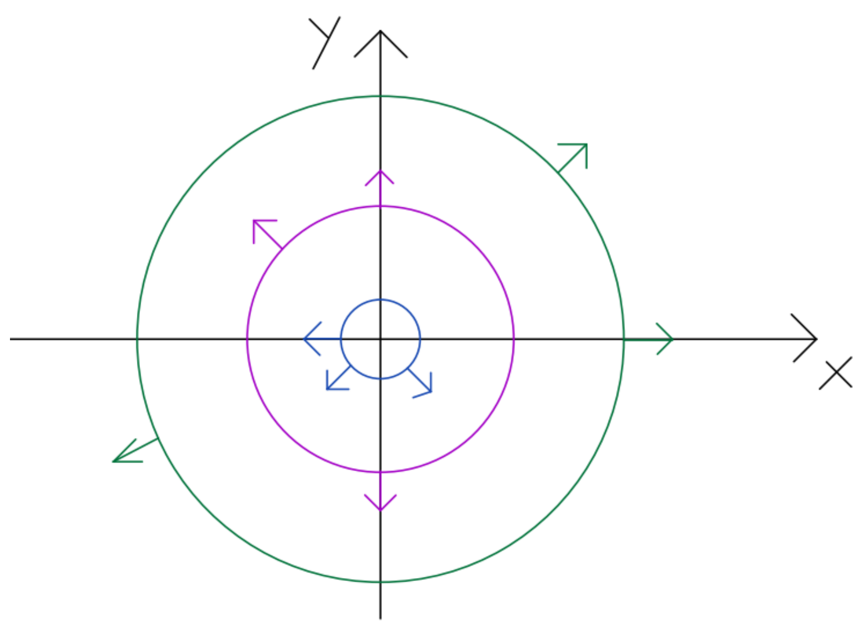
Next, we recall that the level curves of ϕ are the curves of the form $\phi(x,y) = c$ for some constant c. We see that

$$\phi(x,y) = c \Leftrightarrow \sqrt{x^2+y^2} = c \Leftrightarrow x^2+y^2 = c^2, \tag{302}$$

so the level curves of ϕ are circles centered at the origin. We recall that at a given point (x,y) the vector $\nabla\varphi(x,y)$ is perpendicular to the level curve that passes through (x,y), and we also observe that for any (x,y) we have

$$|\nabla\varphi(x,y)| = \sqrt{(\frac{x}{\sqrt{x^2+y^2}})^2 + (\frac{y}{\sqrt{x^2+y^2}})^2} = 1, \tag{303}$$

so we obtain the sketch below of some vectors from the gradient field and some level curves.



Problem 1.18: Below is a contour plot of some function $z = f(x, y)$ along with 4 vectors.

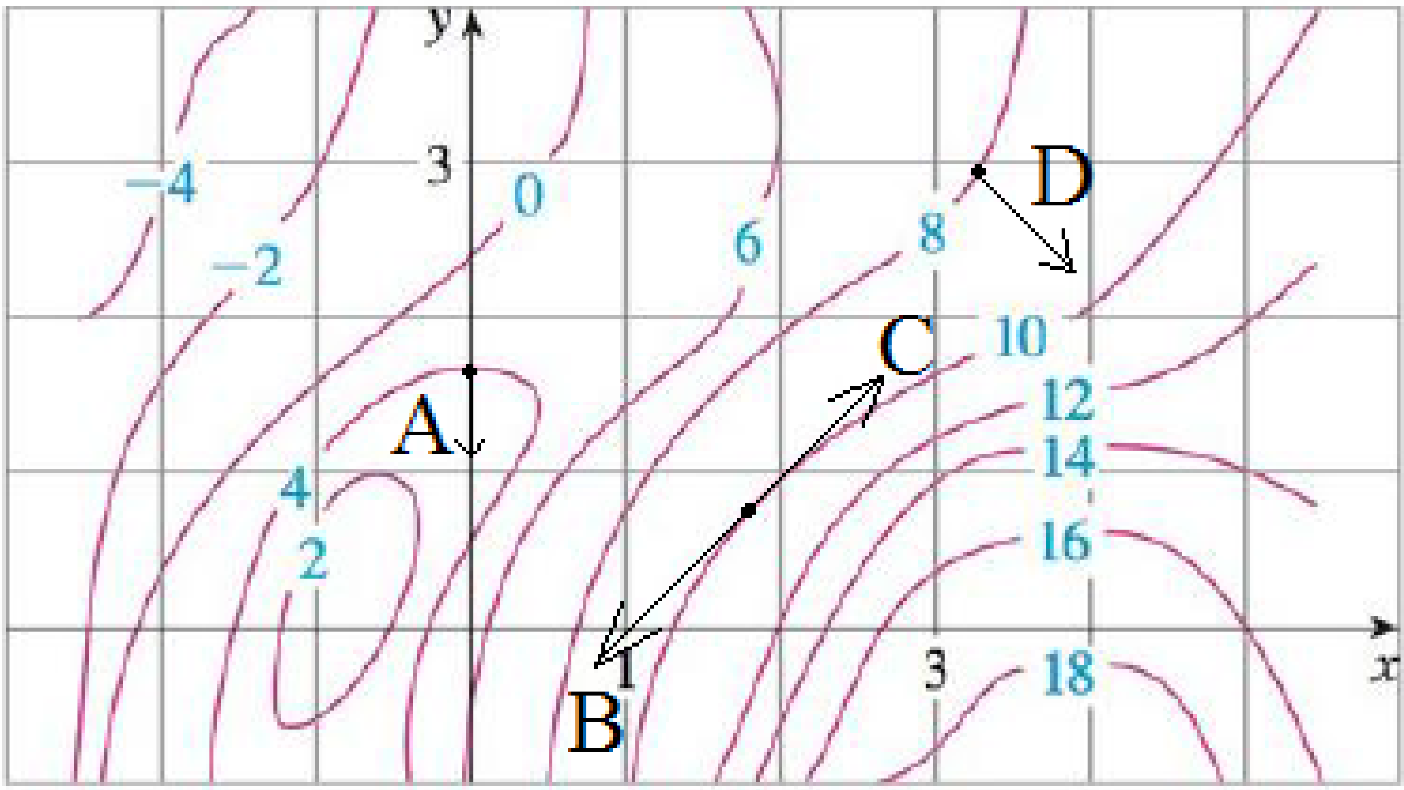


FIGURE 11. Contour plot of $z = f(x, y)$.

Which of the vectors in the above plot could possibly be a gradient vector of the function $f(x, y)$? Please circle all that apply.

(A) (B) (C) (D) (E) None of the given vectors

Explanation: The gradient vector of a function $f(x, y)$ is normal to the level curves (the curves of the form $f(x, y) = c$, with c a constant) and points in the direction of maximum increase. We see that vector A is normal to a level curve of f, but points in the direction of decrease and is therefore not a gradient vector. We see that vectors B and C are tangent to a level curve, not normal to the level curve, so neither of them can be a gradient vector. We see that vector D is normal to a level curve of f and points in the direction of increase, so D could be a gradient vector of f.

Problem 1.20: Imagine a string that is fixed at both ends (for example, a guitar string). When plucked, the string forms a standing wave. The displacement u of the string varies with position x and with time t. Suppose it is given by $u = f(x,t) = 2\sin(\pi x)\sin(\frac{\pi}{2}t)$, for $0 \le x \le 1$ and $t \ge 0$ (see figure 12). At a fixed point in time, the string forms a wave on $[0,1]$. Alternatively, if you focus on a point on the string (fix a value of x), that point oscillates up and down in time.

(a) What is the period of the motion in time?
(b) Find the rate of change of the displacement with respect to time at a constant position (which is the vertical velocity of a point on the string).
(c) At a fixed time, what point on the string is moving fastest?
(d) At a fixed position on the string, when is the strong moving fastest?
(e) Find the rate of change of the displacement with respect to position at a constant time (which is the slope of the string).
(f) At a fixed time, where is the slope of the string greatest?

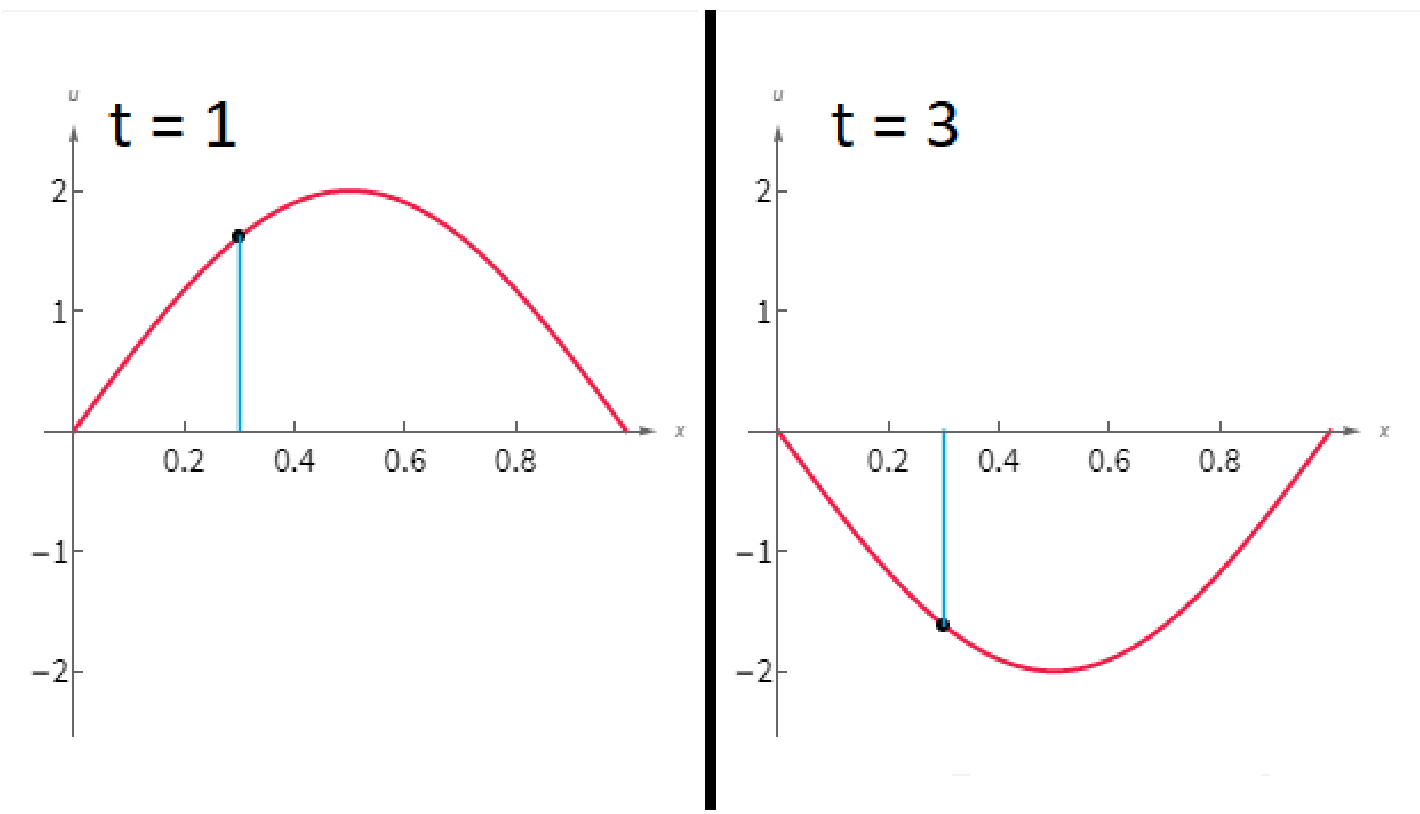


FIGURE 12. Snapshots of the wave at times $t = 1$ and $t = 3$.

Solution to (a): We begin by recalling that the period of sin (as well as cos, tan, csc, sec, and cot) is 2π, i.e., $\sin(y) = \sin(y + 2\pi)$ for all $y \in \mathbb{R}$. We now want to find the smallest $p > 0$ such that

(304) $$\sin(\frac{\pi}{2}(t+p)) = \sin(\frac{\pi}{2}t),$$

which will happen if

(305) $$\frac{\pi}{2}t + \frac{\pi}{2}p = \frac{\pi}{2}t + 2\pi \Rightarrow \boxed{p = 4}.$$

Solution to (b): If we fix a position of $x = x_0$, then $v(x_0, t)$, the rate of change of displacement with respect to time is given by the t partial derivative of f. We now observe that

(306) $$v(x_0, t) = f_t(x_0, t) = \frac{\partial}{\partial t} 2\sin(\pi x_0)\sin(\frac{\pi}{2}t) \overset{*}{=} 2\sin(\pi x_0)\frac{\partial}{\partial t}\sin(\frac{\pi}{2}t)$$

(307) $$= 2\sin(\pi x_0)\Big(\frac{\pi}{2}\cos(\frac{\pi}{2}t)\Big) = \boxed{\pi\sin(\pi x_0)\cos(\frac{\pi}{2}t)},$$

where equation * follows from the fact that $2\sin(\pi x_0)$ is a constant.

Solution to (c): Since speed is just the absolute value of velocity, it suffices to optimize the velocity function $v(x, t)$. In the end, the largest possible speed is going to either be the largest possible velocity, or the absolute value of the smallest possible velocity. Since we are fixing a time $t = t_0$, we seek to optimize the function $h(x) := v(x, t_0)$ with respect to x, which is essentially a single variable calculus optimization problem. Consequently, we begin by finding the the critical point of $h(x)$ in the interval $[0, 1]$. We now see that

(308) $$0 = \frac{d}{dx}h(x) = \frac{\partial}{\partial x}v(x, t_0) = \frac{\partial}{\partial x}\pi\sin(\pi x)\cos(\frac{\pi}{2}t_0)$$

$$\overset{*}{=}\pi\cos(\frac{\pi}{2}t_0)\frac{\partial}{\partial x}\sin(\pi x) = \pi\cos(\frac{\pi}{2}t_0)\Big(\pi\cos(\pi x)\Big)$$

$$= \pi^2\cos(\frac{\pi}{2}t_0)\cos(\pi x)$$

(309) $$\Rightarrow 0 = \cos(\frac{\pi}{2}t_0)\cos(\pi x),$$

where equation * follows from the fact that $\pi\cos(\frac{\pi}{2}t_0)$ is a constant. We now observe that if $\cos(\frac{\pi}{2}t_0) = 0$, then $v(x, t_0) = 0$ for all $x \in [0, 1]$, so in this situation every point on the string is the fastest moving point since every point is moving (or not moving since the velocity is 0) at the same speed. Having fully resolved the situation when $\cos(\frac{\pi}{2}t_0) = 0$, we proceed to the remaining situation in which $\cos(\frac{\pi}{2}t_0) \neq 0$, in which case we are allowed to divide both sides of the right hand equation in (309) by $\cos(\frac{\pi}{2}t_0)$ to see that $0 = \cos(\pi x)$. Recalling that $\cos(y) = 0$ if and only if $y = \frac{\pi}{2} + n\pi$ for some integer n, we see that $x \in \{\cdots - \frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \cdots\}$. Recalling that $x \in [0,1]$ we see that $x = \frac{1}{2}$ is the only critical point. Since the end points of the domain of x are 0 and 1, we observe that $v(0,t_0) = v(1,t_0) = 0$ (which should not be a surprise since the end points of our string are not moving) and that $v(\frac{1}{2},t_0) = 2\cos(\frac{\pi}{2}t_0)$. Lastly, to put together the results of our preceding two cases we recall that $\cos(\frac{\pi}{2}t_0) = 0$ if and only if $\frac{\pi}{2}t_0 = \frac{\pi}{2} + n\pi$ for some integer n, which happens if and only if $t = 2n+1$. Since $t \geq 0$, we see that this happens if and only if t is a positive odd integer. In conclusion, the fastest moving point on the string at time $t = t_0$ is

$$\boxed{\begin{cases} x = \frac{1}{2} & \text{if } t \text{ is not an odd integer} \\ \text{all } x \in [0,1] & \text{else} \end{cases}}. \tag{310}$$

Solution to (d): As in part (c) we begin by optimizing the velocity function $v(x,t)$. Since we are fixing a position $x = x_0$, we seek to optimize the function $q(t) := v(x_0,t)$ with respect to t, which is essentially a single variable calculus optimization problem. Consequently, we begin by finding the the critical point of $q(t)$ over $[0,\infty)$. We now see that

$$0 = \frac{d}{dt}q(t) = \frac{\partial}{\partial t}v(x_0,t) = \frac{\partial}{\partial t}\pi\sin(\pi x_0)\cos(\frac{\pi}{2}t) \overset{*}{=} \pi\sin(\pi x_0)\frac{\partial}{\partial t}\cos(\frac{\pi}{2}t) \tag{311}$$

$$= \pi\sin(\pi x_0)\Big(-\frac{\pi}{2}\sin(\frac{\pi}{2}t)\Big) = -\frac{\pi^2}{2}\sin(\pi x_0)\sin(\frac{\pi}{2}t)$$

$$\Rightarrow 0 = \sin(\pi x_0)\sin(\frac{\pi}{2}t), \tag{312}$$

where equation * follows from the fact that $\pi\sin(\pi x_0)$ is a constant. We now observe that if $\sin(\pi x_0) = 0$ for some $x_0 \in [0,1]$, then $x_0 = 0, 1$. Noting that $v(0,t) = v(1,t) = 0$ for all $t \geq 0$ (which makes sense since the endpoints of the string are fixed) we see that any time $t \geq 0$ results in the fastest velocity if $x_0 = 0, 1$. We now proceed to the situation in which $x_0 \in (0,1)$, so

$\sin(\pi x_0) \neq 0$ and we can divide the right hand equation in (312) by $\sin(\pi x_0)$ to see that $0 = \sin(\frac{\pi}{2}t)$. Recalling that $\sin(y) = 0$ if and only if $y = n\pi$ for some integer n, we see that $t \in \{\cdots - 4, -2, 0, 2, 4, \cdots\}$. Recalling that $t \geq 0$ we see that $\{0, 2, 4, \cdots\}$ are all of the critical points, and this set of critical points coincidentally (luckily) includes the endpoint 0 of our region. Observing that $v(x_0, 2n) = \pi\sin(\pi x_0)\cos(n\pi) = (-1)^n\pi\sin(\pi x_0)$, so the largest speed is $\pi|\sin(\pi x_0)|$. In conclusion, the fastest speed of the point $x = x_0$ on the string is attained at

$$\boxed{\begin{cases} \text{Every } t \geq 0 & \text{if } x_0 = 0, 1 \\ \{0, 2, 4, \cdots\} & \text{else} \end{cases}}. \tag{313}$$

Interestingly, we note that if $t \in \{0, 2, 4, \cdots\}$ then the string is back at equilibrium (every point has 0 displacement) and if $t \in \{1, 3, 5, \cdots\}$ then the string is in an extreme state in which every point has its maximum possible displacement.

Solution to (e): If we fix a time $t = t_0$, then $s(x, t_0)$, the rate of change of the displacement with respect to position (slope of the string) at a constant time is given by the x partial derivative of f. We now observe that

$$s(x, t_0) = \frac{\partial}{\partial x}f(x, t_0) = \frac{\partial}{\partial x}2\sin(\pi x)\sin(\frac{\pi}{2}t_0) \overset{*}{=} 2\sin(\frac{\pi}{2}t_0)\frac{\partial}{\partial x}\sin(\pi x) \tag{314}$$

$$= 2\sin(\frac{\pi}{2}t_0)\Big(\pi\cos(\pi x)\Big) = \boxed{2\pi\cos(\pi x)\sin(\frac{\pi}{2}t_0)}. \tag{315}$$

Solution to (f): As in parts (c) and (d) we begin by optimizing the slope function $s(x, t)$. Since we are fixing a time $t = t_0$, we seek to optimize the function $r(x) := s(x, t_0)$ with respect to x, which is essentially a single variable calculus optimization problem. Consequently, we begin by finding the the critical point of $r(x)$ over $[0, 1]$. We now see that

(316)
$$0 = \frac{d}{dx}r(x) = \frac{\partial}{\partial x}s(x, t_0) = \frac{\partial}{\partial x}2\pi\cos(\pi x)\sin(\frac{\pi}{2}t_0) \overset{*}{=} 2\pi\sin(\frac{\pi}{2}t_0)\frac{\partial}{\partial x}\cos(\pi x)$$

(317)
$$= 2\pi \sin(\frac{\pi}{2}t_0)\Big(-\pi\sin(\pi x)\Big) = -2\pi\sin(\pi x)\sin(\frac{\pi}{2}t_0) \Rightarrow 0 = \sin(\pi x)\sin(\frac{\pi}{2}t_0),$$

where equation * follows from the fact that $2\pi\sin(\frac{\pi}{2}t_0)$ is a constant. As we saw in part (d), $\sin(\frac{\pi}{2}t_0) = 0$ for $t_0 \geq 0$ if and only if $t_0 \in \{0, 2, 4, \cdots\}$, and in this situation we see that $s(x, t_0) = 0$ for all x, so every x attains the greatest slope. We now consider the situation in which $t_0 \notin \{0, 2, 4, \cdots\}$ and divide the right hand equation of 317 by $\sin(\frac{\pi}{2}t_0)$ to see that $0 = \sin(\pi x)$. As before, we deduce that $x \in [0, 1]$ must be an integer, so $x = 0, 1$. Observing that

(318) $$s(0, t_0) = 2\pi\cos(0)\sin(\frac{\pi}{2}t_0) = 2\pi\sin(\frac{\pi}{2}t_0) \text{ and}$$

(319) $$s(1, t_0) = 2\pi\cos(\pi)\sin(\frac{\pi}{2}t_0) = -2\pi\sin(\frac{\pi}{2}t_0),$$

we see that the largest slope in this case is $2\pi|\sin(\frac{\pi}{2}t_0)|$, which occurs at 0 if $\sin(\frac{\pi}{2}t_0) > 0$ and at 1 if $\sin(\frac{\pi}{2}t_0) < 0$. In conclusion, the greatest slope of a point on the string at time $t = t_0$ is attained at

(320) $$\boxed{\begin{cases} \text{Every } x \in [0,1] & \text{if } t_0 \in \{0, 2, 4, \cdots\} \\ x = 0 & \text{if } t_0 \in (2n, 2n+1) \text{ for some integer } n \\ x = 1 & \text{if } t_0 \in (2n+1, 2n+2) \text{ for some integer } n \end{cases}}.$$

Problem 1.21: Let $w = f(x,y,z) = 2x + 3y + 4z$, which is defined for all $(x,y,z) \in \mathbb{R}^3$. Suppose we are interested in the partial derivative w_x on a subset of $\mathbb{R}^3$, such as the plane P given by $z = 4x - 2y$. The point to be made is that the result is not unique unless we specify which variables are considered independent.

(a) We could proceed as follows. On the plane P, consider x and y as the independent variables, which means z depends on x and y, so we write $w = w(x,y) = f(x,y,z(x,y))$. Show that $\frac{\partial}{\partial x}w(x,y) = 18$.

(b) Alternatively, on the plane P, we could consider x and z as the independent variables, which means y depends on x and z, so we write $w = w(x,z) = f(x,y(x,z),z)$. Show that $\frac{\partial}{\partial x}w(x,z) = 8$.

(c) Make a sketch of the plane $z = 4x - 2y$ and interpret the results of parts (a) and (b) geometrically.

Solution to (a): Since $z = 4x - 2y$, we are lucky enough to see that $z(x,y) = 4x - 2y$ without even having to manipulate the original equation. We now see that

$$w = w(x,y) = f(x,y,z(x,y)) = 2x + 3y + 4(4x - 2y) = 18x - 5y \tag{321}$$

$$\Rightarrow \frac{\partial}{\partial x}w(x,y) = 18. \tag{322}$$

Solution to (b): Firstly, we observe that

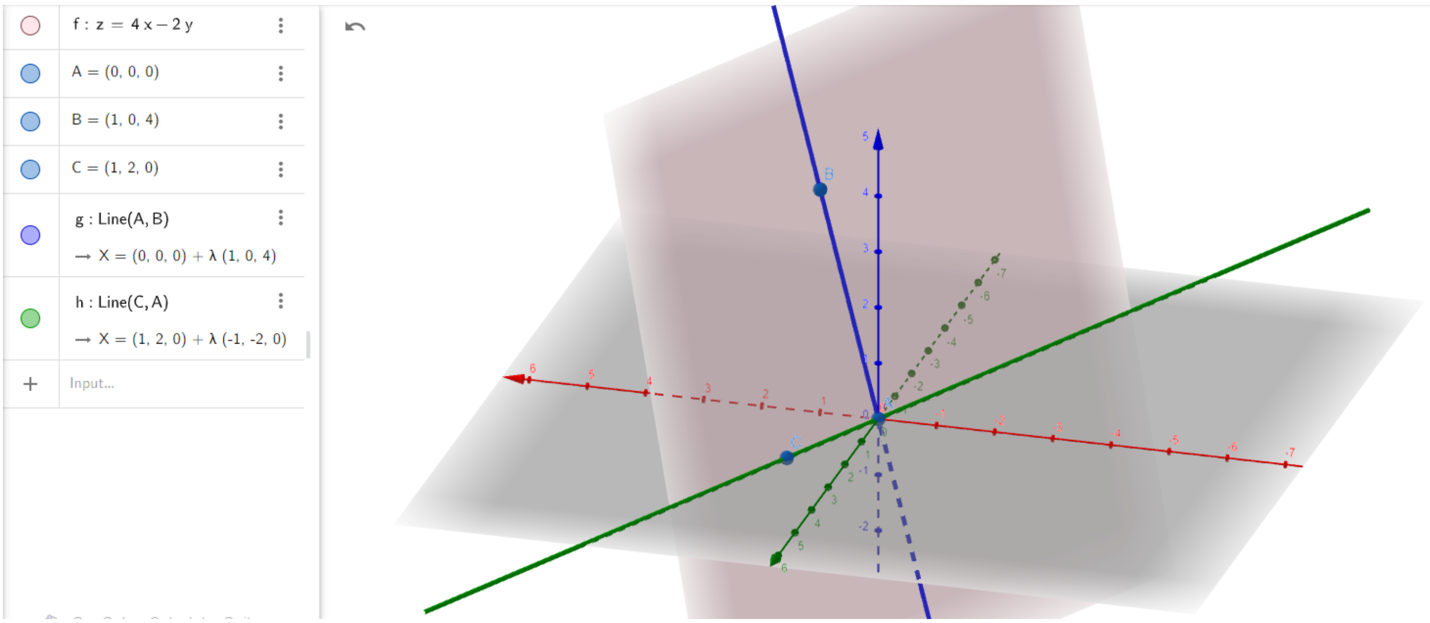
$$z = 4x - 2y \Rightarrow 2y = 4x - z \Rightarrow y = y(x,z) = 2x - \frac{1}{2}z. \tag{323}$$

We now see that

$$w = w(x,z) = f(x,y(x,z),z) = 2x + 3(2x - \frac{1}{2}z) + 4z = 8x + \frac{5}{2}z \tag{324}$$

$$\Rightarrow \frac{\partial}{\partial x}w(x,z) = 8. \tag{325}$$

Solution to (c): In our graph of $z = 4x - 2y$ we have also included graphs of the lines $(0 = 4x - 2y, z = 0)$ and $(z = 4x, y = 0)$, which are the lines residing within $z = 4x - 2y$ when you set $y = 0$ and $z = 0$ respectively. We do this to analyze what happens when calculating $\frac{\partial w}{\partial x}(0,0,0)$ (to have a concrete example) using the methods of parts (a) and (b).



We see that fixing $z = 0$ in part (a) to obtain $w(x,y)$ simply gives us the values of w over the line $(0 = 4x - 2y, z = 0)$. Similarly, fixing $y = 0$ in part (b) to obtain $w(x,y)$ simply gives us the values of w over the line $(z = 4x, y = 0)$. We now see that in part (a) we calculated the directional derivative of w in the direction of the line $(0 = 4x - 2y, z = 0)$ and in part (b) we calculated the directional derivative of w in the direction of the line $(z = 4x, y = 0)$. Said differently, part (a) showed us that $D_{\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\rangle} w(x,y,z) = 18$ and part (b) showed us that $D_{\langle \frac{1}{\sqrt{17}}, 0, \frac{4}{\sqrt{17}}\rangle} w(x,y,z) = 8$.

Problem 1.19: Consider the function $f(x,y) = x^2 + y^2$ and and the point $P = (2,3)$.

(a) Find the unit vector that points in direction of maximum decrease of the function f at the point P.

(b) Calculate the directional derivative of f at the point P in the direction of the vector $\vec{u} = \langle 3, 2\rangle$.

Solution to (a): We see that $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 2x, 2y\rangle$. We see that $-\nabla f(2,3) = \langle -4, -6\rangle$ is a vector that points in the direction of maximum decrease of f at the point P. Since $|\langle -4, -6\rangle| = \sqrt{52} = 2\sqrt{13}$, we see that

$$\frac{\langle -4, -6\rangle}{|\langle -4, -6\rangle|} = \frac{1}{2\sqrt{13}}\langle -4, -6\rangle = \boxed{\langle \frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\rangle} \tag{326}$$

is the direction of maximum decrease of f at the point P.

Solution to (b): We see that $|\vec{u}| = \sqrt{13}$, so

$$\vec{w} = \frac{\vec{u}}{|\vec{u}|} = \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle \tag{327}$$

is the unit vector that points in the same direction as $\vec{u}$, so

$$d_{\vec{w}}f(2,3) = \nabla f(2,3) \cdot \vec{w} = \langle 4, 6\rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle = \boxed{\frac{24}{\sqrt{13}}}. \tag{328}$$

Problem 3.1: Determine all critical points of the function $f(x,y) = x^3 - y^3 + xy$, then classify each of the critical points as a local maximum, local minimum, or saddle point.

Solution: To find the criticial points of f, we simply have to find all (x,y) for which both partial derivatives of f are 0.

$$\begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} 3x^2 + y & = & 0 \\ -3y^2 + x & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} -3x^2 & = & y \\ 3y^2 & = & x \end{array} \tag{329}$$

$$\to x = 3(-3x^2)^2 = 27x^4 \to x = 0, \frac{1}{3} \to (x,y) = \boxed{(0,0), (\frac{1}{3}, -\frac{1}{3})}. \tag{330}$$

We now proceed to calculate all of the second derivatives of f as well as the discriminant function so that we can apply the second derivative test.

$$\begin{array}{rcl} f_{xx}(x,y) & = & 6x \\ f_{yy}(x,y) & = & -6y \\ f_{xy}(x,y) & = & 1 \end{array} \tag{331}$$

$$\to D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = -36xy - 1. \tag{332}$$

Since $D(0,0) = -1 < 0$, we see that $\boxed{(0,0) \text{ is a saddle point}}$.

Since $D(\frac{1}{3}, -\frac{1}{3}) = 3 > 0$ and $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = 2 > 0$ we see that $\boxed{(\frac{1}{3}, -\frac{1}{3}) \text{ is a local minimum}}$.

Problem 3.2: A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that require the least cardboard.

Solution: If the box has a width of w, a length of ℓ and a height of h, then the volume V is given by $V = wh\ell$. We also see from figure 1 that the amount of cardboard it takes to make such a box is $2hw + 2h\ell + wl$.

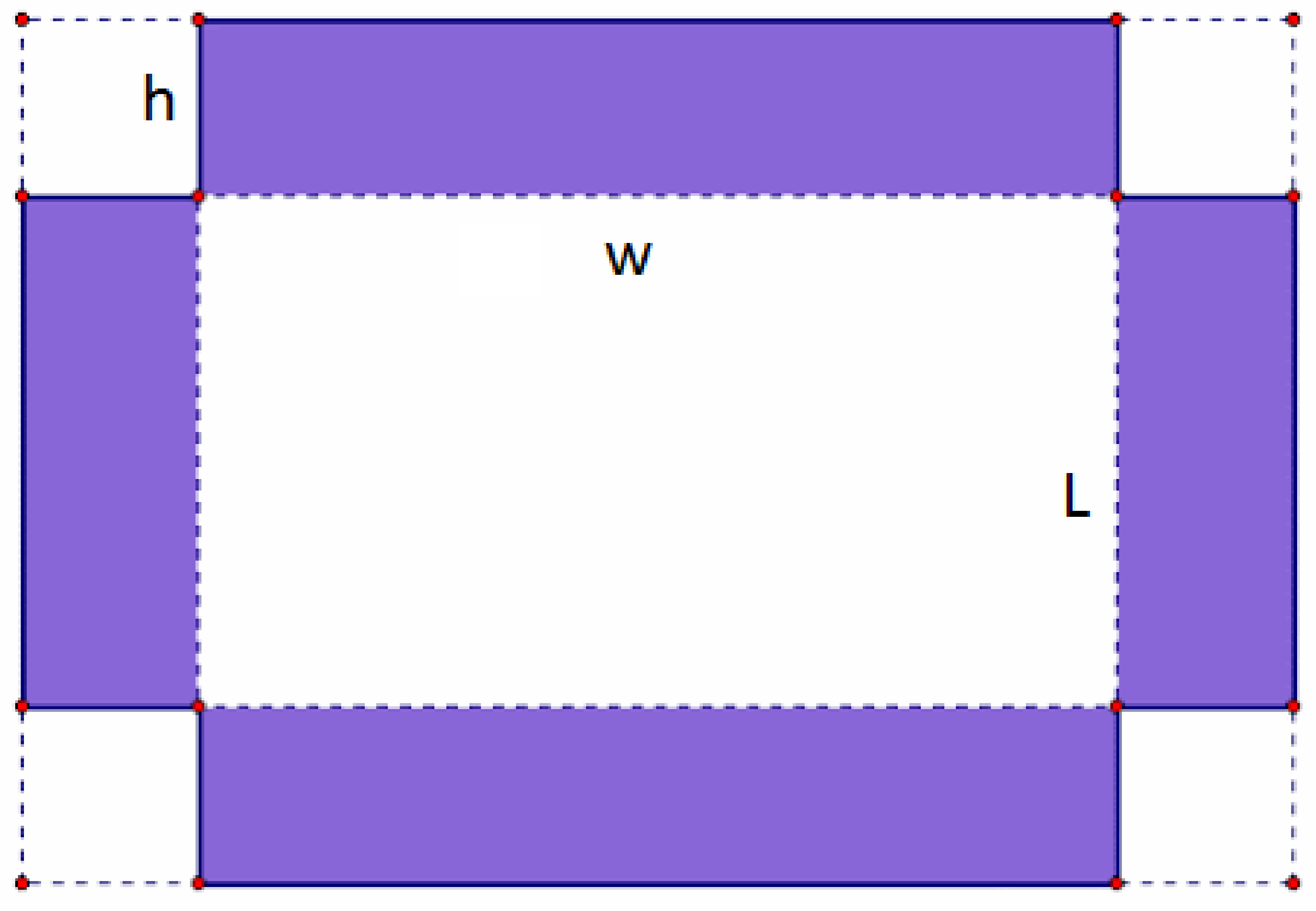


FIGURE 13.

It follows that we are trying to optimize the function

$$f(w, h, \ell) = 2hw + 2h\ell + w\ell \tag{333}$$

subject to the constraint

$$wh\ell = 4. \tag{334}$$

Noting that

(335) $$h = \frac{4}{w\ell},$$

we now want to optimize the function

(336) $$g(w,\ell) = f(w,h,\ell) = f(w,\frac{4}{w\ell},\ell) = 2\frac{4}{w\ell}w+2\frac{4}{w\ell}\ell+w\ell = \frac{8}{\ell}+\frac{8}{w}+w\ell$$

over the first quadrant of $\mathbb{R}^2$. We see that

(337) $$\frac{\partial g}{\partial w} = -\frac{8}{w^2}+\ell \text{ and } \frac{\partial g}{\partial \ell} = -\frac{8}{\ell^2}+w, \text{ so}$$

(338) $$\begin{matrix}\frac{\partial g}{\partial w}(w,\ell)=0 \\ \frac{\partial g}{\partial \ell}(w,\ell)=0\end{matrix} \Leftrightarrow \begin{matrix}-\frac{8}{w^2}+\ell=0 \\ -\frac{8}{\ell^2}+w=0\end{matrix} \Leftrightarrow 8 = w\ell^2 = w^2\ell \overset{*}{\to} w=\ell$$

(339) $$\to 8 = w^3 \to (w,h,\ell) = \boxed{(2,1,2)}.$$

To verify that $g(w,\ell)$ at the very least attain a local minimum value at $(w,\ell) = (2,2)$ we will use the second derivative test. Technically, this step is not needed as discussed in the remark after the proof. We note that

(340) $$\frac{\partial^2 g}{\partial w^2}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial w}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{w^2}+\ell) = \frac{16}{w^3},$$

(341) $$\frac{\partial^2 g}{\partial \ell^2}(w,\ell) = \frac{\partial}{\partial \ell}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial \ell}(-\frac{8}{\ell^2}+w) = \frac{16}{\ell^3}, \text{ and}$$

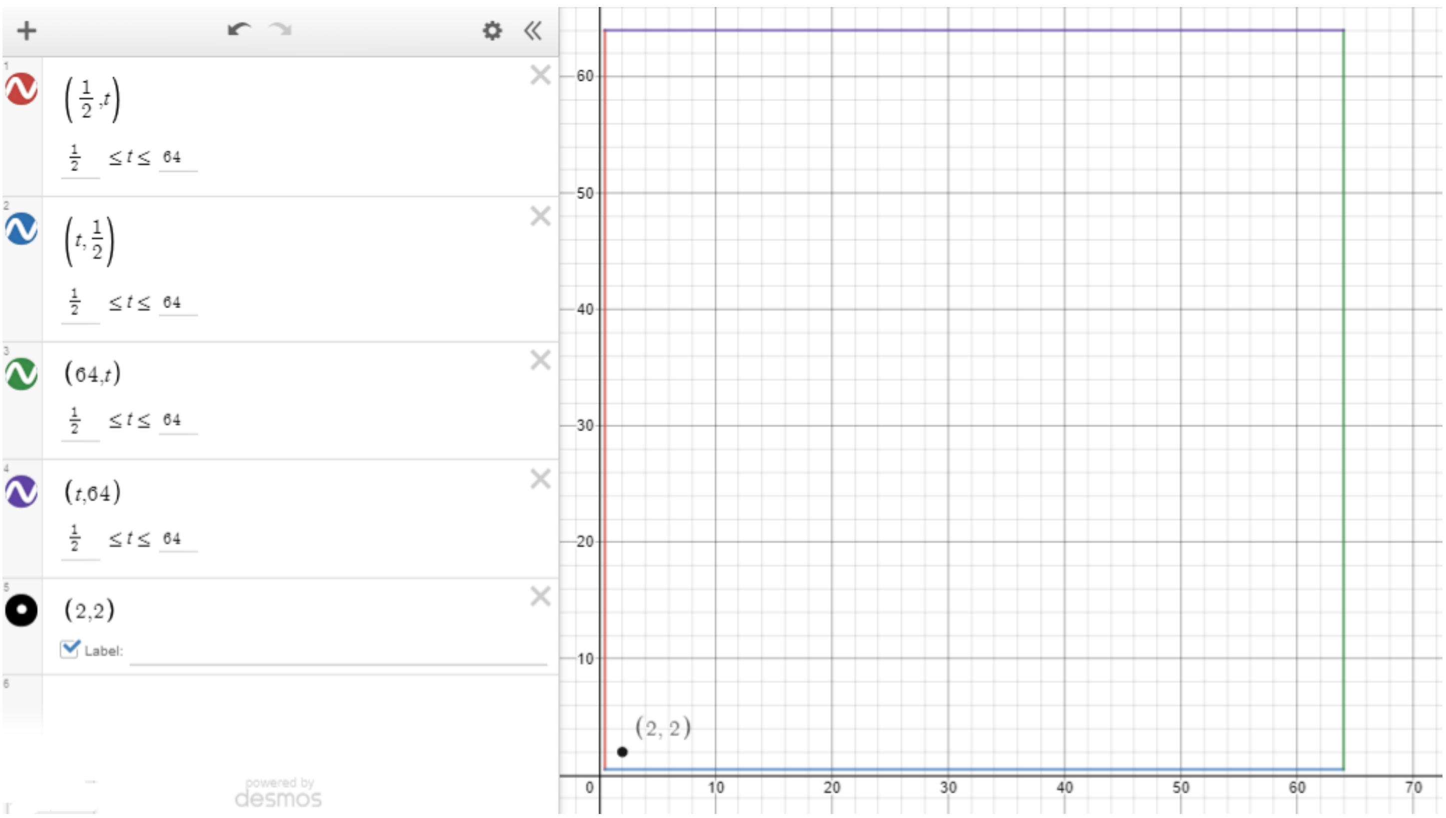
(342) $$\frac{\partial^2 g}{\partial w\partial \ell}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{\ell^2}+w) = 1, \text{ so}$$

(343) $$D(w,\ell) = \frac{\partial^2 g}{\partial w^2}(w,\ell)\frac{\partial^2 g}{\partial \ell^2}(w,\ell) - (\frac{\partial^2 g}{\partial w\partial \ell}(w,\ell))^2 = \frac{16}{w^3}\cdot\frac{16}{\ell^3} - 1^2 = \frac{256}{w^3\ell^3} - 1.$$

Since

$$D(2,2) = \frac{256}{8 \cdot 8} - 1 = 3 > 0 \text{ and } \frac{\partial^2 g}{\partial w^2}(2,2) = \frac{16}{2^3} = 2 > 0, \tag{344}$$

the second derivative test tells us that $g(w,\ell)$ attains a local minimum at the critical point $(2,2)$. We will now verify that $(2,2)$ is actually the global minimum of $g(w,\ell)$ over the first quadrant of $\mathbb{R}^2$. Consider the closed and bounded region region $R = [\frac{1}{2}, 64]^2$.



A picture of R.

We note that $(2,2) \in R$, and that $(2,2)$ is the only critical point of $g(w,\ell)$ in R (because $g(w,\ell)$ only had 1 critial point anyways). We also see that $g(w,\ell) \geq 16 > 12 = g(2,2)$ for (w,ℓ) on the boundary of R (this can easily be checked on each of the 4 sides of the boundary of R separately). By the extreme value theorem, we see that g attains its absolute minimum over R at the point $(2,2)$. Since $g(w,\ell) \geq 16 > 12$ for (w,ℓ) that are in the first quadrant of $\mathbb{R}^2$ but outside of R (this fact is left as a challenge to the reader), we see that $g(w,\ell)$ does indeed attain its global minimum over the first quadrant of $\mathbb{R}^2$ at $(2,2)$.

Remark: We never actually needed to use the second derivative test to verify that the global minimum occurred at $(2, 2)$. The second derivative test was only useful for telling us that $(2, 2)$ was a local minimum, but we never used the fact that $(2, 2)$ was a local minimum in order to conclude that it was actually a global minimum. I only wrote that into the solutions since I permitted you to finish the problem by checking that it is a local minimum instead of a global minimum. Instructors of sophomore level calculus classes usually allow for this simplification.

Problem 3.3: Consider the function $f(x,y) = 3 + x^4 + 3y^4$. Show that $(0,0)$ is a critical point for $f(x,y)$ and show that the second derivative test is inconclusive at $(0,0)$. Then describe the behavior of $f(x,y)$ at $(0,0)$.
*H*int: The product of 2 negative numbers is positive.

Solution: We see that

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 \text{ and } \frac{\partial f}{\partial y}(x,y) = 12y^3, \text{ so} \tag{345}$$

$$\begin{matrix} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4x^3 = 0 \\ 12y^3 = 0 \end{matrix} \Leftrightarrow (x,y) = (0,0). \tag{346}$$

It follows that $(0,0)$ is the only critical point of f in all of $\mathbb{R}^2$. We also note that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(4x^3) = 12x^2, \\ \frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(12y^3) = 36y^2, \text{ and} \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial x}(12y^3) = 0, \text{ so} \end{aligned} \tag{347}$$

$$\begin{aligned} D(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - \left(\frac{\partial^2 f}{\partial x \partial y}(x,y)\right)^2 \\ &= 12x^2 \cdot 36y^2 - 0^2 = 432x^2y^2 \end{aligned}. \tag{348}$$

Since $D(0,0) = 0$, we see that the second derivative test is inconclusive. However, we are still able to describe the behavior of $f(x,y)$ at $(0,0)$. Note that $x^4 \geq 0$ for all $x \in \mathbb{R}$, and $3y^4 \geq 0$ for all $y \in \mathbb{R}$. Furthermore, $x^4 = 0$ if and only if $x = 0$, and $3y^4 = 0$ if and only if $y = 0$. It follows that $x^4 + 3y^4 \geq 0$ for all $(x,y) \in \mathbb{R}^2$, and $x^4 + 3y^4 = 0$ if and only if $(x,y) = (0,0)$. From this we are able to see that $f(x,y) = 3 + x^4 + 3y^4$ attains an absolute minimum at $(0,0)$.

Problem 3.4: Show that the second derivative test is inconclusive when applied to the function $f(x,y) = x^4y^2$ at the point $(0,0)$. Show that $f(x,y)$ has a local minimum at $(0,0)$ by direct analysis.
Hint: The product of 2 negative numbers is positive.

Solution: We will first verify that $(0,0)$ is a critical point. We see that

$$\frac{\partial f}{\partial x}(x,y) = 4x^3y^2 \text{ and } \frac{\partial f}{\partial y}(x,y) = 2x^4y, \text{ so} \tag{349}$$

$$\begin{matrix} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4x^3y^2 = 0 \\ 2x^4y = 0 \end{matrix} \Leftrightarrow x = 0 \text{ or } y = 0. \tag{350}$$

It follows that the critical points of f are precisely those points which are on either the x-axis or the y-axis, and $(0,0)$ is certainly such a point. Next, we notice that

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(4x^3y^2) = 12x^2y^2, \\ \frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(2x^4y) = 2x^4, \text{ and} \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial x}(2x^4y) = 8x^3y, \text{ so} \end{aligned} \tag{351}$$

$$\begin{aligned} D(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - (\frac{\partial^2 f}{\partial x \partial y}(x,y))^2 \\ &= 12x^2y^2 \cdot 2x^4 - (8x^3y)^2 = -40x^6y^2. \end{aligned} \tag{352}$$

Since $D(x,y) = 0$ whenever $x = 0$ or $y = 0$, we see that the second derivative test is inconclusive for every critical point of f (which includes (0,0)). However, we are still able to describe the behavior of $f(x,y)$ at any of its critical points by using a direct analysis. Note that $x^4y^2 \geq 0$ for all $(x,y) \in \mathbb{R}^2$ (use the hint if this is not obvious to you), and that $x^4y^2 = 0$ whenever $x = 0$ or $y = 0$. It follows that f attains its absolute minimum at any of its critical points.

Problem 3.5: Find the absolute minimum and absolute maximum values of the function $f(x, y) = xy$ over the region $R = \{(x, y) \mid (x-1)^2 + y^2 \leq 1\}$.

Solution: Since R is a closed and bounded region, and f is a continuous function, the Extreme Value Theorem tells us that f will attain its absolute minimum and absolute maximum values over the region R. Furthermore, we know that the extreme values of f will either be attained on the boundary of R, or at a critical point of f in the interior of R.

We will begin by finding all critical points in the interior of R. Since $f_x(x, y) = y$ and $f_y(x, y) = x$, we immediately see that $(0, 0)$ is the only critical point of f, and it is on the boundary (not interior) of the region R, but it is still a candidate for where f can attain one of its extreme values. We note that $f(0, 0) = 0$.

We will now proceed to find the absolute minimum and absolute maximum values of f on the boundary of R. Since the boundary of R is given by $\partial R = \{(x, y) \mid (x-1)^2 + y^2 = 1\}$, we will use the method of Lagrange Multipliers to optimize the function $f(x, y) = xy$ subject to the constraint $g(x, y) = (x-1)^2 + y^2 - 1 = 0$. We note that

$$\nabla f(x, y) = \langle y, x\rangle \text{ and } \nabla g(x, y) = \langle 2x - 2, 2y\rangle, \tag{353}$$

so the method of Lagrange Multipliers results in the following system of equations for us to solve:

$$\begin{array}{rcl} g(x, y) & = & 0 \\ \nabla f(x, y) & = & \lambda \nabla g(x, y) \end{array} \Leftrightarrow \begin{array}{rcl} (x-1)^2 + y^2 & = & 1 \\ y & = & \lambda(2x - 2) \\ x & = & \lambda 2y \end{array} \tag{354}$$

$$\rightarrow \lambda x(2x - 2) = xy = \lambda 2y^2 \rightarrow 0 = 2\lambda(y^2 - x^2 + x). \tag{355}$$

By the zero-product property, we see that we must have $\lambda = 0$ or $y^2 - x^2 + x = 0$, so we will consider both cases separately.

Case 1: For our first case let us assume that $\lambda = 0$. In this case we see that the last 2 equations from (354) tell us that $x = y = 0$, since $g(0, 0) = 0$, we see that we reobtain the critical point $(x, y) = (0, 0)$.

Case 2: For our next case let us assume that $y^2 - x^2 + x = 0$, so $y^2 = x^2 - x$. We see that

$$1 = y^2 + (x-1)^2 = x^2 - x + (x-1)^2 = 2x^2 - 3x + 1 \tag{356}$$

$$\rightarrow 2x^2 - 3x = 0 \rightarrow x = 0, \frac{3}{2} \rightarrow (x,y) = (0,0), (\frac{3}{2}, \frac{\sqrt{3}}{2}), (\frac{3}{2}, -\frac{\sqrt{3}}{2}). \tag{357}$$

Making a table of our critical points and corresponding values of f, we see that

(x,y)	$f(x,y)$
$(0,0)$	0
$(\frac{3}{2}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{4}$
$(\frac{3}{2}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{4}$

so f attains its absolute maximum value of $\frac{3\sqrt{3}}{4}$ at the point $(\frac{3}{2}, \frac{\sqrt{3}}{2})$ and f attains its absolute minimum value of $-\frac{3\sqrt{3}}{4}$ at the point $(\frac{3}{2}, -\frac{\sqrt{3}}{2})$.

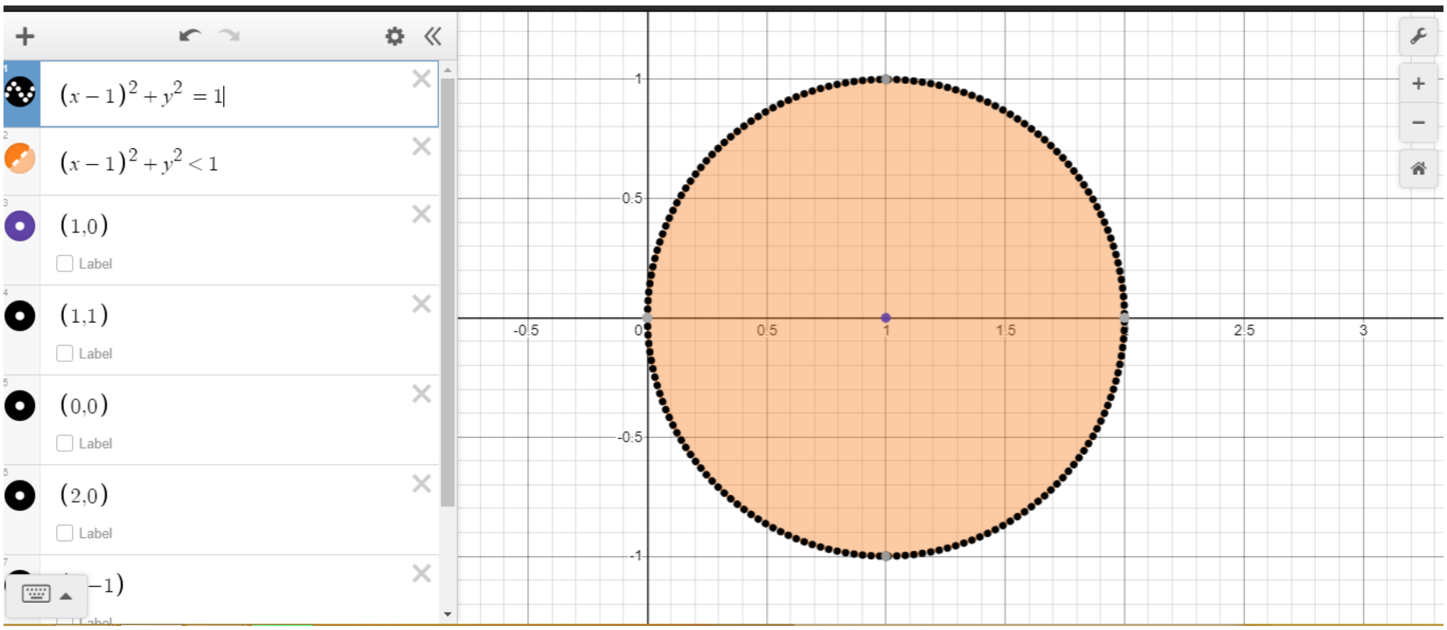
Problem 3.6: Find the absolute minimum and maximum value of the function

$$f(x,y) = 2x^2 - 4x + 3y^2 + 2 \tag{358}$$

over the region

$$R := \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}. \tag{359}$$

Hint: There is an easy solution to this problem that doesn't use calculus if you write $f(x,y)$ in a more convenient form.



Solution: Note that the interior of R is given by

$$R^\circ = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1\} \tag{360}$$

and the boundary of R is given by

$$\partial R = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}. \tag{361}$$

We will first find all critical points in the interior of R. We note that

$$\frac{\partial f}{\partial x} = 4x - 4 \text{ and } \frac{\partial f}{\partial y} = 6y, \text{ so} \tag{362}$$

$$\begin{array}{c} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{array} \Leftrightarrow \begin{array}{c} 4x - 4 = 0 \\ 6y = 0 \end{array} \Leftrightarrow (x,y) = (1,0). \tag{363}$$

We see that $(1,0)$ is the only critical point of f in all of $\mathbb{R}^2$. Since $(1,0) \in R$, we have to take this critical point into consideration when searching for our absolute minimum and maximum values. Now that we have addressed the interior of R, we will proceed to address the boundary of R. We note that ∂R can be parameterized by $\vec{r}(t)$, where

$$\vec{r}(t) = (1+\cos(t), \sin(t)), \quad 0 \le t \le 2\pi, \tag{364}$$

so on ∂R we have

$$\begin{aligned} f(x,y) &= f(\vec{r}(t)) = f(1+\cos(t), \sin(t)) \\ &= 2(1+\cos(t)-1)^2 + 3\sin^2(t) = 2\cos^2(t) + 3\sin^2(t) = 2 + \sin^2(t). \end{aligned} \tag{365}$$

We may now use the (single variable) first derivative test to optimize $f(\vec{r}(t)) = 2+\sin^2(t)$ on the interval $[0, 2\pi]$, but we may also directly notice that the maximum is attained for $t \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ which corresponds to $(x,y) \in \{(1,1),(1,-1)\}$ and the minimum is attained for $t \in \{0, \pi, 2\pi\}$ which corresponds to $(x,y) \in \{(0,0),(2,0)\}$. We now evaluate f at all of the critical points that we have found so far to determine the absolute minimum and maximum values. Noting that

(x,y)	f(x,y)
(1,0)	0
(1,1)	3
(1,-1)	3
(0,0)	2
(2,0)	2

so $f(x,y)$ attains a minimum value of 0 at $(1,0)$, and $f(x,y)$ attains a maximum value of 3 at any of $\{(1,1),(1,-1)\}$.

Remark: In this problem, one may also try to address the boundary of R by noting that $(x-1)^2 = 1 - y^2$ on the boundary, so $f(x,y) = 2(x-1)^2 + 3y^2 = 2 + y^2$ on the boundary.

Problem 3.7: Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function

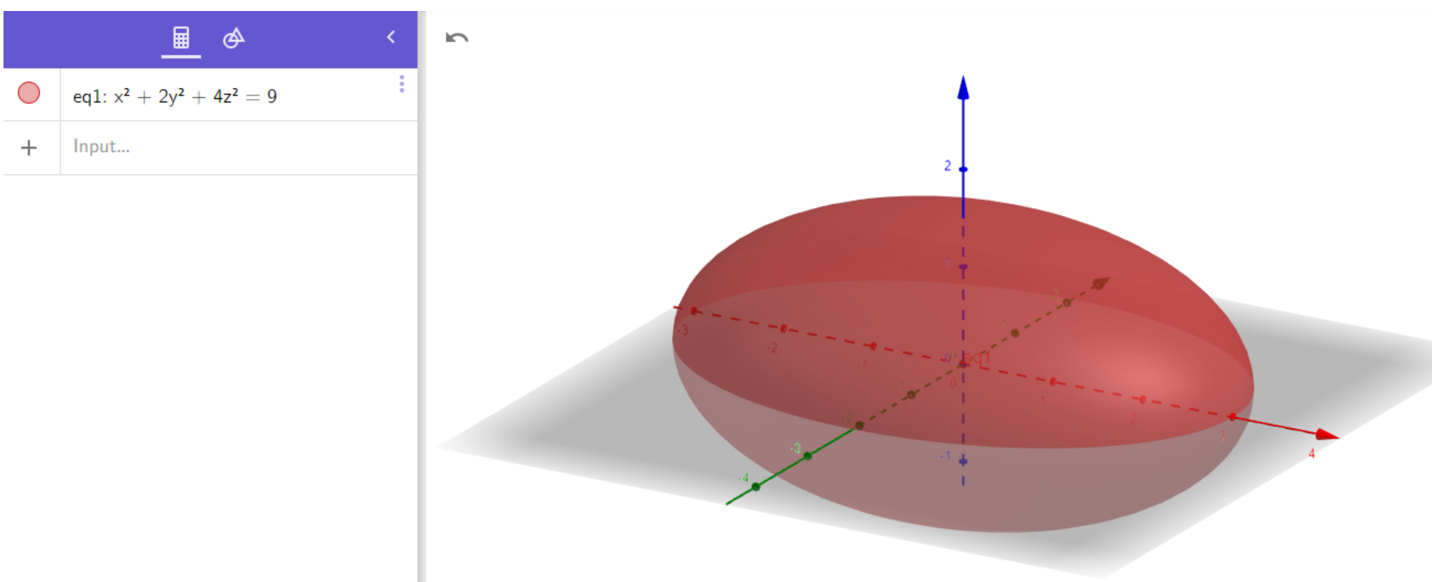
$$f(x, y, z) = xyz \tag{366}$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 9. \tag{367}$$

Solution: We will present two different solutions to this problem. The method of setting up the system of equations from the method of Lagrange multipliers is the same in both solutions, but the method of solving the resulting system will be different.

We see that the region defined by the constraint is a closed and bounded region with no boundary, so the method of Lagrange multipliers will give us the complete list of critical points that we need to check in order to determine the absolute minimum and absolute maximum values of f subject to the constraint.



We see that

$$x^2 + 2y^2 + 4z^2 = 9 \Leftrightarrow x^2 + 2y^2 + 4z^2 - 9 = 0, \tag{368}$$

so we may take our constraint function to be $g(x, y, z) = x^2 + 2y^2 + 4z^2 - 9$. We see that

(369) $\vec{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle yz, xz, xy\rangle,$ and

(370) $$\vec{\nabla} g(x,y,z) = \langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z) = \langle 2x, 4y, 8z\rangle.$$

We now want to find all (x, y, z, λ) (although we don't really care about the value of λ) such that

(371) $$\begin{aligned} g(x,y,z) &= 0 \\ \vec{\nabla} f(x,y,z) &= \lambda\vec{\nabla} g(x,y,z) \end{aligned}$$

. . . .

(372) $$\Leftrightarrow \begin{aligned} x^2+2y^2+4z^2-9 &= 0 \\ \langle yz, xz, xy\rangle &= \lambda\langle 2x, 4y, 8z\rangle \end{aligned}$$

. . . .

(373) $$\Leftrightarrow \begin{aligned} x^2+2y^2+4z^2-9 &= 0 \\ yz &= 2\lambda x \\ xz &= 4\lambda y \\ xy &= 8\lambda z \end{aligned}$$

Finish 1: We will now use the method of cross multiplication to solve the system of equations in (373). This method will be computationally intensive, but is 'standard' and does not require any 'tricky insights'. By cross multiplying the second and third equations in (373) we see that

(374) $$4\lambda y^2 z = 2\lambda x^2 z \to 0 = 4\lambda y^2 z - 2\lambda x^2 z = 2\lambda z(2y^2 - x^2),$$

so by the zero product property we see that either $\lambda = 0$, $z = 0$, or $2y^2 - x^2 = 0$. We will handle each case separately.

Case 1 ($\lambda = 0$): By plugging $\lambda = 0$ back into (373) we see that

(375)
$$\begin{aligned} x^2+2y^2+4z^2-9 &= 0 \\ yz &= 0 \\ xz &= 0 \\ xy &= 0 \end{aligned}.$$

Using the zero product property once again on the second, third, and fourth equations of (375), we see that 2 of x, y, and z must be 0. In conjunction with the first equation of (373) (the constraint equation) we see that $(x, y, z, \lambda) \in \{(0, 0, \pm\frac{3}{2}, 0), (0, \pm\frac{3}{\sqrt{2}}, 0, 0), (\pm 3, 0, 0, 0)\}$ are the solutions that we obtain from this case.

Case 2 (z = 0): By plugging $z = 0$ back into (373) we see that

(376)
$$\begin{aligned} x^2+2y^2-9 &= 0 \\ 0 &= 2\lambda x \\ 0 &= 4\lambda y \\ xy &= 0 \end{aligned}.$$

Since we are done with case 1, we may also assume that $\lambda \neq 0$. It now follows from the second and third equations in (376) that $x = y = 0$, but this contradicts the first equation in (376), so we obtain no additional solutions in this case.

Case 3 ($2y^2 - x^2 = 0$): In this case we see that $x^2 = 2y^2$ so $x = \pm\sqrt{2}y$, which means that we have 2 subcases to handle. For our first subcase, we plug $x = \sqrt{2}y$ back into (373) to obtain

(377)
$$\begin{aligned} 2y^2+2y^2+4z^2-9 &= 0 \\ yz &= 2\sqrt{2}\lambda y \\ \sqrt{2}yz &= 4\lambda y \\ \sqrt{2}y^2 &= 8\lambda z \end{aligned}.$$

By cross-multiplying the third and fourth equations in (377) we see that

(378)
$$8\sqrt{2}\lambda yz^2 = 4\sqrt{2}\lambda y^3 \to 0 = 8\sqrt{2}\lambda yz^2 - 4\sqrt{2}\lambda y^3 = 4\sqrt{2}\lambda y(2z^2 - y^2).$$

Since we are no longer in case 1, we may assume that $\lambda \neq 0$, so either $y = 0$ or $2z^2 - y^2 = 0$. If $y = 0$, then $x = \sqrt{2}y = 0$, and we reobtain the solution $(x, y, z) = (0, 0, \frac{3}{2})$. If $2z^2 - y^2 = 0$, then $y^2 = 2z^2$. Plugging this back into the first equation of (377) yields

$$12z^2 = 9 \to z = \pm\frac{\sqrt{3}}{2}, \tag{379}$$

so we obtain the solutions

$$(x, y, z) \in \{(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \tag{380}$$

For our second subcase we let $x = -\sqrt{2}y$ and a similar calculation yields the additional solutions

$$(x, y, z) \in \{(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \tag{381}$$

Now that we have found all solutions to the system of equations in (373), we see that

(x,y,z)	f(x,y,z)		(x,y,z)	f(x,y,z)
$(0,0,\frac{3}{2})$	0		$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,\frac{3}{\sqrt{2}},0)$	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
(3,0,0)	0		$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,0,-\frac{3}{2})$	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,-\frac{3}{\sqrt{2}},0)$	0		$(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
(-3,0,0)	0		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$		$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$

In conclusion, we see that the absolute minimum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $-\frac{3\sqrt{3}}{2\sqrt{2}}$ and the absolute maximum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $\frac{3\sqrt{3}}{2\sqrt{2}}$.

. . . .

Finish 2: We will now use the symmetry that appears in the system of equations in (373) in order to solve the system more quickly. Observe that

$$\begin{array}{rcl} x^2+2y^2+4z^2-9 &=& 0 \\ yz &=& 2\lambda x \\ xz &=& 4\lambda y \\ xy &=& 8\lambda z \end{array} \rightarrow \begin{array}{rcl} x^2+2y^2+4z^2-9 &=& 0 \\ xyz &=& 2\lambda x^2 \\ xyz &=& 4\lambda y^2 \\ xyz &=& 8\lambda z^2 \end{array} \tag{382}$$

$$\rightarrow \lambda x^2 = 2\lambda y^2 = 4\lambda z^2. \tag{383}$$

We now have 2 cases to consider based on whether or not $\lambda = 0$.

Case 1 ($\lambda = 0$): In this case, we plug $\lambda = 0$ into the system of equations appearing in the left hand portion of (382) (the original system of equations that we started with) to see that

$$(384)\qquad \begin{aligned} x^2+2y^2+4z^2-9 &= 0\\ yz &= 0\\ xz &= 0\\ xy &= 0 \end{aligned} \rightarrow (x,y,z)\in\{(x,0,0),(0,y,0),(0,0,z)\}.$$

$$(385)\qquad \rightarrow (x,y,z)\in\{(\pm 3,0,0),(0,\pm\frac{3}{\sqrt{2}},0),(0,0,\pm\frac{3}{2})\}.$$

Case 2 ($\lambda \neq 0$): In this case, we see that we can divide the equations appearing in (383) by λ and plug to result back into our constraint equation to obtain

$$(386)\qquad x^2=2y^2=4z^2\rightarrow 9=x^2+2y^2+4z^2=3x^2\rightarrow x=\pm\sqrt{3}, \text{ and}$$

$$(387)\qquad (x,y,z)\in\{(x,\frac{x}{\sqrt{2}},\frac{x}{2}),(x,-\frac{x}{\sqrt{2}},\frac{x}{2}),(x,\frac{x}{\sqrt{2}},-\frac{x}{2}),(x,-\frac{x}{\sqrt{2}},-\frac{x}{2})\}.$$

Putting together all of our results from cases 1 and 2, we once again find all solutions to the system of equations in (382) as

(x,y,z)	f(x,y,z)		(x,y,z)	f(x,y,z)
$(0,0,\frac{3}{2})$	0		$(\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,\frac{3}{\sqrt{2}},0)$	0		$(\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
(3,0,0)	0		$(\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,0,-\frac{3}{2})$	0		$(-\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(0,-\frac{3}{\sqrt{2}},0)$	0		$(-\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
(-3,0,0)	0		$(-\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$		$(-\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$

In conclusion, we see that the absolute minimum value of $f(x,y,z)$ subject to $g(x,y,z)=0$ is $-\frac{3\sqrt{3}}{2\sqrt{2}}$ and the absolute maximum value of $f(x,y,z)$ subject to $g(x,y,z)=0$ is $\frac{3\sqrt{3}}{2\sqrt{2}}$.

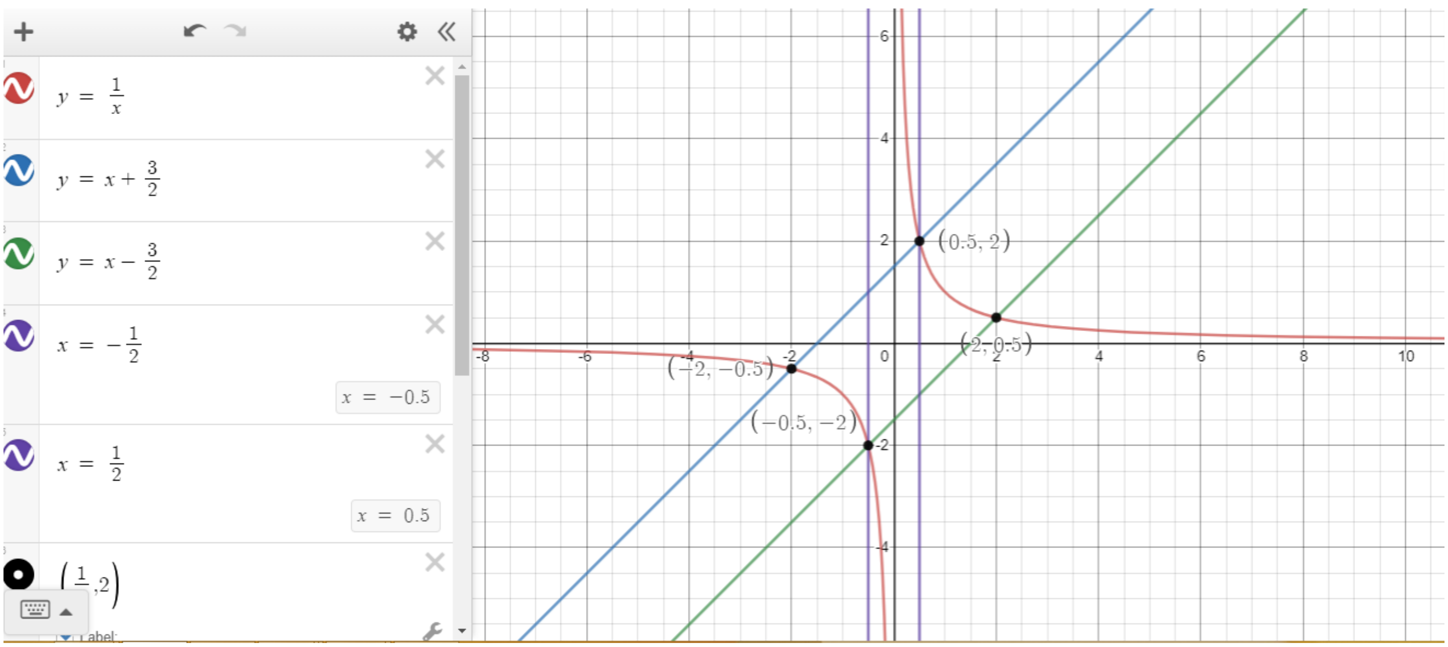
Problem 4.6: Let R be the region that is bounded by both branches of $y = \frac{1}{x}$, the line $y = x + \frac{3}{2}$, and the line $y = x - \frac{3}{2}$.

(a) Find the area of R.

(b) Evaluate

(388) $$\iint_R xy\,dA.$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x + \frac{3}{2}$ to see that

(389) $$\begin{aligned} y &= \frac{1}{x} \\ y &= x + \frac{3}{2} \end{aligned} \rightarrow \frac{1}{x} = x + \frac{3}{2} \rightarrow x^2 + \frac{3}{2}x - 1 = 0$$

(390) $$\rightarrow x = -2, \frac{1}{2} \rightarrow (x, y) = (-2, -\frac{1}{2}), (\frac{1}{2}, 2).$$

Similarly, we solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x - \frac{3}{2}$ to see that

(391) $$\begin{aligned} y &= \frac{1}{x} \\ y &= x - \frac{3}{2} \end{aligned} \rightarrow \frac{1}{x} = x - \frac{3}{2} \rightarrow x^2 - \frac{3}{2}x - 1 = 0$$

(392) $$\rightarrow x = -\frac{1}{2}, 2 \rightarrow (x,y) = (-\frac{1}{2}, -2), (2, \frac{1}{2}).$$

We now see that the area of R is

(393) $$\iint_R 1dA = \iint_R 1dydx$$

(394) $$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} 1dydx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1dydx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} 1dydx$$

(395) $$= \int_{-2}^{-\frac{1}{2}} \left(y\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(y\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}} \right) dx + \int_{\frac{1}{2}}^{2} \left(y\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}} \right) dx$$

(396) $$= \int_{-2}^{-\frac{1}{2}} \left(x + \frac{3}{2} - \frac{1}{x} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x + \frac{3}{2} \right) dx$$

(397) $$\left(\frac{1}{2}x^2 + \frac{3}{2}x - \ln|x| \right) \Big|_{-2}^{-\frac{1}{2}} + 3x\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \left(\ln|x| - \frac{1}{2}x^2 + \frac{3}{2}x \right) \Big|_{\frac{1}{2}}^{2}$$

(398) $$= (1 + 2\ln(2) - \frac{5}{8}) + 3 + (1 + 2\ln(2) - \frac{5}{8}) = \boxed{\frac{15}{4} + 4\ln(2)}.$$

Solution to (b): Using our diagram from part (a) we see that

(399) $$\iint_R xydA = \iint_R xydydx$$

(400) $$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} xydydx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} xydydx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} xydydx$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}xy^2 \Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}xy^2 \Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}} \right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}xy^2 \Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}} \right) dx \tag{401}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(\frac{1}{x})^2 \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(x-\frac{3}{2})^2 \right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}x(\frac{1}{x})^2 - \frac{1}{2}x(x-\frac{3}{2})^2 \right) dx \tag{402}$$

$$= \frac{1}{2}\int_{-2}^{-\frac{1}{2}} \left(x^3 + 3x^2 + \frac{9}{4}x - \frac{1}{x} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3x^2 dx + \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x^3 + 3x^2 - \frac{9}{4}x \right) dx \tag{403}$$

$$= \frac{1}{2}\left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 - \ln|x| \right) \Big|_{-2}^{-\frac{1}{2}} + x^3 \Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{2}\left(\ln|x| - \frac{1}{4}x^4 + x^3 - \frac{9}{8}x^2 \right) \Big|_{\frac{1}{2}}^{2} \tag{404}$$

$$= \boxed{2\ln(2) - \frac{5}{64}} \tag{405}$$

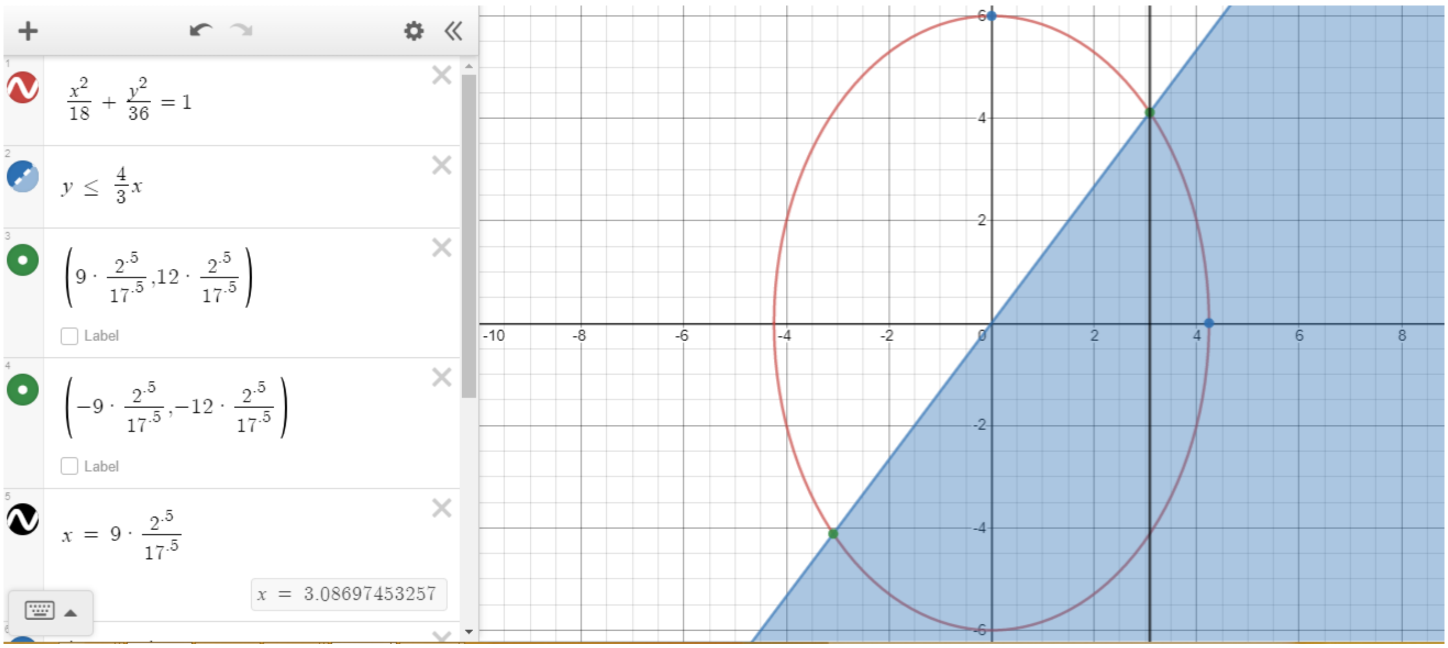
Problem 4.7: Let R be the region inside of the ellipse $\frac{x^2}{18}+\frac{y^2}{36}=1$ for which we also have $y \le \frac{4}{3}x$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xydA. \tag{406}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $\frac{x^2}{18}+\frac{y^2}{36}=1$ and $y=\frac{4}{3}x$. We see that

$$\begin{matrix} \frac{x^2}{18}+\frac{y^2}{36} & = & 1 \\ y & = & \frac{4}{3}x \end{matrix} \to \frac{x^2}{18}+\frac{\frac{16}{9}x^2}{36}=1 \tag{407}$$

$$\to x = \pm\frac{9\sqrt{2}}{\sqrt{17}} \to (x,y) = \left(-\frac{9\sqrt{2}}{\sqrt{17}}, -\frac{12\sqrt{2}}{\sqrt{17}}\right), \left(\frac{9\sqrt{2}}{\sqrt{17}}, \frac{12\sqrt{2}}{\sqrt{17}}\right). \tag{408}$$

We now see that the area of R is

$$\iint_R 1dA = \iint_R 1dydx \tag{409}$$

. . . .

$$(410)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} 1dydx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} 1dydx$$

. . . .

$$(411)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} y\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x}dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} y\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}}dx$$

. . . .

$$(412)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36-2x^2}\right)dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx$$

Since

$$(413)\qquad \int \sqrt{1-x^2} = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) + C, \quad (\text{substitute } x = \sin(\theta))$$

we see that

$$(414)\qquad \int \sqrt{36-2x^2}dx = \int 6\sqrt{1-(\frac{x}{3\sqrt{2}})^2}dx \overset{y=\frac{x}{3\sqrt{2}}}{=} \int 18\sqrt{2}\sqrt{1-y^2}dy$$

. . . .

$$(415)\qquad = 9\sqrt{2}y\sqrt{1-y^2} + 9\sqrt{2}\sin^{-1}(y) = \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}}).$$

Applying this result to equation (412), we see that

$$(416)\qquad \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{4}{3}x + \sqrt{36-2x^2}\right)dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx$$

. . . .

$$(417) \quad = \left(\frac{2}{3}x^2 + \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right)\Big|_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} + \left(x\sqrt{36-2x^2} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}$$

. . . .

$$(418) \quad 2\left(\frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}$$

. . . .

$$(419) \quad x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} - x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{3\sqrt{2}} - 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}$$

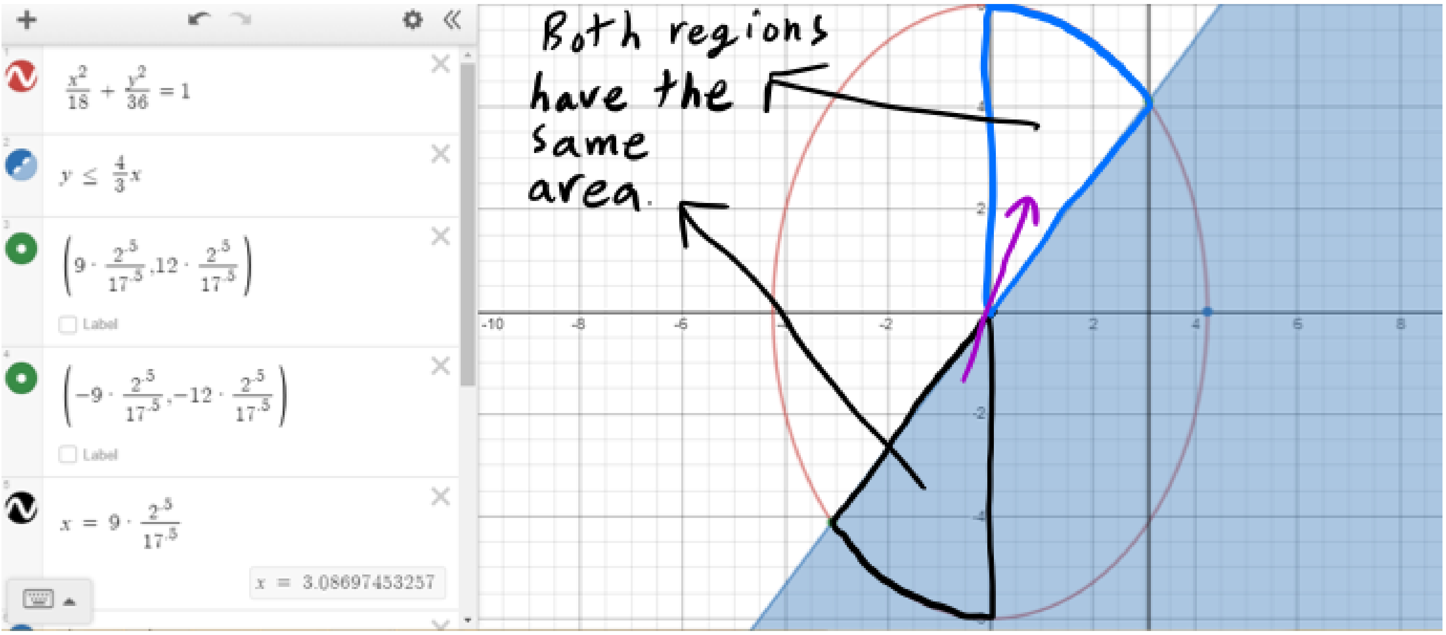
. . . .

$$(420) \quad = x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}\left(\frac{x}{3\sqrt{2}}\right)\Big|_{3\sqrt{2}}$$

. . . .

$$(421) \quad = 0 + 18\sqrt{2}\sin^{-1}(1) = \boxed{9\sqrt{2}\pi}.$$

Remark: For the ellipse $\frac{y^2}{36} + \frac{x^2}{18} = 1$ we see that the major radius is 6 and the minor radius is $3\sqrt{2}$, so the area of the ellipse is $6 \cdot 3\sqrt{2} \cdot \pi = 18\sqrt{2}\pi$. We now see that our region R has half the area of the ellipse containing it. In fact, we can prove this directly with symmetry and no calculus at all! We just have to remember that when we reflect the point (x, y) across the origin we get the point $(-x, -y)$, and that reflection across the origin (or reflection across any other point) preserves area as shown in the picture below.



Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xy dA = \iint_R xy dy dx \tag{422}$$

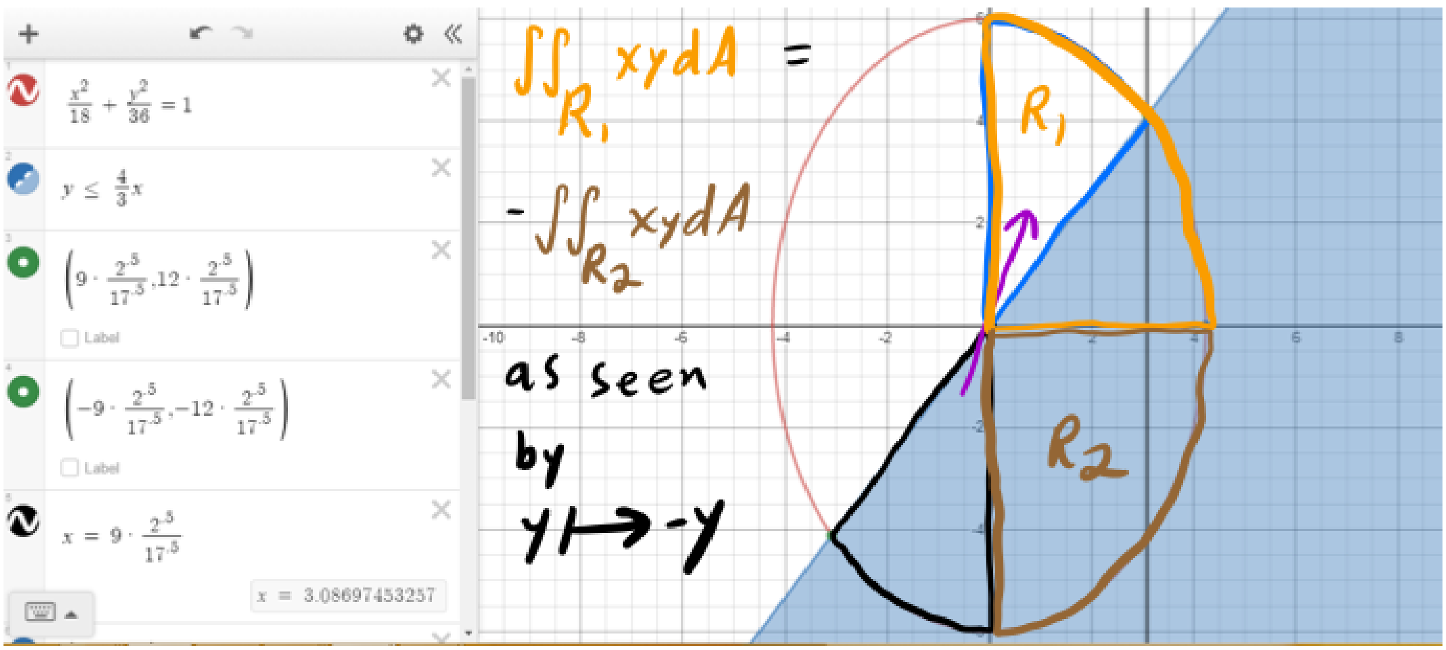
$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} xy dy dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} xy dy dx \tag{423}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx \tag{424}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}x(\frac{4}{3}x)^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}x(\sqrt{36-2x^2})^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx \tag{425}$$

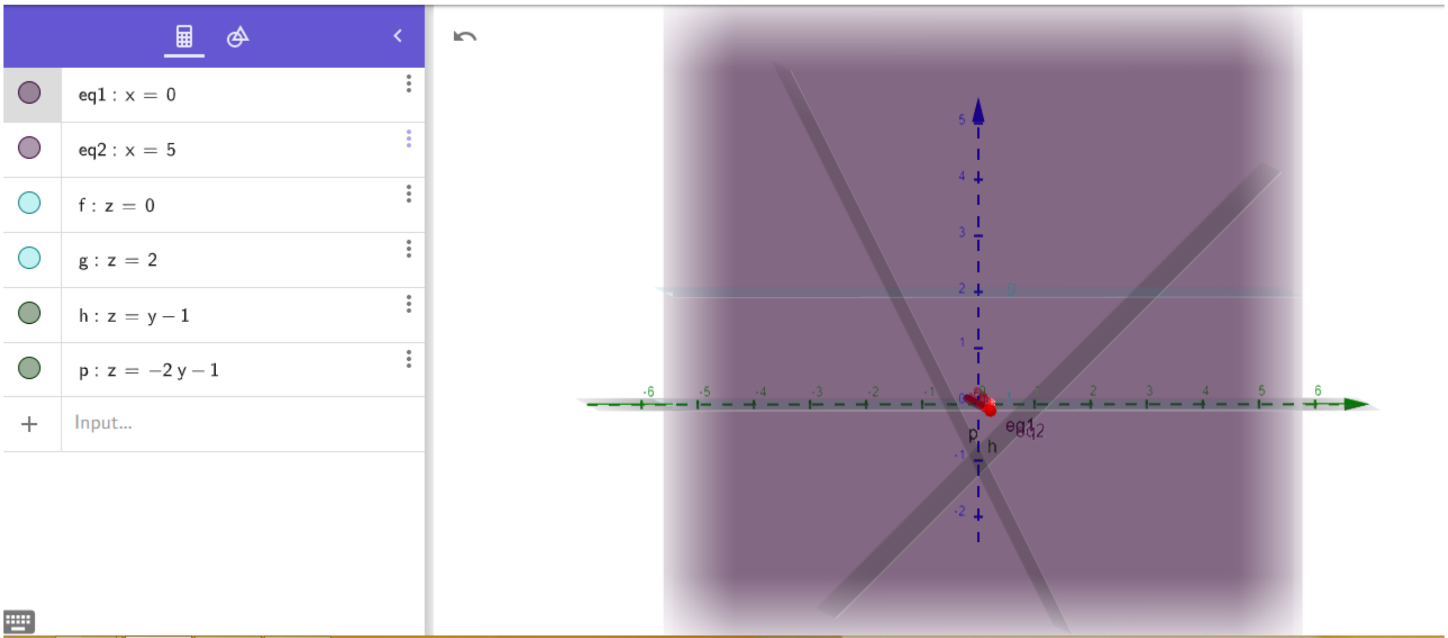
$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{16}{9}x^3 - 18x + x^3\right) dx = \boxed{0}. \tag{426}$$

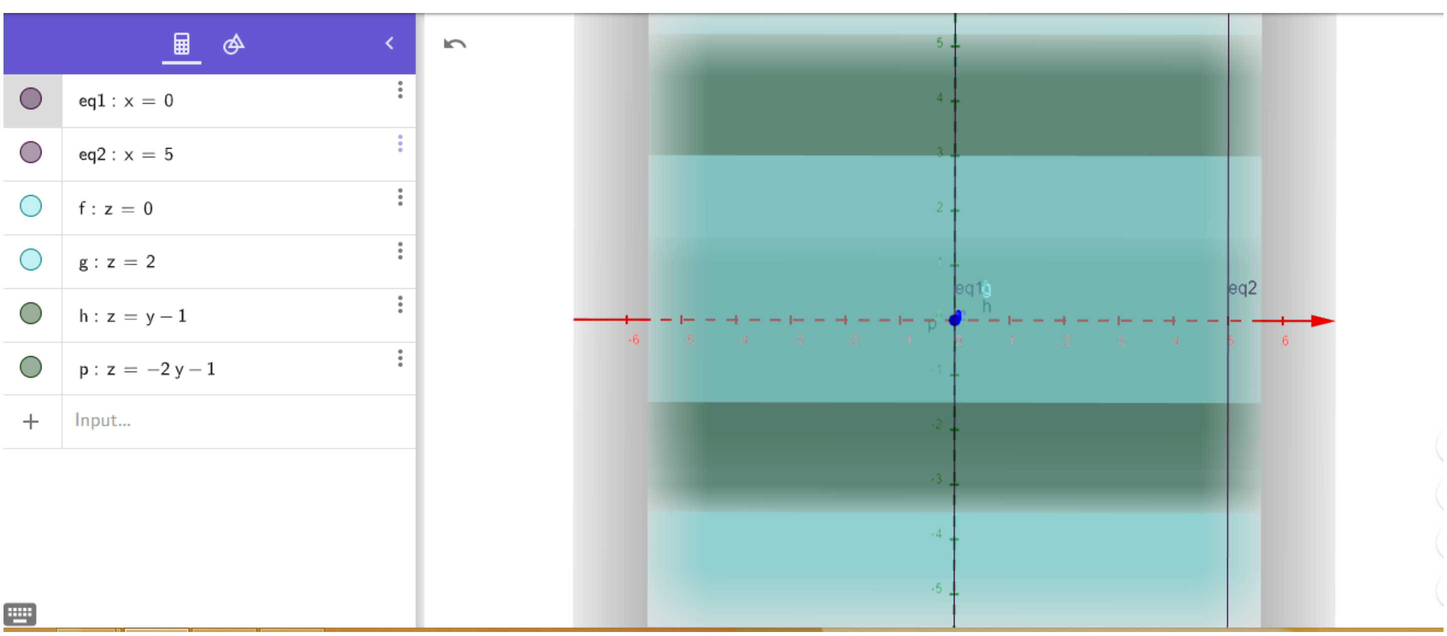
Remark: We see that both integrals appearing in equation (423) are 0. It turns out that this can also be shown directly with symmetry instead of evaluating the integrals! Firstly, we recall that (x, y) turns into $(-x, -y)$ when reflected across the origin and that reflection across the origin preserves area. We also note that $xy = (-x)(-y)$, so we can rewrite our double integral as a double integral that takes place over the right (or left) half of the ellipse instead of the region R. We then notice that $x(-y) = -(xy)$, so the integrals over the top right and lower right quarters of the ellipse cancel each other out to yield 0 as shown in the picture below.

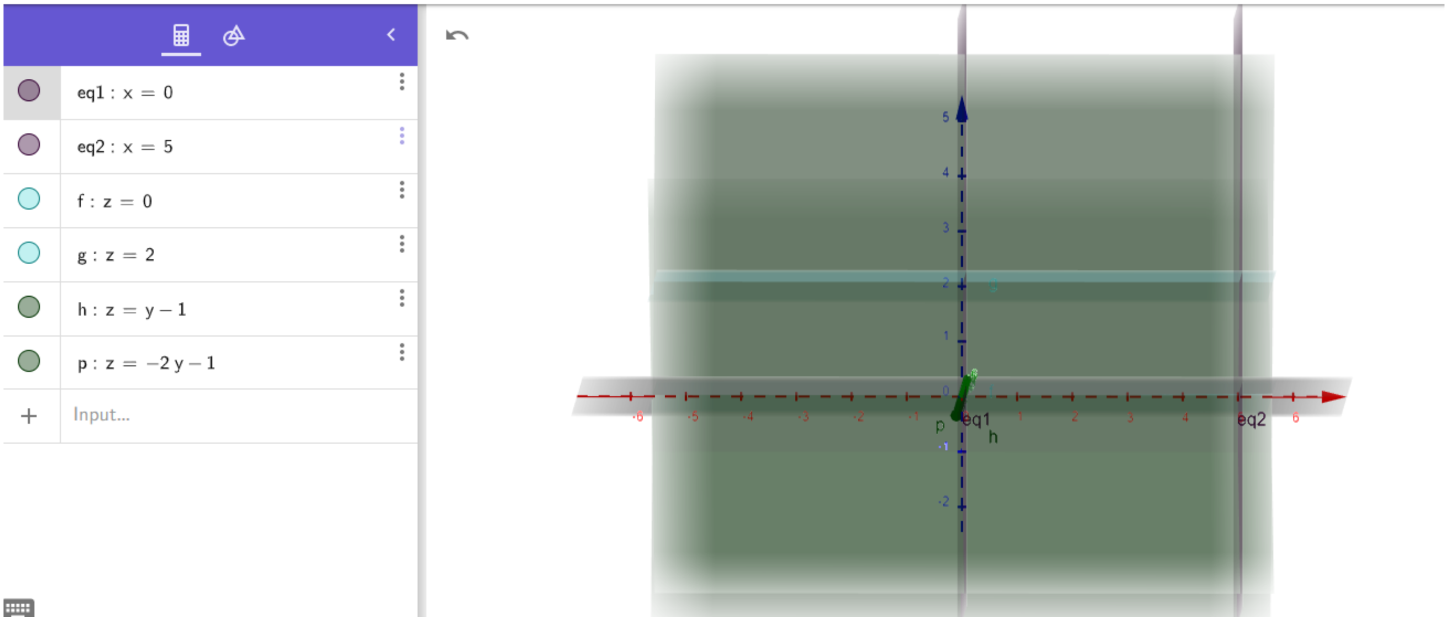


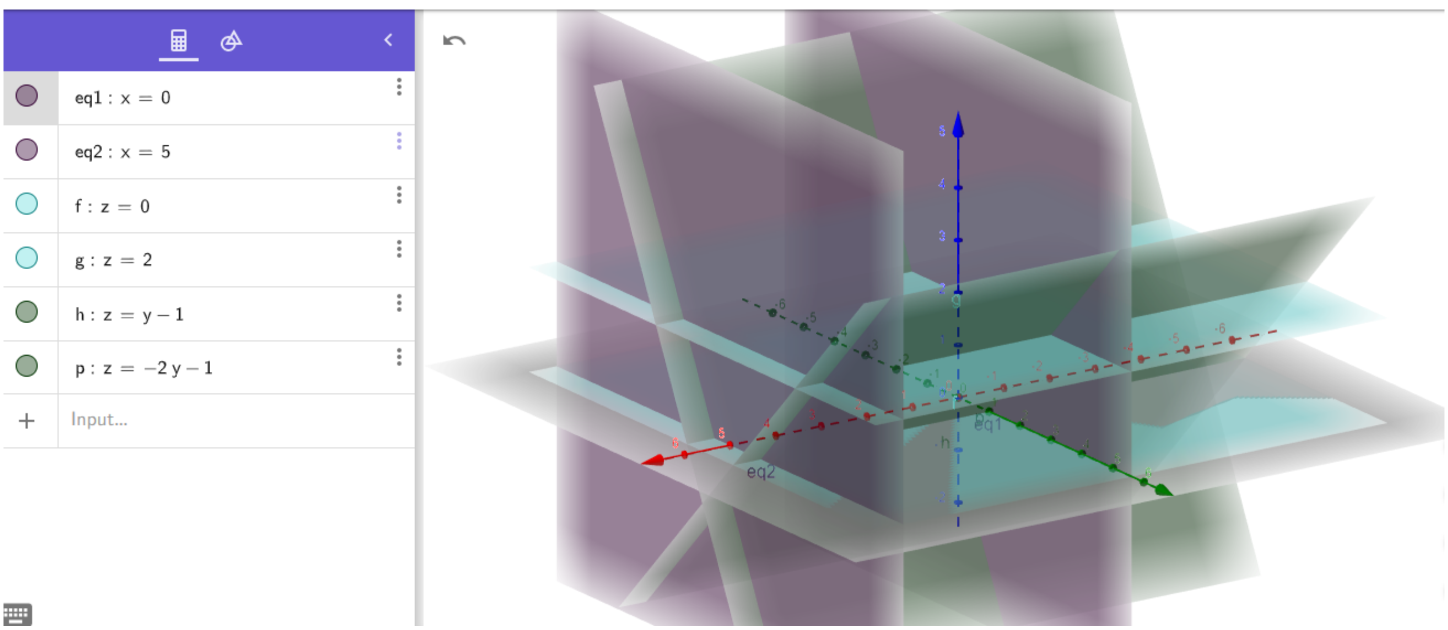
Problem 4.8: Find the volume of the solid bounded by the planes $x = 0$, $x = 5$, $z = y - 1$, $z = -2y - 1$, $z = 0$, and $z = 2$.

Solution: Let us first examine our solid from a few different angles.









Due to the third and fourth pictures, we will choose to view the 'base' of our solid in the xz-plane so that it is simply the rectangle $R = \{(x, z) \in \mathbb{R}^2 \mid 0 \leq x \leq 5,\ 0 \leq z \leq 2\}$. We could also come to this decision simply by examining the x and z bounds without drawing any diagrams. We then see that the 'heights' of our solid are along the y-axis. Solving for y in terms of x and z we see that $y = z + 1$ and $y = -\frac{z+1}{2}$ are the surfaces bounding the 'heights' of our solid. By examining the values of y for some $(x, z) \in R$ (such as $(0, 0)$), we see that $y = z + 1$ is the upper bound for our heights and $y = \frac{z+1}{2}$ is the lower bound for our heights. We now see that the volume V of our solid is given by

$$V = \iint_R (y_{\text{top}} - y_{\text{bottom}})\, dA = \iint_R \left(z + 1 - (-\frac{z+1}{2}) \right) dA \tag{427}$$

$$= \int_0^5 \int_0^2 3\frac{z+1}{2} dz dx = \int_0^5 \left(\frac{3}{4}z^2 + \frac{3}{2}z \right) \Big|_{z=0}^{2} dx = \int_0^5 6 dx = \boxed{30}. \tag{428}$$

Problem 4.9: Let R be the region in the xy-plane that is bounded by the spiral $r = \theta$ for $0 \leq \theta \leq \pi$ and the x-axis. Find the volume of the 3-dimensional solid S that lies above the region R and underneath the surface $z = x^2 + y^2$.

Solution: Below is a picture of the region R, which is the base of our solid S.

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bot.}})dA = \iint_R \underbrace{(x^2+y^2) - 0}_{r^2} \underbrace{dA}_{rdrd\theta} \tag{429}$$

$$= \int_0^\pi \int_0^\theta r^3 drd\theta = \int_0^\pi \frac{1}{4}r^4\Big|_{r=0}^\theta d\theta = \int_0^\pi \frac{1}{4}\theta^4 d\theta \tag{430}$$

$$= \frac{1}{20}\theta^5\Big|_0^\pi = \boxed{\frac{\pi^5}{20}}. \tag{431}$$

Problem 4.10: The limaçon $r = b + a\cos(\theta)$ has an inner loop if $b < a$ and no inner loop if $b > a$.

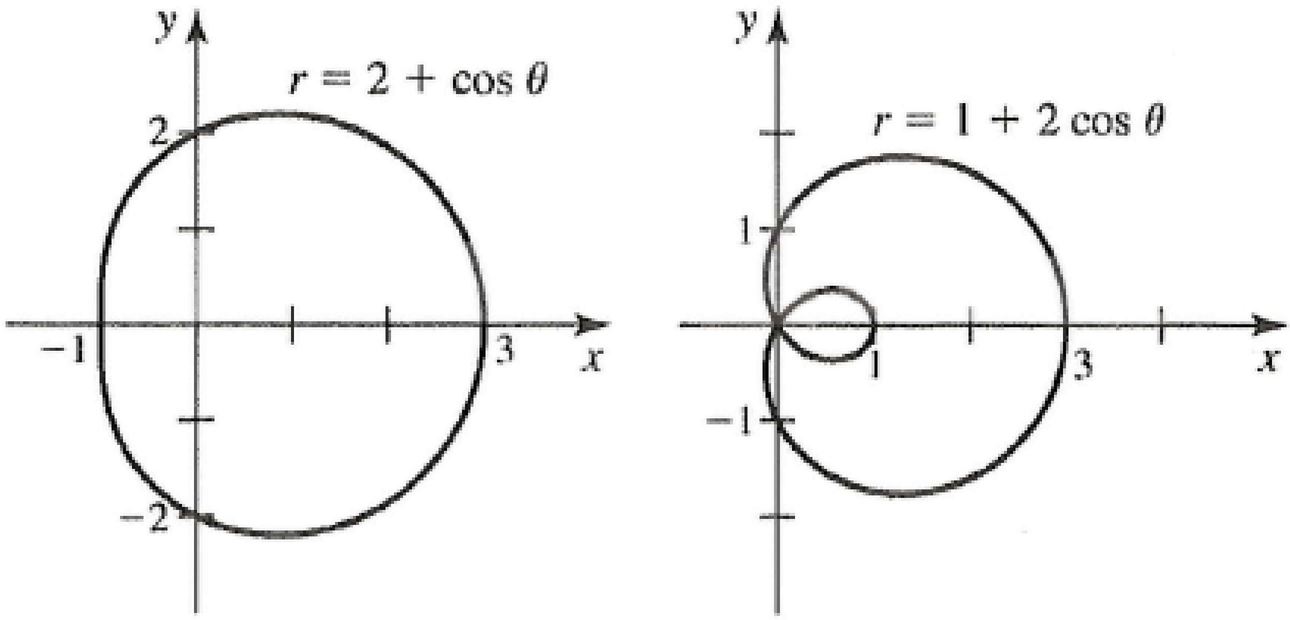


FIGURE 14. From page 139 of the course textbook.

(a) Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.
(b) Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.
(c) Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.

Solution to (a): Letting R denote the interior of the limaçon $r = 2+\cos(\theta)$, we see that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_0^{2\pi}\int_0^{2+\cos(\theta)} rdrd\theta \tag{432}$$

$$= \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{2+\cos(\theta)} d\theta = \int_0^{2\pi} \frac{1}{2}(2+\cos(\theta))^2 d\theta \tag{433}$$

$$= \int_0^{2\pi} (2+2\cos(\theta)+\frac{1}{2}\cos^2(\theta))d\theta = \int_0^{2\pi} (2+2\cos(\theta)+\frac{1}{4}\cos(2\theta)+\frac{1}{4})d\theta \tag{434}$$

$$(\frac{9}{4}\theta + 2\sin(\theta) + \frac{1}{8}\sin(2\theta))\Big|_0^{2\pi} = \boxed{\frac{9}{2}\pi}. \tag{435}$$

Solution to (c): Let R denote the region inside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$. We see that the inner loop of the limaçon begins and ends when $r = 0$, which occurs when $\cos(\theta) = -\frac{1}{2}$, which occurs when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. It follows that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_0^{1+2\cos(\theta)} rdrd\theta \tag{436}$$

. . . .

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}r^2\Big|_{r=0}^{1+2\cos(\theta)} d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1 + 2\cos(\theta))^2 d\theta \tag{437}$$

. . . .

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + 2\cos^2(\theta))d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + \cos(2\theta) + 1)d\theta \tag{438}$$

. . . .

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{\pi - \frac{3}{2}\sqrt{3}}. \tag{439}$$

Solution to (b): Letting R' denote the region inside of the outer loop and outside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$, we see that

$$\text{Area}(R') + 2\text{Area}(R) = \int_0^{2\pi} \int_0^{1+2\cos(\theta)} rdrd\theta \tag{440}$$

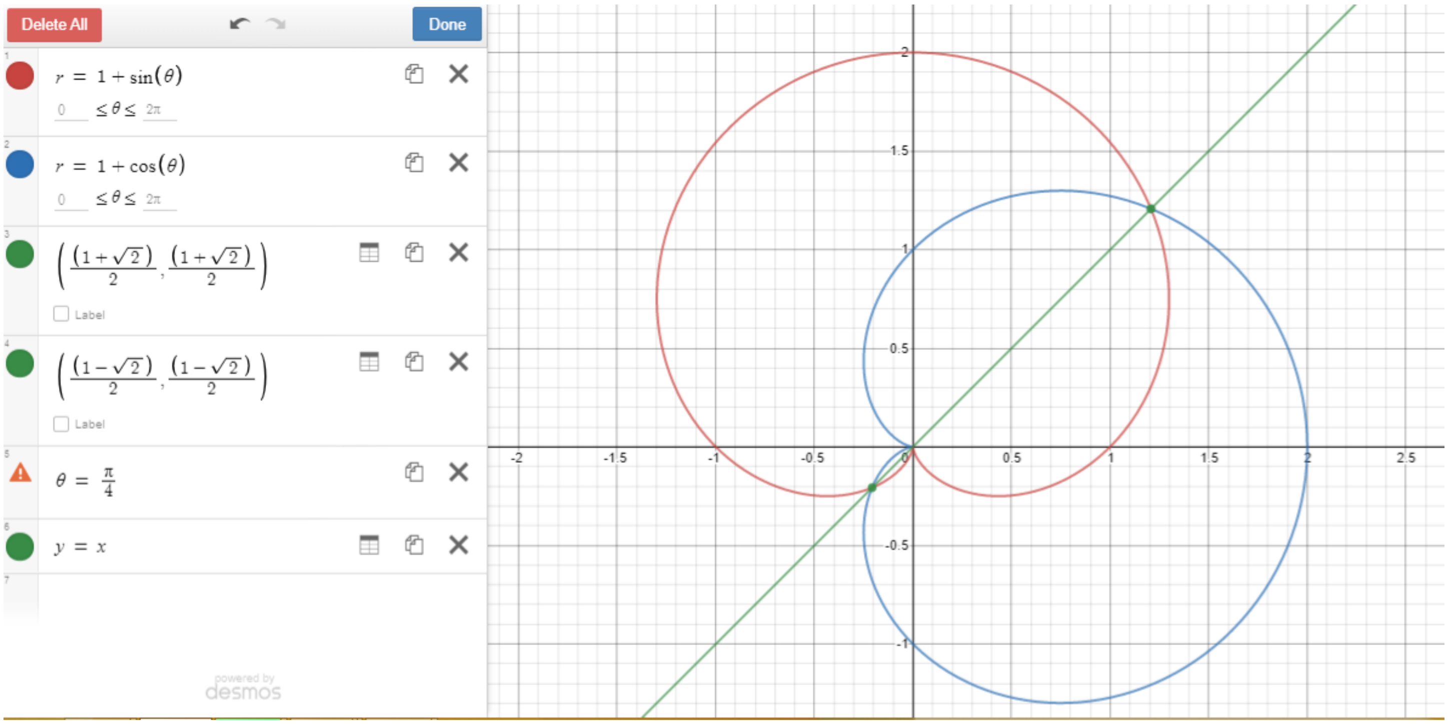
$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_0^{2\pi} = 3\pi. \tag{441}$$

Using our answer from part (c), we see that

$$\text{Area}(R') = 3\pi - 2\text{Area}(R) = 3\pi - 2(\pi - \frac{3}{2}\sqrt{3}) = \boxed{\pi + 3\sqrt{3}}. \tag{442}$$

Problem 4.11: Let R be the region inside both the cardiod $r = 1 + \sin(\theta)$ and the cardiod $r = 1 + \cos(\theta)$. Sketch a picture of the region R, or create an image of the region R using a graphing program, then use double integration to find the area of R.

Solution: We begin by drawing a picture of the region R.



We see that the 2 cardiods intersect when $\sin(\theta) = \cos(\theta)$, which occurs when $\theta = \frac{\pi}{4}, -\frac{3\pi}{4}$. We now see that when $-\frac{3\pi}{4} \leq \theta \leq \frac{\pi}{4}$ we have $1 + \sin(\theta) \leq 1 + \cos(\theta)$ and when $\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}$ we have $1 + \cos(\theta) \leq 1 + \sin(\theta)$. It follows that

$$\text{(443)} \quad \text{Area}(R) = \iint_R 1dA = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_0^{1+\sin(\theta)} rdrd\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{1+\cos(\theta)} rdrd\theta$$

$$\text{(444)} \quad = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\sin(\theta)} d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}r^2\Big|_{r=0}^{1+\cos(\theta)} d\theta$$

$$\text{(445)} \quad = \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \sin^2(\theta))d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \cos^2(\theta))d\theta$$

. . . .

$$= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(1 + 2\sin(\theta) + \frac{1-\cos(2\theta)}{2})d\theta + \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{2}(1 + 2\cos(\theta) + \frac{1+\cos(2\theta)}{2})d\theta \tag{446}$$

. . . .

$$= \left(\frac{3}{4}\theta - \cos(\theta) + \frac{-\sin(2\theta)}{4}\Big|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}}\right) + \left(\frac{3}{4}\theta + \sin(\theta) + \frac{\sin(2\theta)}{4}\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\right) \tag{447}$$

. . . .

$$= \boxed{\frac{3\pi}{2} - 2\sqrt{2}}. \tag{448}$$

Problem 4.12: Evaluate

$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} dydx \tag{449}$$

by changing the order of integration.

Hint: Start by drawing a picture of the region of integration.

Solution: We change the order of integration as shown in the pictures below.

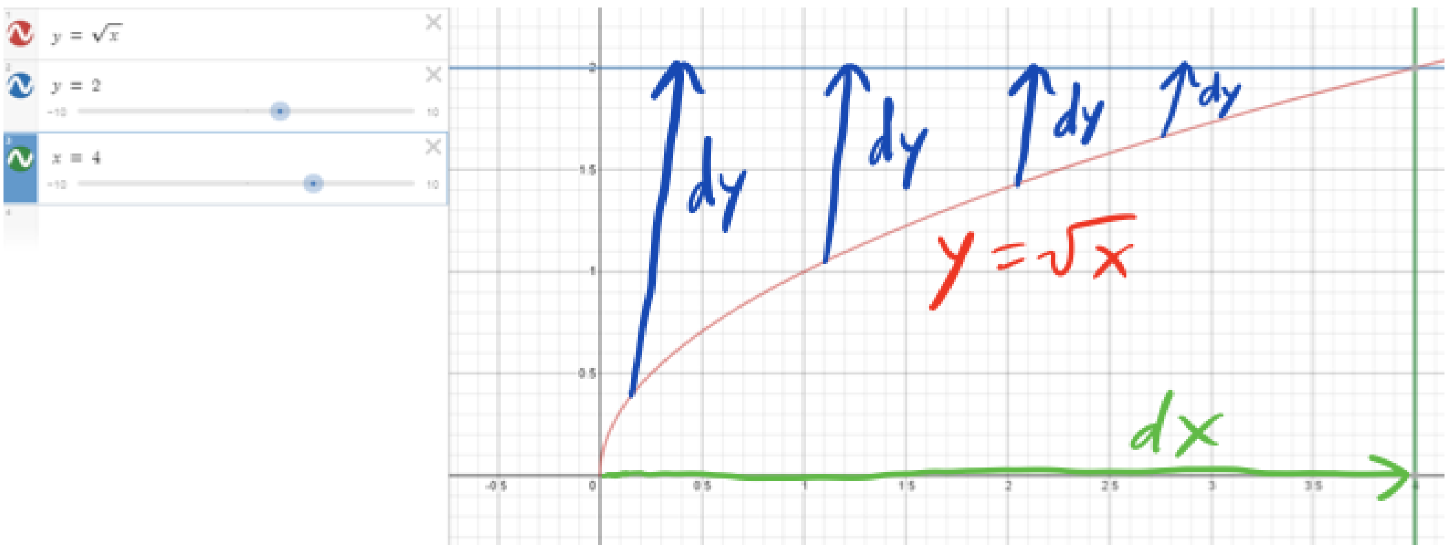


FIGURE 15. Traversing the region of integration when dA = dydx.

. . . .

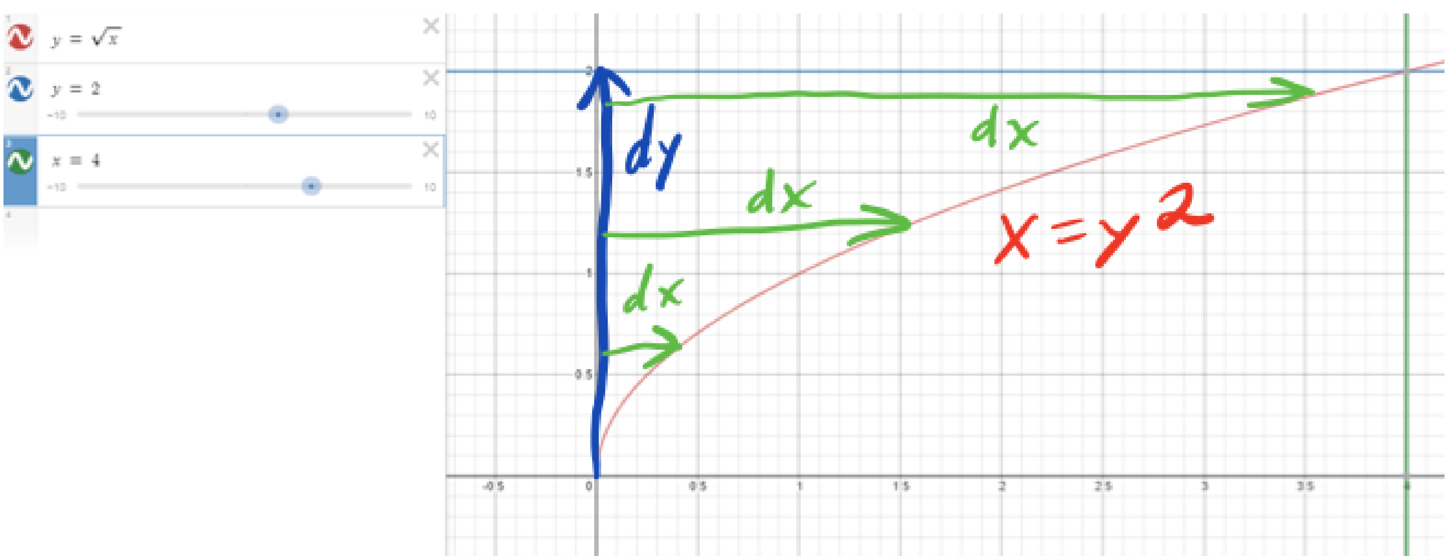


FIGURE 16. Traversing the region of integration when dA = dxdy.

(450)
$$\int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5+1} dydx = \int_0^2 \int_0^{y^2} \frac{x}{y^5+1} dxdy = \int_0^2 \left(\frac{x^2}{2(y^5+1)} \Big|_{x=0}^{y^2} \right) dy$$

(451)
$$= \int_0^2 \frac{y^4}{2y^5+2} dy \overset{u=y^5}{=} \int_{y=0}^2 \frac{1}{2u+2} \frac{du}{5} = \frac{1}{10} \ln(u+1) \Big|_{y=0}^2$$

(452)
$$= \frac{1}{10} \ln(y^5+1) \Big|_{y=0}^2 = \boxed{\frac{\ln(33)}{10}}.$$

Problem 6.2: Let R be the region in the first quadrant bounded by the hyperbolas $xy = 1$ and $xy = 4$ and the lines $y = x$ and $y = 3x$. Evaluate

$$\iint_R y^4 dA. \tag{453}$$

Note that you can also solve this problem in Cartesian coordinates and polar coordinates, not just a change of variables. Try solving it with all three methods and compare their difficulties!

Solution 1: Our first solution will use a change of variables. Noting that the line $y = x$ can be rewritten as $\frac{y}{x} = 1$ and the line $y = 3x$ can be rewritten as $\frac{y}{x} = 3$, we decide to use the change of variables $u = xy$ and $v = \frac{y}{x}$ in order to make our new region of integration in the uv-place a rectangle. In particular, we see that $R' = \{(u, v) : 1 \leq u \leq 4, 1 \leq v \leq 3\}$ is the new region of integration.

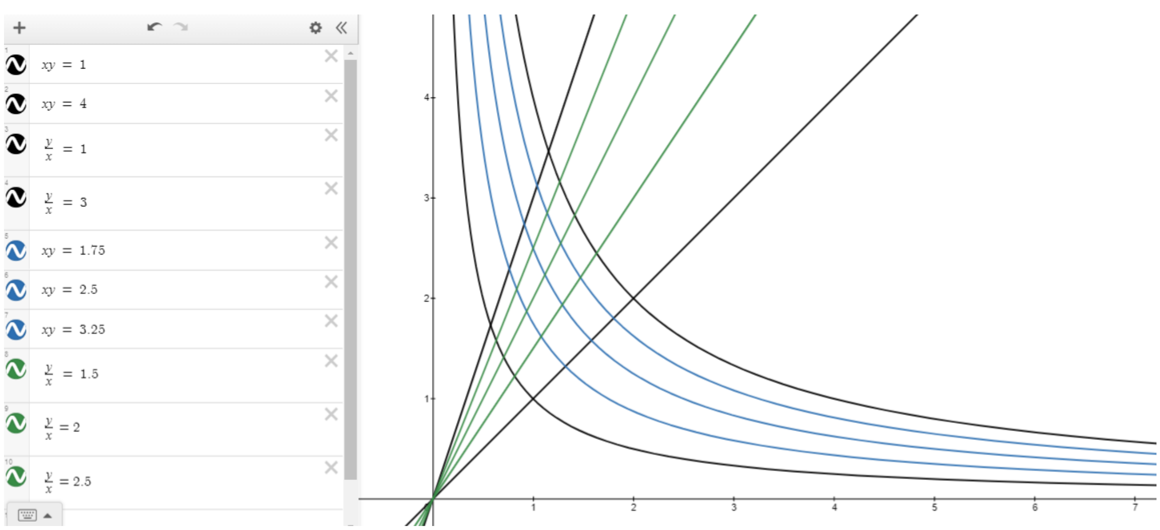


FIGURE 17. The original region of integration in the xy-plane R.

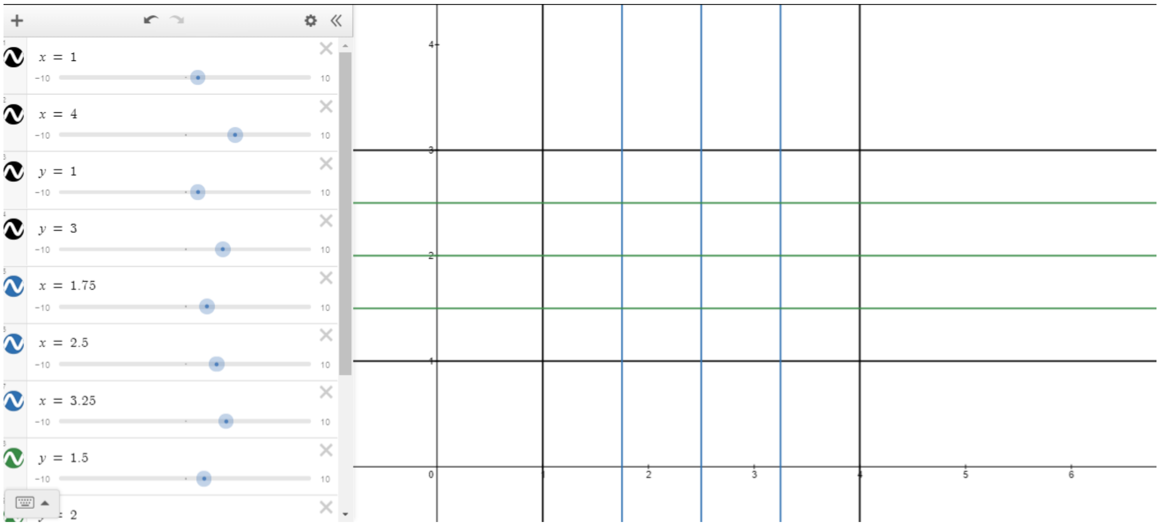


FIGURE 18. The new region of integration in the uv-plane R'.

In order to calculate the Jacobian $J(u,v)$ we must first solve for x and y in terms of u and v. To that end, we see that

$$\begin{aligned} u &= xy \\ v &= \frac{y}{x} \end{aligned} \rightarrow x = (x^2)^{\frac{1}{2}} = (\frac{u}{v})^{\frac{1}{2}} = u^{\frac{1}{2}}v^{-\frac{1}{2}} \text{ and } y = (y^2)^{\frac{1}{2}} = u^{\frac{1}{2}}v^{\frac{1}{2}}. \tag{454}$$

We note that we took the positive square roots above since we are working in the first quadrant of the xy-place, so x and y are both nonnegative. We now see that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} \\ \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix} \tag{455}$$

$$= \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} \cdot \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} - (-\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}}) \cdot \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} = \frac{1}{2}v^{-1}. \tag{456}$$

Since $1 \leq v \leq 3$ in our new region of integration R', we see that $\frac{1}{2}v^{-1} \geq 0$ on R', so $|J(u,v)| = J(u,v)$ on R'. We now see that

$$\iint_R y^4 dA = \iint_{R'} (u^{\frac{1}{2}}v^{\frac{1}{2}})^4 |J(u,v)| dA = \int_1^4 \int_1^3 u^2v^2 \cdot \frac{1}{2}v^{-1} dvdu \tag{457}$$

$$= \frac{1}{2}\int_1^4 \int_1^3 u^2 v dvdu = \frac{1}{2}\int_1^4 \frac{1}{2}u^2v^2\Big|_{v=1}^3 du \tag{458}$$

$$= \int_1^4 2u^2 du = \frac{2}{3}u^3\Big|_1^4 = \boxed{42}. \tag{459}$$

Solution 2: Our next solution will use polar coordinates. We begin by observing that in the first quadrant we have

$$y = x \Leftrightarrow r\sin(\theta) = r\cos(\theta) \Leftrightarrow \sin(\theta) = \cos(\theta) \Leftrightarrow \theta = \frac{\pi}{4}, \tag{460}$$

$$y = 3x \Leftrightarrow r\sin(\theta) = 3r\cos(\theta) \Leftrightarrow \tan(\theta) = 3 \Leftrightarrow \theta = \tan^{-1}(3), \tag{461}$$

$$1 = xy = r^2\cos(\theta)\sin(\theta) \Leftrightarrow r = \sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}, \text{ and} \tag{462}$$

$$4 = xy = r^2\cos(\theta)\sin(\theta) \Leftrightarrow r = \sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}. \tag{463}$$

It follows that

$$\iint_R y^4 dA = \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \int_{\sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}}^{\sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}} (r\sin(\theta))^4 r\,dr\,d\theta \tag{464}$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \int_{\sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}}^{\sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}} r^5\sin^4(\theta)\,dr\,d\theta \tag{465}$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \left(\frac{1}{6}r^6\sin^4(\theta)\Big|_{\sqrt{\frac{1}{\cos(\theta)\sin(\theta)}}}^{\sqrt{\frac{4}{\cos(\theta)\sin(\theta)}}}\right) d\theta \tag{466}$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \frac{21}{2}\frac{\sin(\theta)}{\cos^3(\theta)}d\theta = \frac{21}{2}\int_{\frac{\pi}{4}}^{\tan^{-1}(3)} \underbrace{\frac{\sin(\theta)}{\cos(\theta)}}_{u=\tan(\theta)} \cdot \underbrace{\frac{1}{\cos^2(\theta)}d\theta}_{du=\sec^2(\theta)d\theta} \tag{467}$$

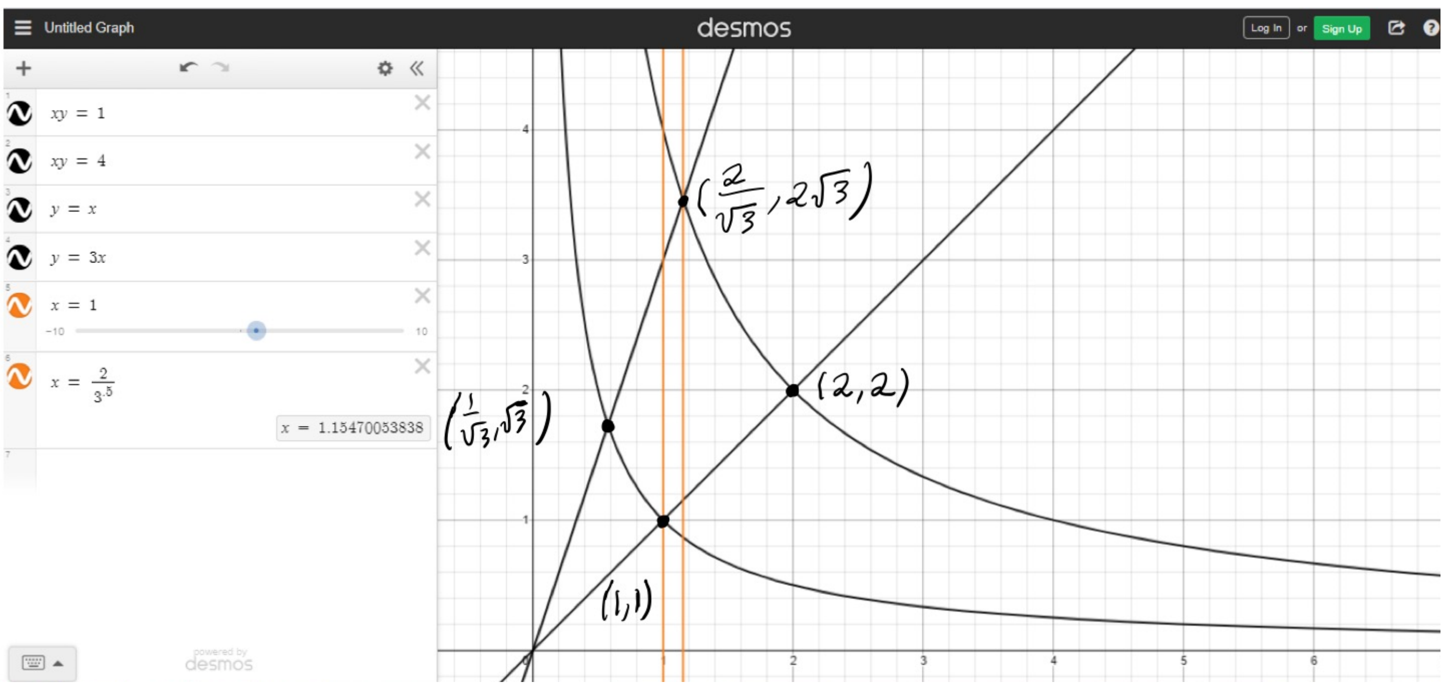
. . . .

$$\frac{21}{2}\left(\frac{1}{2}tan^2(\theta)\Big|_{\frac{\pi}{4}}^{\tan^{-1}(3)}\right) = \frac{21}{4}(3^2 - 1^2) = \boxed{42}. \tag{468}$$

Solution 3: Our last solution will use Cartesian coordinates. We begin by observing that in the first quadrant we have

$$\begin{array}{lllllllll} xy & = & 1 & \text{and} & y & = & x & \rightarrow & (x,y) & = & (1,1) \\ xy & = & 1 & \text{and} & y & = & 3x & \rightarrow & (x,y) & = & (\frac{1}{\sqrt{3}}, \sqrt{3}) \\ xy & = & 4 & \text{and} & y & = & x & \rightarrow & (x,y) & = & (2,2) \\ xy & = & 4 & \text{and} & y & = & 3x & \rightarrow & (x,y) & = & (\frac{2}{\sqrt{3}}, 2\sqrt{3}) \end{array} \tag{469}$$

Now that we have identified the 'corners' of our region as shown in the picture below, we are able to set up and evaluate the desired double integral.



$$\iint_R y^4 dA = \int_{\frac{1}{\sqrt{3}}}^{1}\int_{\frac{1}{x}}^{3x} y^4 dydx + \int_{1}^{\frac{2}{\sqrt{3}}}\int_{x}^{3x} y^4 dydx + \int_{\frac{2}{\sqrt{3}}}^{2}\int_{x}^{\frac{4}{x}} y^4 dydx \tag{470}$$

. . . .

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{5}y^5\Big|_{y=\frac{1}{x}}^{3x} dx + \int_{1}^{\frac{2}{\sqrt{3}}} \frac{1}{5}y^5\Big|_{y=x}^{3x} dx + \int_{\frac{2}{\sqrt{3}}}^{2} \frac{1}{5}y^5\Big|_{y=x}^{\frac{4}{x}} dx \tag{471}$$

. . . .

$$= \frac{1}{5}\Big(\int_{\frac{1}{\sqrt{3}}}^{1}(243x^5 - x^{-5})dx + \int_{1}^{\frac{2}{\sqrt{3}}} 242x^5dx + \int_{\frac{2}{\sqrt{3}}}^{2}(1024x^{-5} - x^5)dx\Big) \tag{472}$$

. . . .

$$= \frac{1}{5}\Big((\frac{81}{2}x^6 + \frac{1}{4}x^{-4}\Big|_{\frac{1}{\sqrt{3}}}^{1}) + (\frac{121}{3}x^6\Big|_{1}^{\frac{2}{\sqrt{3}}}) + (-256x^{-4} - \frac{1}{6}x^6\Big|_{\frac{2}{\sqrt{3}}}^{2})\Big) \tag{473}$$

. . . .

$$\frac{1}{5}\Big((\frac{81}{2} + \frac{1}{4} - \frac{3}{2} - \frac{9}{4}) + (\frac{121}{3}\cdot(\frac{64}{27} - 1)) + (-16 - \frac{32}{3} + 144 + \frac{32}{81})\Big) \tag{474}$$

. . . .

$$\frac{1}{5}\Big(37 + \frac{121\cdot 37}{81} + 128 - \frac{26\cdot 32}{81}\Big) = \boxed{42}. \tag{475}$$

Problem 4.13: Find the volume of the solid S bounded by the paraboloid $z = 8 - x^2 - 3y^2$ and the hyperbolic paraboloid $z = x^2 - y^2$.

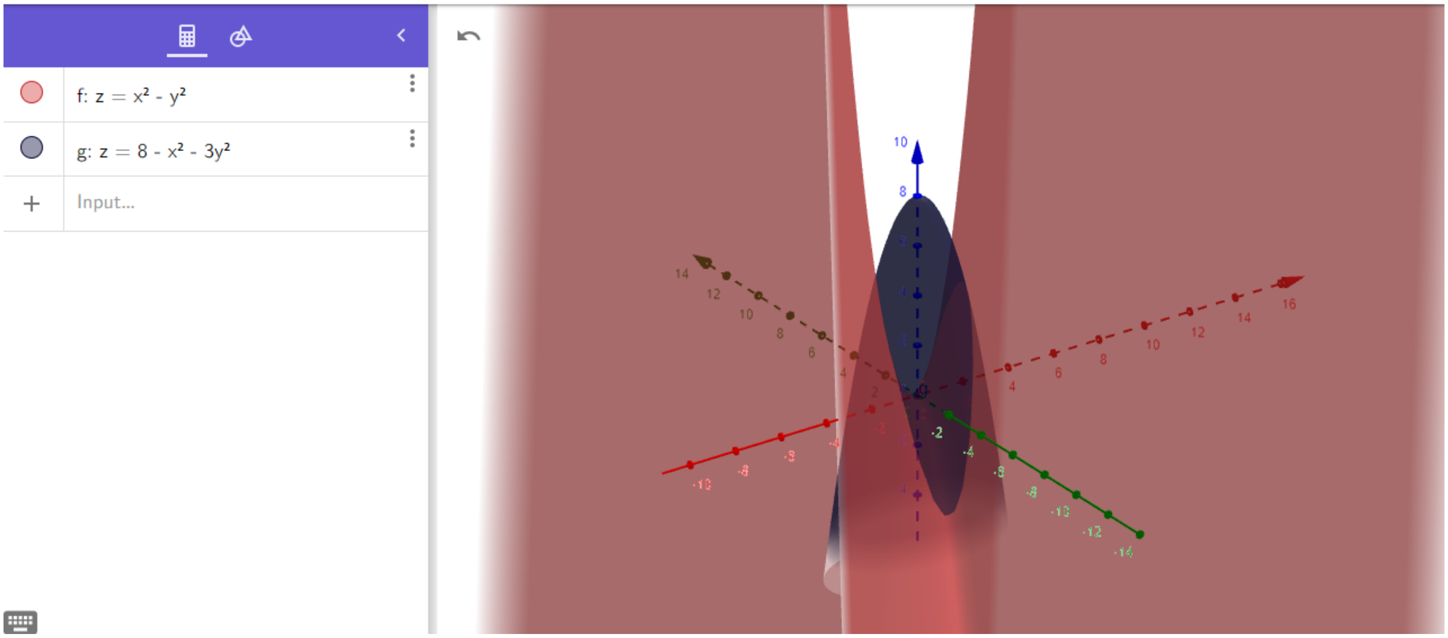


FIGURE 19. A view of the solid S whose volume we are calculating.

Solution: We begin by finding the (x, y)-coordinates of the curves of intersection of the 2 given surfaces. We see that

$$8 - x^2 - 3y^2 = z = x^2 - y^2 \rightarrow 2x^2 + 2y^2 = 8 \rightarrow x^2 + y^2 = 4, \tag{476}$$

so the (x, y)-coordinates of the curve of intersection is simply the circle of radius 2 centered at the origin.

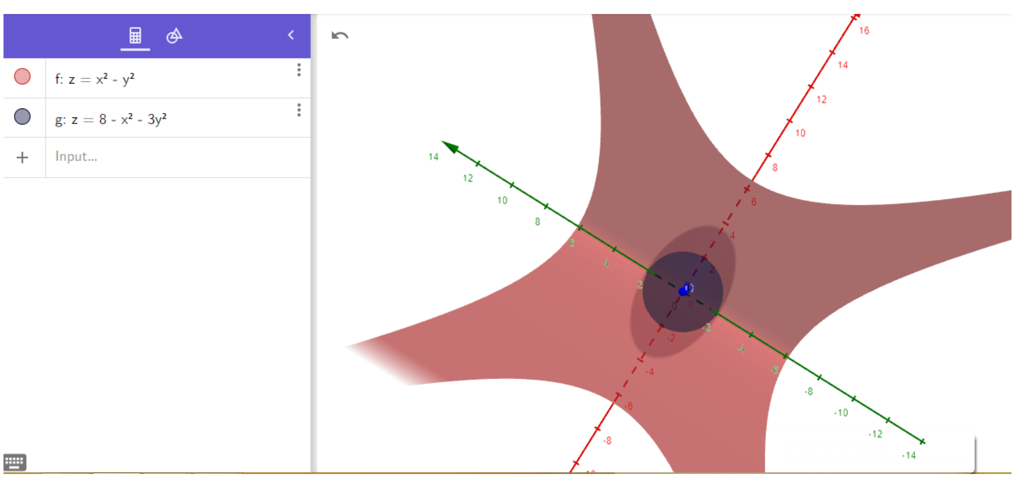


FIGURE 20. A bird's eye view of the solid S that is used to find the region of integration R.

Noting that $8-0^2-3\cdot 0^2 = 8 > 0 = 0^2-0^2$, we see that the curve $z = 8-x^2-3y^2$ lies above the curve $z = x^2 - y^2$ for all (x, y) inside of R, the disc of radius 2 centered at the origin. We now see that

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bot.}})dA = \iint_R \Big((8-x^2-3y^2)-(x^2-y^2)\Big)dA \tag{477}$$

$$= \iint_R (8 - 2x^2 - 2y^2)dA = \int_0^{2\pi}\int_0^2 (8 - 2r^2)r\,dr\,d\theta \tag{478}$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^2 (8r - 2r^3)dr\right) = (2\pi)\left(4r^2 - \frac{1}{2}r^4\Big|_0^2\right) = \boxed{16\pi}. \tag{479}$$

Problem 5.1: Write an iterated integral for $\iiint_D f(x,y,z)dV$, where D is a sphere of radius 9 centered at $(0,0,1)$. Use the order $dV = dzdydx$.

Hint: Start by finding the equation of the the surface of the sphere of radius 9 *centered at* $(0,0,1)$.

Solution: We recall that the equation of the sphere of radius R centered at (a,b,c) is given by

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2, \tag{480}$$

so the equation of the sphere of radius 9 centered at $(0,0,1)$ is given by

$$x^2+y^2+(z-1)^2=81. \tag{481}$$

Since we are considering

$$\iiint_D f(x,y,z)dV=\int_?^?\int_?^?\int_?^? f(x,y,z)dzdydx, \tag{482}$$

we begin by observing that the smallest possible value of x in our region D is -9, and the largest possible value of x in our region D is 9, so $-9 \leq x \leq 9$. We then observe that for each $x \in [-9,9]$, we have

$$y^2+(z-1)^2=81-x^2, \tag{483}$$

so the smallest possible value of y in our region D (corresponding to our chosen value of x) is $-\sqrt{81-x^2}$ and the largest possible value of y in our region D (corresponding to our chosen value of x) is $\sqrt{81-x^2}$, so $-\sqrt{81-x^2} \leq y \leq \sqrt{81-x^2}$. Lastly, we observe that for each $x \in [-9,9]$ and each $y \in [-\sqrt{81-x^2},\sqrt{81-x^2}]$, we have

$$(z-1)^2=81-x^2-y^2 \to z=1\pm\sqrt{81-x^2-y^2}, \tag{484}$$

so the smallest possible value of z in our region D (corresponding to our chosen values of x and y) is $1-\sqrt{81-x^2-y^2}$ and the largest possible value of y in our region D (corresponding to our chosen values of x and y) is $1+\sqrt{81-x^2-y^2}$, so $1-\sqrt{81-x^2-y^2} \leq z \leq 1+\sqrt{81-x^2-y^2}$. It follows that we can describe our region D as

$$D = \Big\{(x,y,z) : -9 \leq x \leq 9, -\sqrt{81-x^2} \leq y \leq \sqrt{81-x^2}, 1-\sqrt{81-x^2-y^2} \leq z \leq 1+\sqrt{81-x^2-y^2}\Big\}, \text{ so} \tag{485}$$

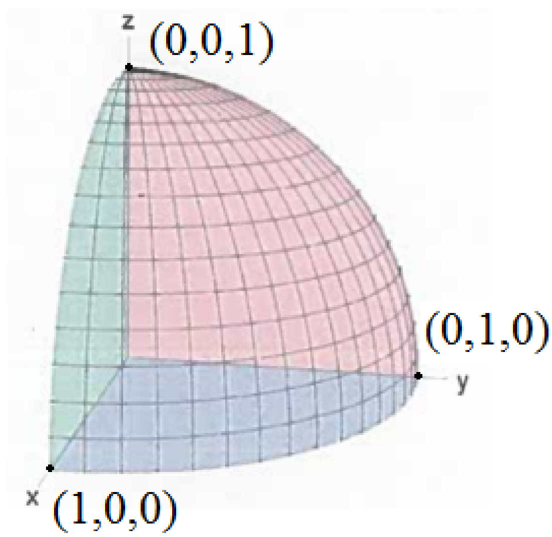
$$\iiint_D f(x,y,z)dV = \boxed{\int_{-9}^{9}\int_{-\sqrt{81-x^2}}^{\sqrt{81-x^2}}\int_{1-\sqrt{81-x^2-y^2}}^{1+\sqrt{81-x^2-y^2}} f(x,y,z)dzdydx}. \tag{486}$$

Problem 5.2: Sketch by hand or graph with a computer program the region of integration for the integral

$$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^{\sqrt{1-y^2-z^2}} f(x,y,z)dxdydz. \tag{487}$$

Note: You may also describe the region of integration in writing instead. If you choose to do this, please write complete sentences and provide a thorough description.

Solution: Since $x^2+y^2+z^2=1$ is the equation of the unit sphere (centered at (0,0,0)), we may repeat the steps of the previous problem to see that the region of integration is related to the unit sphere. The key difference here is that the smallest possible values of x, y, and z are always 0, so our region of integration ends up being the portion of the unit sphere within the first octant.



Problem 5.3: Evaluate

(488)
$$\int_1^{\ln(8)}\int_1^{\sqrt{z}}\int_{\ln(y)}^{\ln(2y)} e^{x+y^2-z}dxdydz.$$

Solution: We see that

(489)
$$\int_1^{\ln(8)}\int_1^{\sqrt{z}}\int_{\ln(y)}^{\ln(2y)} e^{x+y^2-z}dxdydz = \int_1^{\ln(8)}\int_1^{\sqrt{z}} e^{x+y^2-z}\Big|_{x=\ln(y)}^{\ln(2y)} dydz$$

(490)
$$= \int_1^{\ln(8)}\int_1^{\sqrt{z}}(e^{\ln(2y)+y^2-z} - e^{\ln(y)+y^2-z})dydz$$

(491)
$$= \int_1^{\ln(8)}\int_1^{\sqrt{z}}(2ye^{y^2-z} - ye^{y^2-z})dydz$$

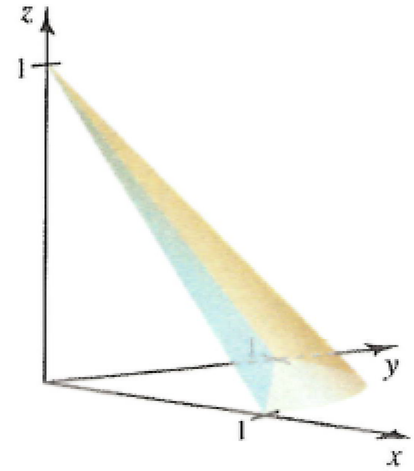
(492)
$$= \int_1^{\ln(8)}\int_1^{\sqrt{z}} ye^{y^2-z}dydz \overset{u=y^2}{=} \int_1^{\ln(8)} \frac{1}{2}e^{y^2-z}\Big|_{y=1}^{\sqrt{z}}$$

(493)
$$\frac{1}{2}\int_1^{\ln(8)}(e^0 - e^{1-z})dz = \frac{1}{2}(z + e^{1-z}\Big|_1^{\ln(8)})$$

(494)
$$= \frac{1}{2}(\ln(8) + e^{1-\ln(8)} - (e^{1-1}+1))$$

(495)
$$= \frac{1}{2}(\ln(8) + e^1\cdot e^{-\ln(8)} - e^0 - 1) = \frac{1}{2}(\ln(8) + \frac{e}{e^{\ln(8)}} - 2) = \boxed{\frac{1}{2}\ln(8) + \frac{e}{16} - 1}.$$

Problem 5.4: Find the volume of the solid S in the first octant that is bounded by the cone $z = 1 - \sqrt{x^2+y^2}$ and the plane $x+y+z=1$.



Solution 1: We see that

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^1 \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz \tag{496}$$

$$= \int_0^1 \int_0^{1-z} x\Big|_{1-z-y}^{\sqrt{(1-z)^2-y^2}} dydz \tag{497}$$

$$= \int_0^1 \int_0^{1-z} \left(\sqrt{(1-z)^2-y^2} - (1-z-y)\right) dydz. \tag{498}$$

We see that evaluating (the difficult part of) the inner integral in (498) is tantamount to evaluating

$$\int \sqrt{1-y^2}dy, \tag{499}$$

which is certainly possible, but it is difficult and computationally intensive, so we will evaluate the volume by an alternative method. If we more closely examine the integrals in (496), then we see that

$$\int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy \tag{500}$$

calculates the area of the cross section C_z shown in figure 21.

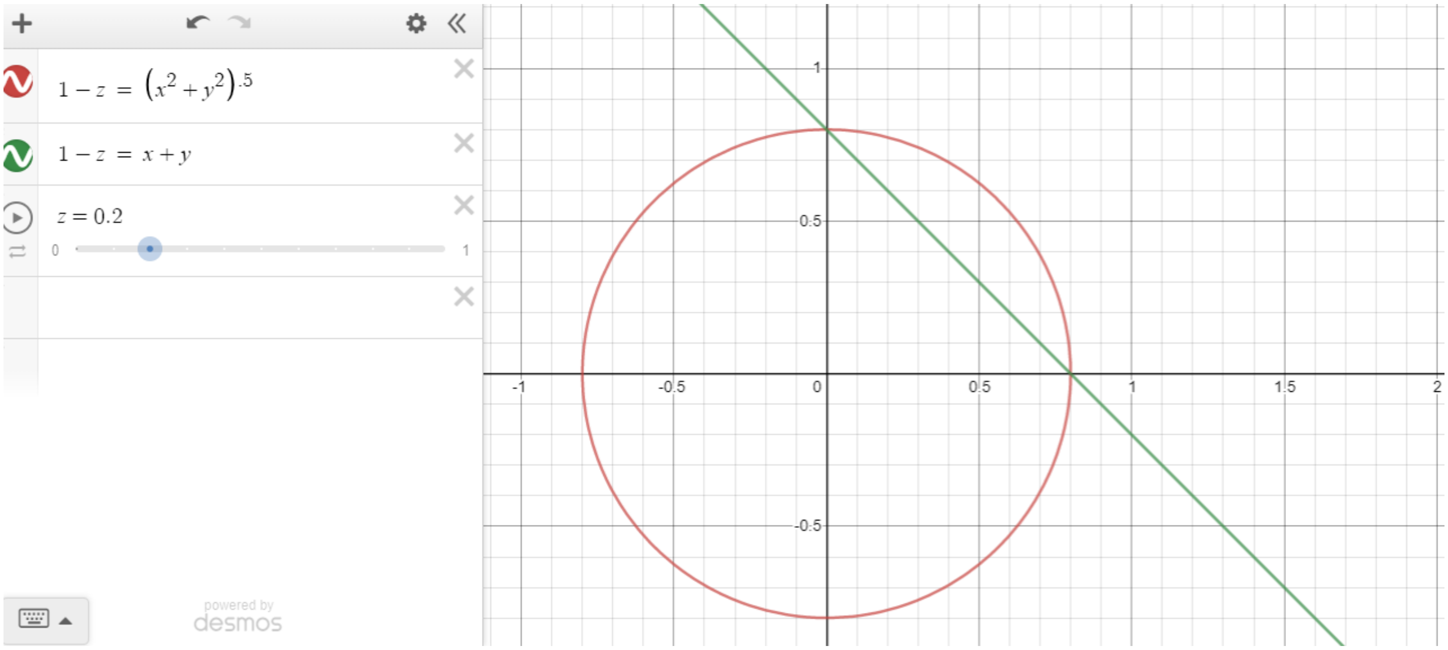


FIGURE 21. The cross section of S at a particular height z.

Using elementary Euclidena geometry, we see that

$$\int_0^{1-z}\int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy = \text{Area}(C_z)$$
$$= \frac{1}{4}\pi(1-z)^2 - \frac{1}{2}(1-z)^2 = \frac{\pi-2}{4}(1-z)^2. \tag{501}$$

It follows that

$$\int_0^1\int_0^{1-z}\int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz = \int_0^1 \frac{\pi-2}{4}(1-z)^2dz$$
$$= -\frac{\pi-2}{12}(1-z)^3\Big|_0^1 = \boxed{\frac{\pi-2}{12}}. \tag{502}$$

Solution 2: Let C be the portion of the cone $z = 1 - \sqrt{x^2+y^2}$ that is in the first quadrant and let T be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. We see that S is simply the solid C with the solid T removed from it. Recalling that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$, and that the volume of a tetrahedron with height h and a base of area b is $\frac{1}{3}bh$, we see that

$$\text{(503)}\quad \text{Vol}(S) = \text{Vol}(C) - \text{Vol}(T) = \frac{1}{3}\pi \cdot 1^2 \cdot 1 \underbrace{\cdot\frac{1}{4}}_{QI} - \frac{1}{3}\cdot \underbrace{(\frac{1}{2}\cdot 1\cdot 1)}_{\text{Area of base}} \cdot 1 = \boxed{\frac{\pi - 2}{12}}.$$

Solution 3: We proceed as we did in Solution 2, but we will now derive the formula for the volume of C and T by using a triple integral in cylindrical coordinates for C and a triple integral in Cartesian coordinates for T. Recalling that the Cartesian equation $z = 1 - \sqrt{x^2 + y^2}$ is rewritten as $z = 1 - r$ in cylindrical coordinates, we see that

$$\text{(504)}\quad \text{Vol}(S) = \text{Vol}(C) - \text{Vol}(T)$$

$$\text{(505)}\quad = \int_0^{\frac{\pi}{2}}\int_0^1\int_0^{1-r} r\,dz\,dr\,d\theta - \int_0^1\int_0^{1-z}\int_0^{1-z-y} dx\,dy\,dz$$

$$\text{(506)}\quad = \int_0^{\frac{\pi}{2}}\int_0^1 rz\Big|_0^{1-r} dr\,d\theta - \int_0^1\int_0^{1-z} x\Big|_0^{1-z-y} dy\,dz$$

$$\text{(507)}\quad = \int_0^{\frac{\pi}{2}}\int_0^1 (r - r^2)\,dr\,d\theta - \int_0^1\int_0^{1-z}(1 - z - y)\,dy\,dz$$

$$\text{(508)}\quad = \int_0^{\frac{\pi}{2}}(\frac{1}{2}r^2 - \frac{1}{3}r^2\Big|_0^1)d\theta - \int_0^1((1-z)y - \frac{1}{2}y^2\Big|_{y=0}^{1-z})dydz$$

$$\text{(509)}\quad = \int_0^{\frac{\pi}{2}}\frac{1}{6}d\theta - \int_0^1\frac{1}{2}(1-z)^2dz$$

$$= \frac{1}{6}\theta\Big|_0^{\frac{\pi}{2}} - \frac{1}{6}(1-z)^3\Big|_0^1 = \boxed{\frac{\pi - 2}{12}}. \tag{510}$$

Problem 5.5: Use triple integration in Cartesian coordinates to find the volume of the tetrahedron S that has its vertices at $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, where $a,b,c>0$.
Hint: One of the faces of the tetrahedron lies on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Solution: We see that an alternative description of S is that it is the solid bound between the planes $x=0$, $y=0$, $z=0$, and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

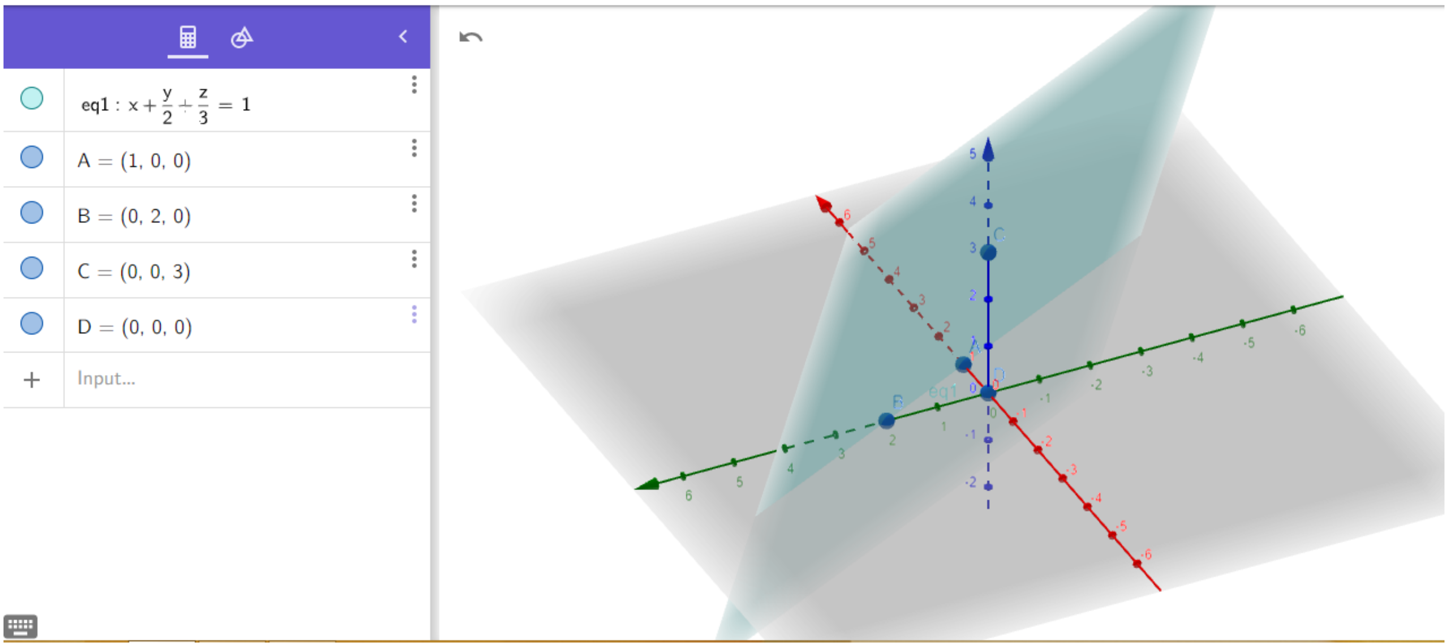


FIGURE 22. A picture of the solid S when $a=1$, $b=2$, and $c=3$.

$$\text{Volume of } S = \iiint_S 1dV = \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{z}{c}-\frac{y}{b})} 1dxdydz \tag{511}$$

$$= \int_0^c \int_0^{b(1-\frac{z}{c})} a(1-\frac{z}{c}-\frac{y}{b})dydz = a\int_0^c (y-\frac{z}{c}y-\frac{1}{2b}y^2\Big|_{y=0}^{b(1-\frac{z}{c})})dz \tag{512}$$

$$= a\int_0^c \Big(\underbrace{b(1-\frac{z}{c})-\frac{z}{c}b(1-\frac{z}{c})}_{b(1-\frac{z}{c})^2}-\frac{1}{2b}b^2(1-\frac{z}{c})^2\Big)dz = \frac{ab}{2}\int_0^c (1-\frac{z}{c})^2dz \tag{513}$$

$$= \frac{ab}{2}(-\frac{c}{3}(1-\frac{z}{c})^3\Big|_{z=0}^{c}) = \boxed{\frac{abc}{6}}. \tag{514}$$

Problem 5.6: Evaluate

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}} dydxdz. \tag{515}$$

Hint: A different order of integration can make the problem easier, even though it is not necessary.

Solution: We see that trying to evaluate the inner integral in the current order of integration is tantamount to evaluating

$$\int c_1 \sin(c_2\sqrt{y})dy, \tag{516}$$

which is very difficult, so we decide to change the order of integration in hopes that the inner integral becomes easier to evaluate. We see that integrating with respect to z in the inner integral is not any easier since z and y are symmetric in the integrand, so we decide to integrate with respect to x in the inner integral in our new order of integration. Since z and y are symmetric in the integrand, the difficulty of the integrations doesn't seem to change if we use $dxdydz$ or $dxdzdy$, so we will use the order $dxdydz$ in order to reduce our workload by only changing the order of dx and dy instead of changing the order of dx, dy, and dz. We see that the bounds that we have in (515) tell us that

$$\begin{array}{ccccc} 1 & \le & z & \le & 4 \\ z & \le & x & \le & 4z \\ 0 & \le & y & \le & \pi^2 \end{array} \to \begin{array}{ccccc} 1 & \le & z & \le & 4 \\ 0 & \le & y & \le & \pi^2 \\ z & \le & x & \le & 4z \end{array}. \tag{517}$$

Thankfully, we didn't have to do any work to interchange the order of dx and dy since the bounds for y in the first order of integration were independent of x. We now see that

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}} dydxdz = \int_1^4 \int_0^{\pi^2} \int_z^{4z} \sin(\sqrt{yz})x^{-\frac{3}{2}} dxdydz \tag{518}$$

. . . .

(519) $$= \int_1^4 \int_0^{\pi^2} -2\sin(\sqrt{yz})x^{-\frac{1}{2}}\Big|_{x=z}^{4z} dydz$$

. . . .

(520) $$= \int_1^4 \int_0^{\pi^2} \left(-2\sin(\sqrt{yz})(4z)^{-\frac{1}{2}} + 2\sin(\sqrt{yz})z^{-\frac{1}{2}}\right) dydz$$

. . . .

(521)
$$= \int_1^4 \int_0^{\pi^2} \left(-\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} + 2\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}\right) dydz = \int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz.$$

We see that evaluating the inner integral at the end of (521) is again tantamount to evaluating the integral in (516), so we decide to change the order of integration once again. Note that this is equivalent to having decided to use the order $dxdzdy$ from the beginning, but we were not able to see that $dxdzdy$ was the best order of integration until now. Nonetheless, our initial change in the order of integration did allow us to make progress despite not being the best possible order of integration.

(522) $$\int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz = \int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy.$$

Recalling that y does not change when evaluating the inner integral with respect to z, we treat y as a constant (relative to z) to perform the u-substituion

(523) $$u = \sqrt{yz}, du = \frac{\sqrt{y}}{2\sqrt{z}}dz, dz = \frac{2\sqrt{z}}{\sqrt{y}}du.$$

We now see that

(524) $$\int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy = \int_0^{\pi^2} \int_{z=1}^4 \frac{2\sin(u)}{\sqrt{y}} dudy$$

. . . .

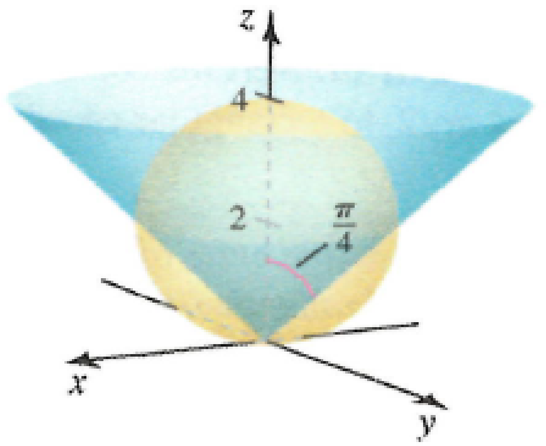
$$= \int_0^{\pi^2} \frac{-2\cos(u)}{\sqrt{y}}\Big|_{z=1}^{4} dy = \int_0^{\pi^2} \frac{-2\cos(\sqrt{yz})}{\sqrt{y}}\Big|_{z=1}^{4} dy \tag{525}$$

$$= \int_0^{\pi^2} \left(\frac{-2\cos(\sqrt{4y})}{\sqrt{y}} + \frac{2\cos(\sqrt{y})}{\sqrt{y}}\right) dy \tag{526}$$

$$\overset{u=\sqrt{y}}{=} \int_{y=0}^{\pi^2} (-4\cos(2u) + 4\cos(u))\, du = (-2\sin(2u) + 4\sin(u))\Big|_{y=0}^{\pi^2} \tag{527}$$

$$= (-2\sin(2\sqrt{y}) + 4\sin(\sqrt{y}))\Big|_{y=0}^{\pi^2} = \boxed{0}. \tag{528}$$

Problem 5.7: Find the volume of the solid region S outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.



First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. Remember that the angle φ is measured from the z-axis and satisfies $0 \le \varphi \le \pi$, not $0 \le \varphi \le 2\pi$. Also remember that this cross section is showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

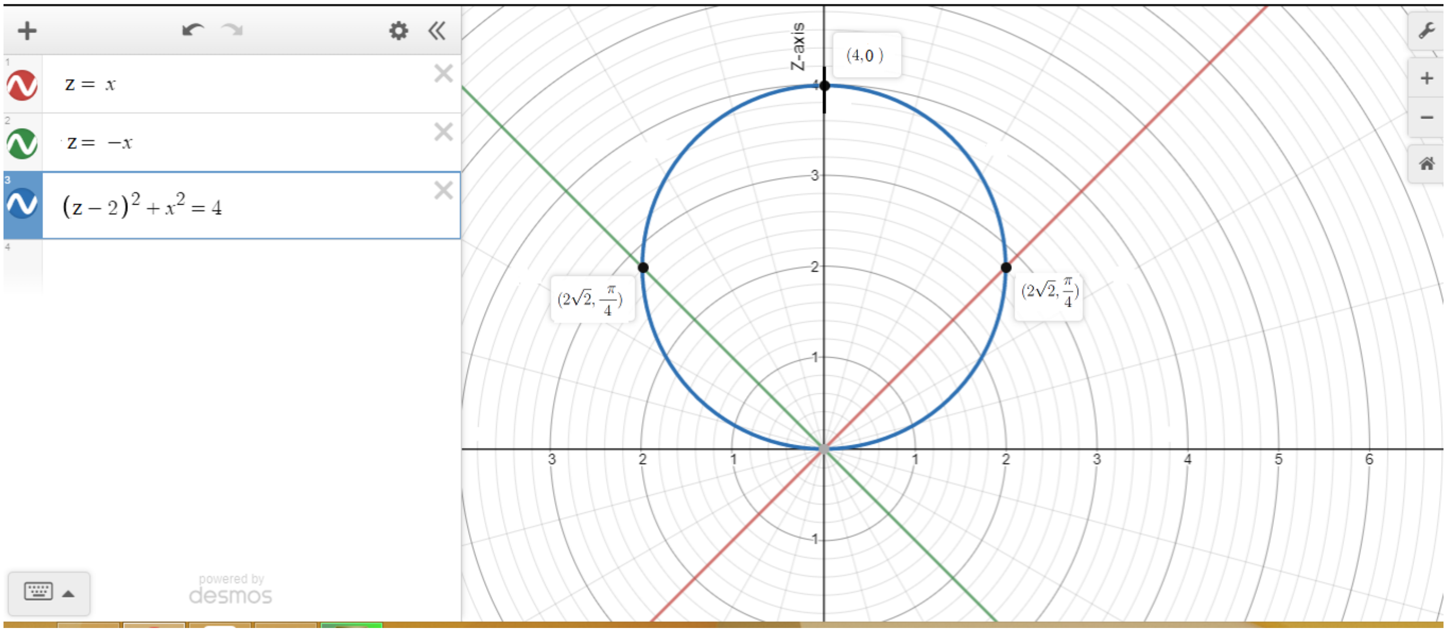


FIGURE 23. The xz-plane cross section in spherical coordinates.

We now see that for any $\theta \in [0, 2\pi)$ we have that $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$. Recalling that the blue circle is defined by $\rho = 4\cos(\varphi)$, we see that once φ is also chosen we have that $0 \leq \rho \leq 4\cos(\varphi)$. We now see that the volume of the solid is given by

$$\text{(529)} \qquad \text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta$$

$$\text{(530)} \qquad = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=0}^{4\cos(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64}{3}\underbrace{\cos^3(\varphi)}_{u^3}\underbrace{\sin(\varphi)d\varphi}_{-du}\, d\theta$$

$$\text{(531)} \qquad = -\frac{64}{3}\int_0^{2\pi}\int_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} u^3 du d\theta = -\frac{64}{3}\int_0^{2\pi}\frac{1}{4}u^4\Big|_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta$$

$$\text{(532)} \qquad = -\frac{64}{3}\int_0^{2\pi}\frac{1}{4}\cos^4(\varphi)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = -\frac{64}{3}\int_0^{2\pi} -\frac{1}{16}d\theta = -\frac{64}{3}\cdot 2\pi\cdot\frac{-1}{16} = \boxed{\frac{8\pi}{3}}.$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). Remember that this cross section is also showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

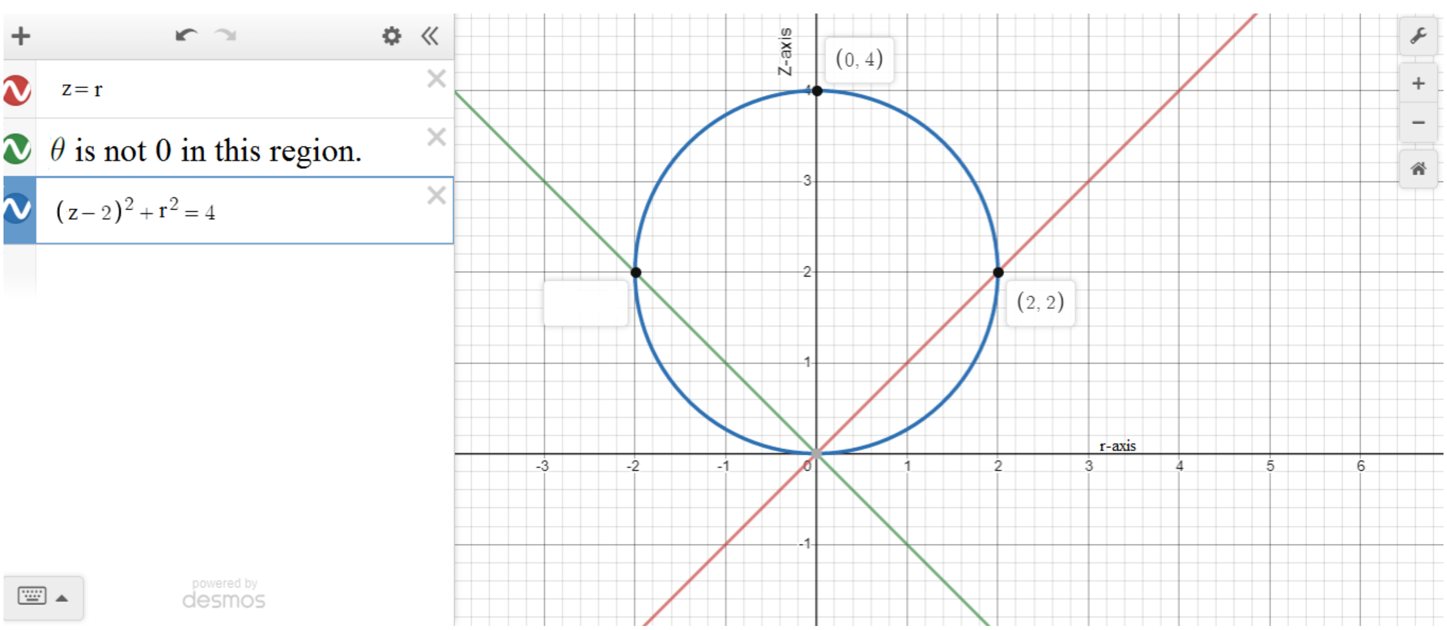


Figure 24. The xz-plane cross section in cylindrical coordinates.

We now see that for any $0 \leq \theta < 2\pi$ we have that $0 \leq z \leq 2$. Noting that we have $r = \sqrt{4-(z-2)^2} = \sqrt{4z-z^2}$ on the blue circle, we see that once z is chosen we have $z \leq r \leq \sqrt{4z-z^2}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_0^2\int_z^{\sqrt{4z-z^2}} rdrdzd\theta \tag{533}$$

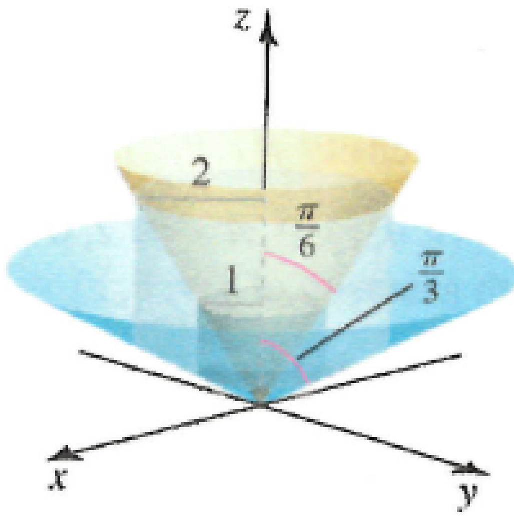
. . . .

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^2\Big|_z^{\sqrt{4z-z^2}} dzd\theta = \int_0^{2\pi}\int_0^2 (2z-z^2)dzd\theta \tag{534}$$

. . . .

$$\int_0^{2\pi}(z^2 - \frac{1}{3}z^3)\Big|_0^2 d\theta = \int_0^{2\pi}\frac{4}{3}d\theta = \boxed{\frac{8\pi}{3}}. \tag{535}$$

Problem 5.8: Find the volume of the solid region S that is bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.



First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

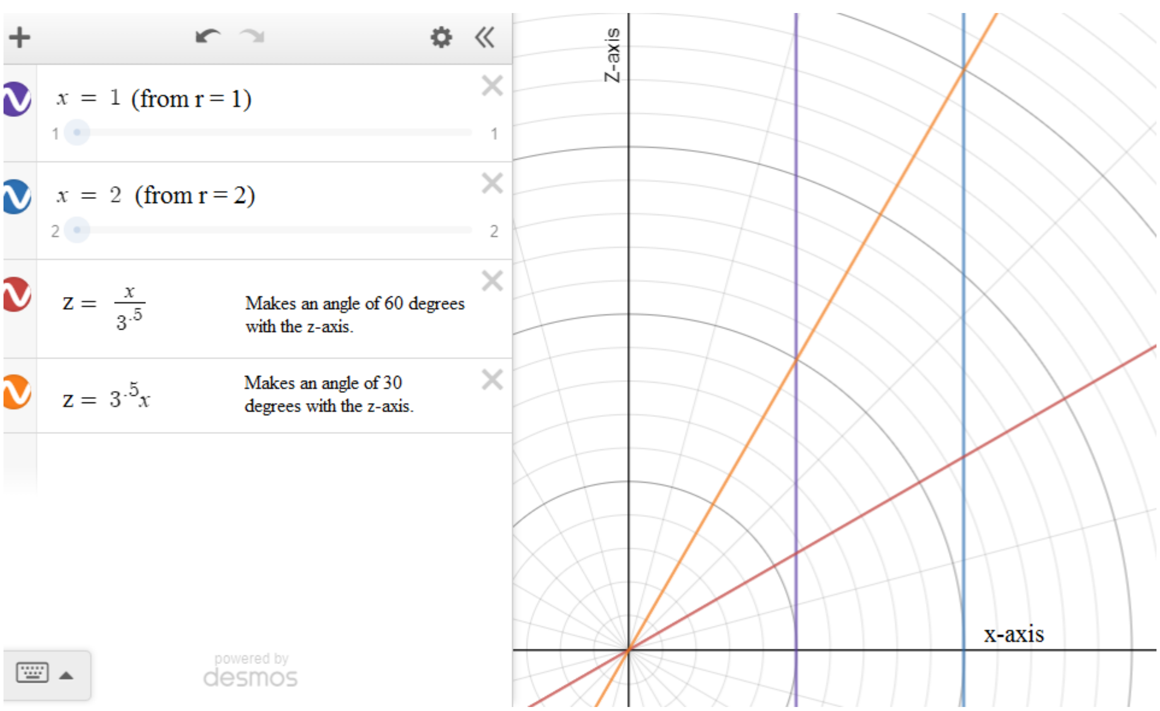


Figure 25. The xz-plane cross section in spherical coordinates.

We see that for any $0 \le \theta < 2\pi$ we have $\frac{\pi}{6} \le \varphi \le \frac{\pi}{3}$. Noting that $r = \rho\sin(\varphi)$, we see that when $r = 1$ we have $\rho = \csc(\varphi)$ and when $r = 2$ we have $\rho = 2\csc(\varphi)$. It follows that once φ is also chosen we have $\csc(\varphi) \le \rho \le 2\csc(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc(\varphi)}^{2\csc(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{536}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=\csc(\varphi)}^{2\csc(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{7}{3}\csc^2(\varphi)d\varphi d\theta \tag{537}$$

$$= \int_0^{2\pi} -\frac{7}{3}\cot(\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \int_0^{2\pi} \frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{538}$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

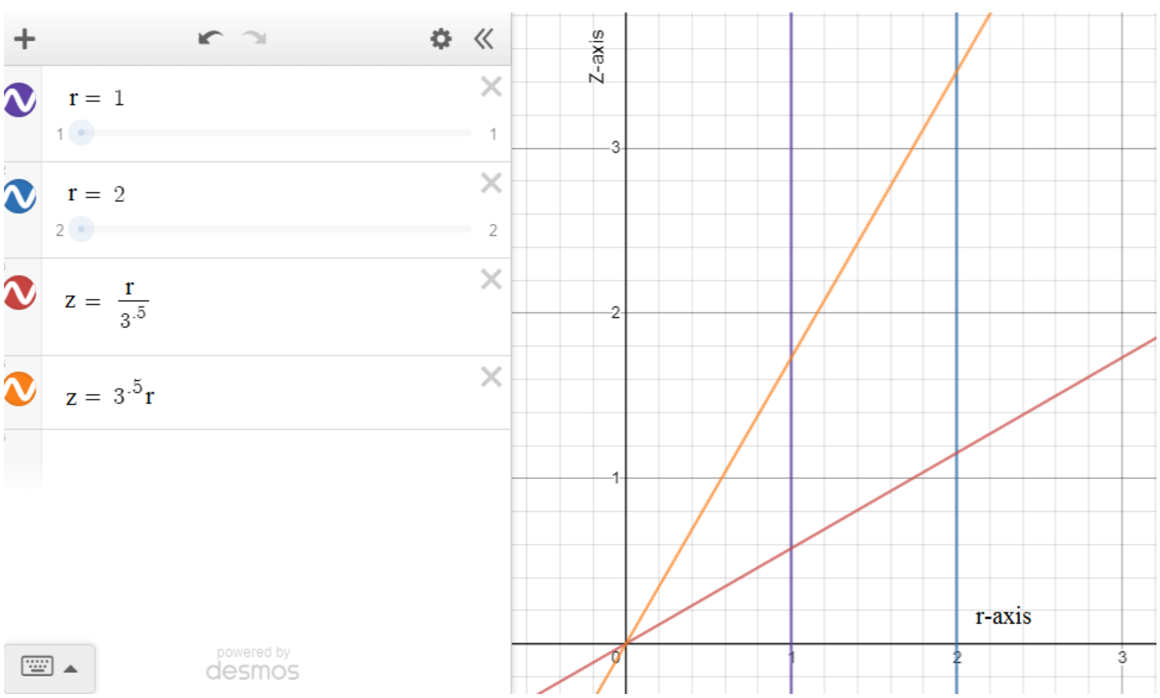


FIGURE 26. The xz-plane cross section in cylindrical coordinates.

We note that for any $0 \le \theta < 2\pi$ we have $1 \le r \le 2$. Once r is also chosen, we see that $\frac{1}{\sqrt{3}}r \le z \le r\sqrt{3}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_1^2\int_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} rdzdrd\theta \tag{539}$$

$$= \int_0^{2\pi}\int_1^2 rz\Big|_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} drd\theta = \int_0^{2\pi}\int_1^2 \frac{2}{\sqrt{3}}r^2 drd\theta = \int_0^{2\pi} \frac{2}{3\sqrt{3}}r^3\Big|_1^2 d\theta \tag{540}$$

$$= \int_0^{2\pi} \frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{541}$$

Problem 3.8: What point on the plane $x+y+4z=8$ is closest to the origin? Give an argument showing that you have found an absolute minimum of the distance function.

Solution: Note that for any (x,y,z) on the plane $x+y+4z=8$ we have

$$z = 2 - \frac{1}{4}x - \frac{1}{4}y, \tag{542}$$

from which we see that

$$\mathrm{d}((x,y,z),(0,0,0)) = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2} \tag{543}$$

$$= \sqrt{x^2+y^2+(2-\frac{1}{4}x-\frac{1}{4}y)^2} = \sqrt{4-x-y+\frac{1}{8}xy+\frac{17}{16}x^2+\frac{17}{16}y^2}. \tag{544}$$

We recall that if $f(x,y)$ is any nonnegative function, then $f(x,y)$ and $f^2(x,y)$ have their (local and global) minimums and maximums occur at the same values of (x,y). It follows that we want to optimize the function

$$f(x,y) = 4-x-y+\frac{1}{8}xy+\frac{17}{16}x^2+\frac{17}{16}y^2. \tag{545}$$

Since any global minimum of $f(x,y)$ is also a local minimum, we see that the global minimum of f (if it exists) is at a critical point. We now begin finding the critical points of f. We see that

$$\begin{matrix} 0 = f_x(x,y) = \frac{17}{8}x+\frac{1}{8}y-1 \\ 0 = f_y(x,y) = \frac{17}{8}y+\frac{1}{8}x-1 \end{matrix} \rightarrow 0 = (\frac{17}{8}x+\frac{1}{8}y-1)-(\frac{17}{8}y+\frac{1}{8}x-1) \tag{546}$$

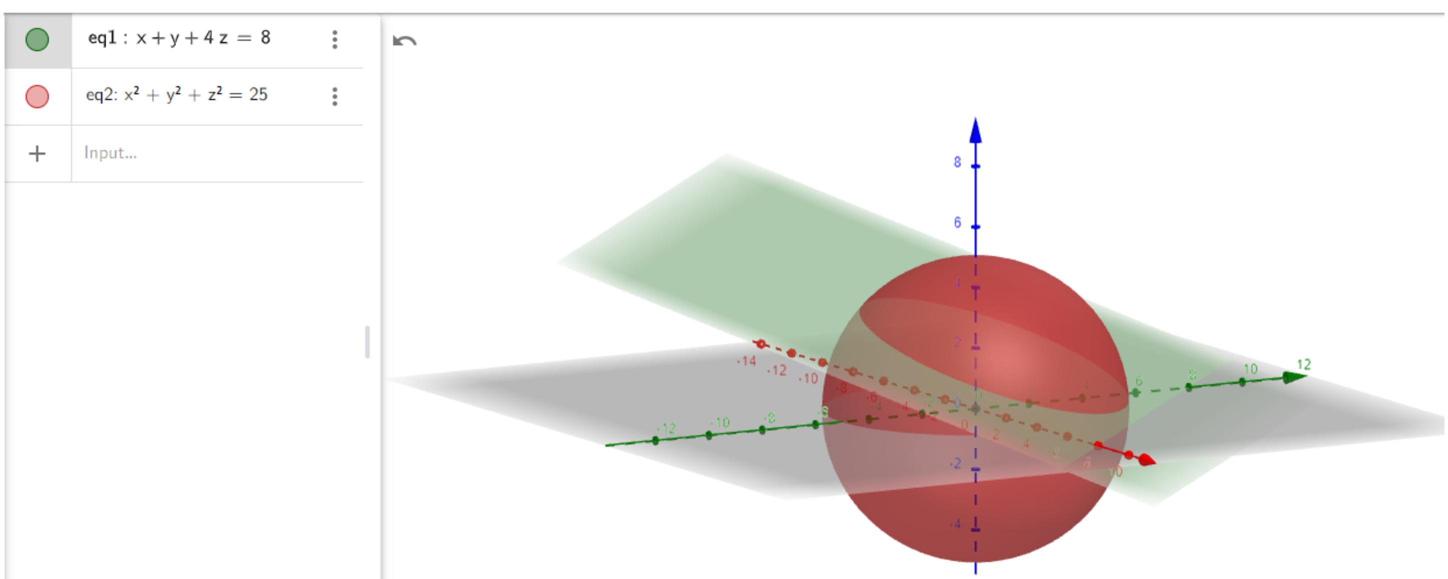
$$= 2x-2y \rightarrow x=y \rightarrow x=y=\frac{4}{9}. \tag{547}$$

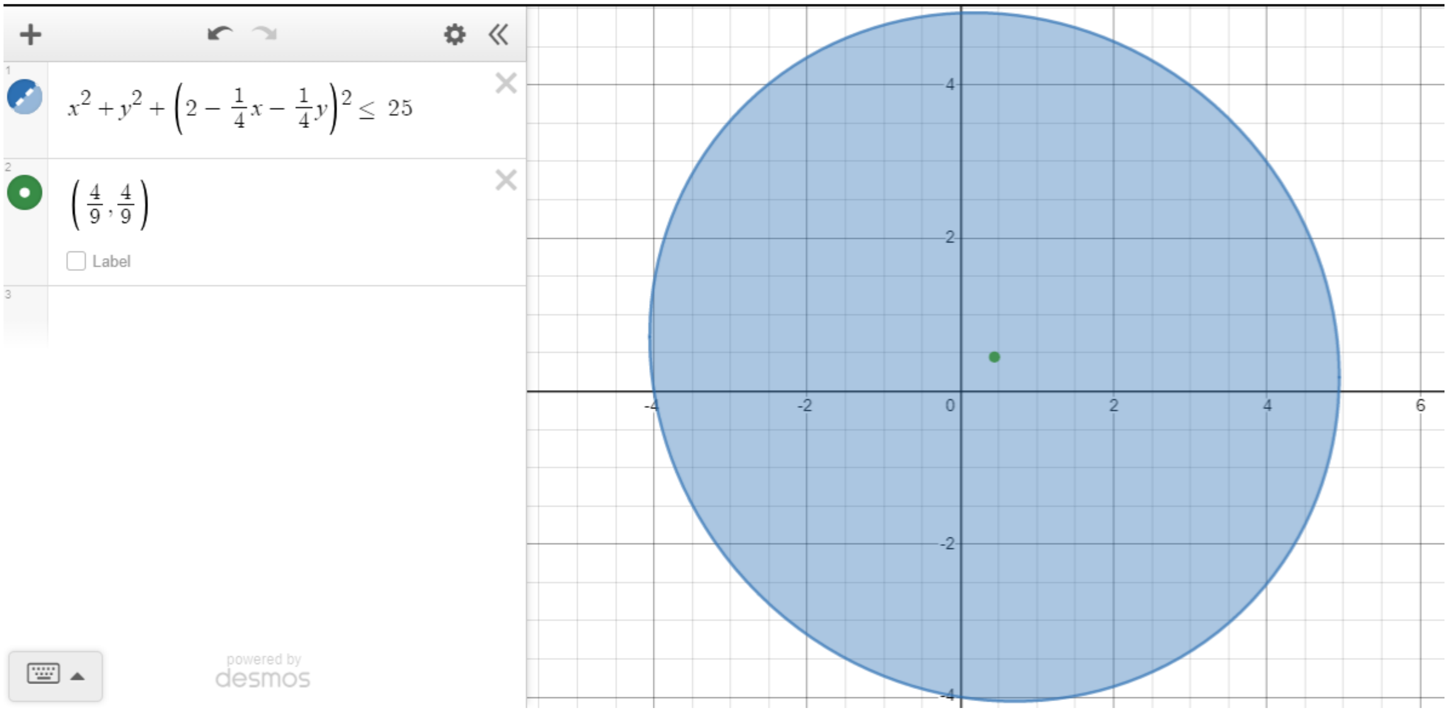
We see that $(\frac{4}{9},\frac{4}{9})$ is the only critical point. We will now use the second derivative test to verify that $(\frac{4}{9},\frac{4}{9})$ is a local minimum. We see that

$$\begin{aligned} f_{xx}(x,y) &= \tfrac{17}{8} \\ f_{yy}(x,y) &= \tfrac{17}{8} \rightarrow D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 \\ f_{xy}(x,y) &= \tfrac{1}{8} \end{aligned} \tag{548}$$

$$= \frac{17}{8}\cdot\frac{17}{8} - (\frac{1}{8})^2 = \frac{9}{2} \rightarrow D(\frac{4}{9},\frac{4}{9}) = \frac{9}{2} > 0. \tag{549}$$

Since we also see that $f_{xx}(\frac{4}{9},\frac{4}{9}) = \frac{17}{8} > 0$, the second derivative test tells us that $(\frac{4}{9},\frac{4}{9})$ is indeed a local minimum of $f(x,y)$. It remains to show that $f(x,y)$ attains its global minimum at $(\frac{4}{9},\frac{4}{9})$. Firstly, we note that $f(\frac{4}{9},\frac{4}{9}) = \frac{32}{9}$. Since $\frac{32}{9} < 25$ (I picked 25 randomly, I just needed some larger number), let us consider the region R of (x,y) for which $(x,y,\underbrace{2-\frac{1}{4}x-\frac{1}{4}y}_{z})$ has a distance of at most 5 from the origin. This is the same as $R = \{(x,y) \mid f(x,y) \le 25\}$.





Since R is a closed and bounded region, and $f(x, y)$ is a continuous function function, we know that f attains an absolute minimum on R. The point $(\frac{4}{9}, \frac{4}{9})$ is inside of R, so the minimum of f is not attained on the boundary of R (as that is where the distance to the origin is exactly 5). Since the minimum of f on R is attained on the interior, we see that it must be obtained at a critical point of $f(x, y)$, so it is attained at $(\frac{4}{9}, \frac{4}{9})$. For any point (x, y) outside of R, we have $f(x, y) > 25$ (by the very definition of R), so the global minimum of $f(x, y)$ is $\frac{32}{9}$ and is attained at $(\frac{4}{9}, \frac{4}{9})$. It follows that the point on the plane $x + y + 4z = 8$ that is closest to the origin is $\boxed{(\frac{4}{9}, \frac{4}{9}, \frac{16}{9})}$.

Problem 3.9: Find the point on the plane $x + y + z = 4$ nearest the point $P(0, 3, 6)$. Remember to justify why your answer is a global minimum and not just a local minimum.
*N*ote: You may solve this problem using geometry instead of calculus and still receive full credit as long as you show all of your work.

Solution: Note that for any (x, y, z) on the plane $x + y + z = 4$ we have

(550) $$z = 4 - x - y,$$

from which we see that

(551) $$\mathrm{d}((x, y, z), (0, 3, 6)) = \sqrt{(x-0)^2 + (y-3)^2 + (z-6)^2}$$

(552) $$= \sqrt{x^2 + y^2 - 6y + 9 + (-2 - x - y)^2} = \sqrt{2x^2 + 2y^2 + 2xy + 4x - 2y + 13}.$$

We recall that if $f(x, y)$ is any nonnegative function, then $f(x, y)$ and $f^2(x, y)$ have their (local and global) minimums and maximums occur at the same values of (x, y). It follows that we can instead optimize the function

(553) $$f(x, y) = 2x^2 + 2y^2 + 2xy + 4x - 2y + 13.$$

Since any global minimum of $f(x, y)$ is also a local minimum, we see that the global minimum of f (if it exists) is at a critical point. We now begin finding the critical points of f. We see that

(554) $$\begin{matrix} 0 = f_x(x, y) = 4x + 2y + 4 \\ 0 = f_y(x, y) = 4y + 2x - 2 \end{matrix} \rightarrow 0 = (4x + 2y + 4) - 2(4y + 2x - 2)$$

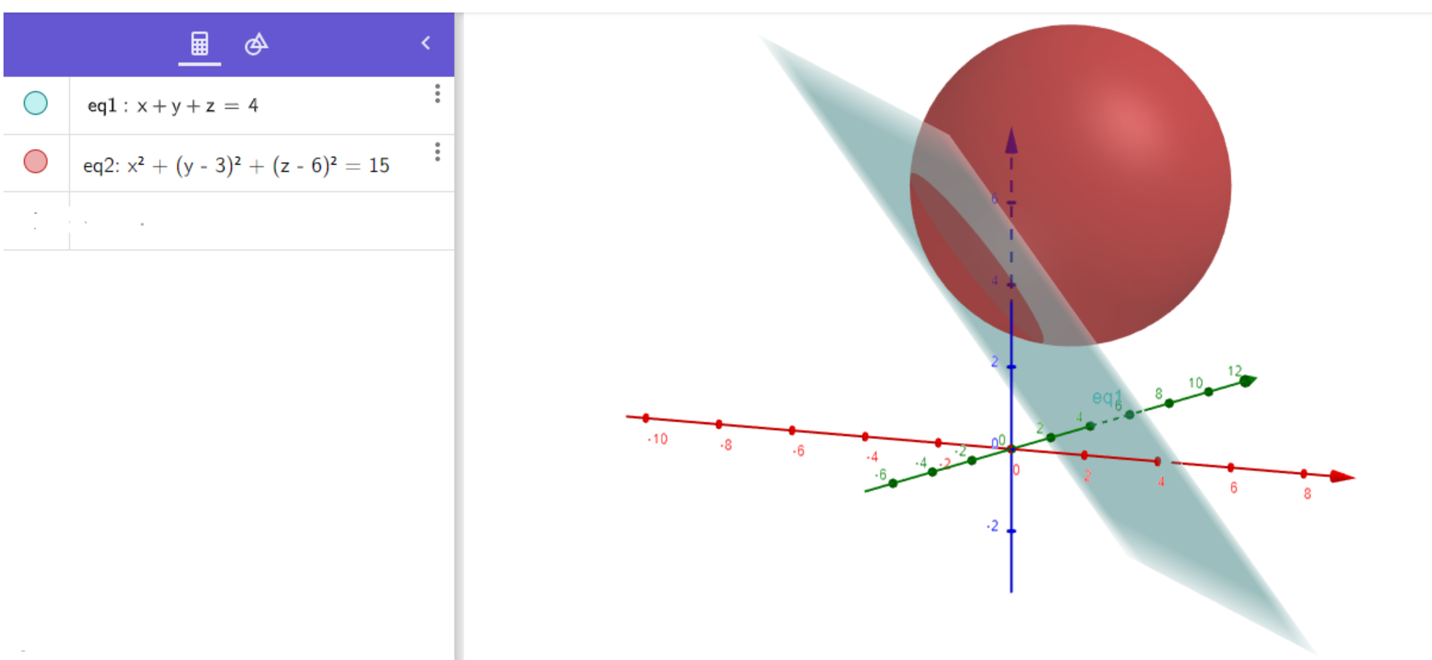
(555) $$= -6y + 8 \rightarrow y = \frac{4}{3} \rightarrow x = -\frac{5}{3}.$$

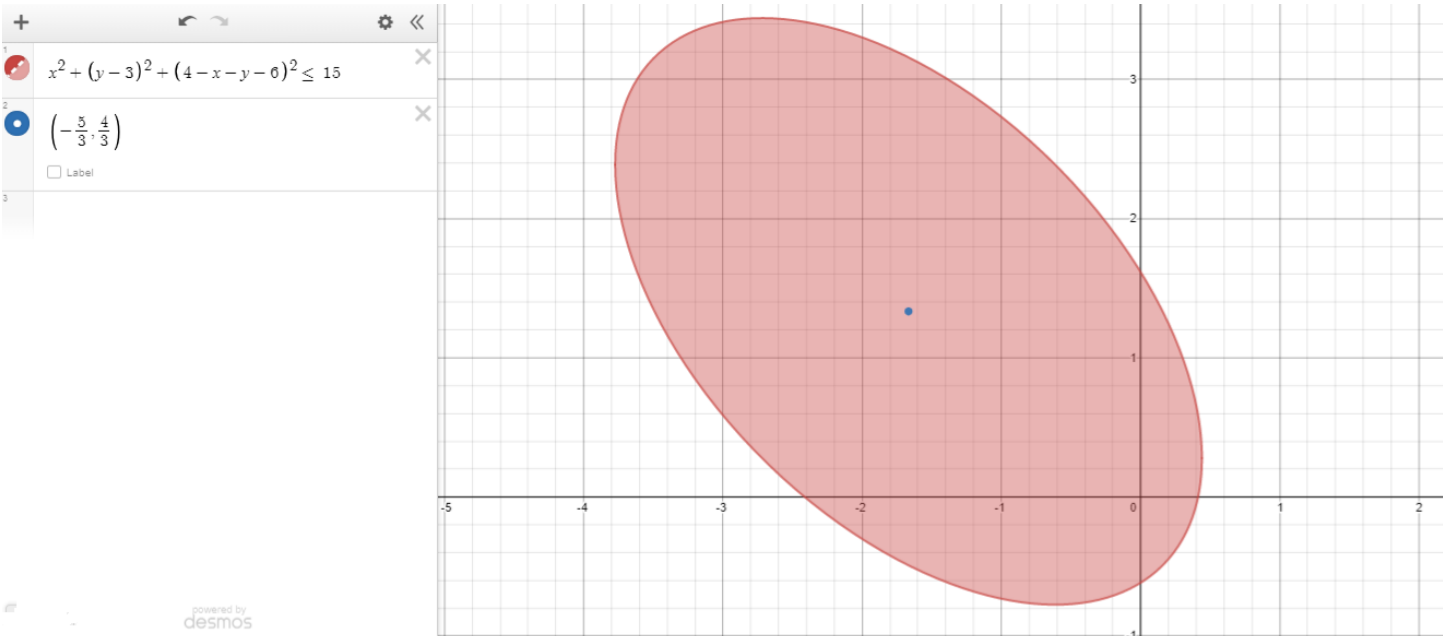
We see that $(-\frac{5}{3}, \frac{4}{3})$ is the only critical point. We will now use the second derivative test to verify that $(-\frac{5}{3}, \frac{4}{3})$ is a local minimum. We see that

$$(556)\qquad \begin{aligned} f_{xx}(x,y) &= 4 \\ f_{yy}(x,y) &= 4 \rightarrow D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 \\ f_{xy}(x,y) &= 2 \end{aligned}$$

$$(557)\qquad = 4\cdot 4 - 2^2 = 12 \rightarrow D(-\frac{5}{3},\frac{4}{3}) = 12 > 0.$$

Since we also see that $f_{xx}(-\frac{5}{3},\frac{4}{3}) = 4 > 0$, the second derivative test tells us that $(-\frac{5}{3},\frac{4}{3})$is indeed a local minimum of $f(x,y)$. It remains to show that $f(x,y)$ attains its global minimum at $(-\frac{5}{3},\frac{4}{3})$. Firstly, we note that $f(-\frac{5}{3},\frac{4}{3}) = \frac{25}{3}$. Since $\frac{25}{3} < 15$ (I picked 15 randomly, I just needed some larger number), let us consider the region R of (x,y) for which $(x,y,\underbrace{4-x-y}_{z})$ has a distance of at most $\sqrt{15}$ from the point $(0,3,6)$. This is the same as $R = \{(x,y) \mid f(x,y) \leq 15\}$.





Since R is a closed and bounded region, and $f(x, y)$ is a continuous function function, we know that g attains an absolute minimum on R. The point $(-\frac{5}{3}, \frac{4}{3})$ is inside of R, so the minimum of g is not attained on the boundary of R (as that is where the squared distance to the origin is exactly 15). Since the minimum of g on R is attained on the interior, we see that it must be obtained at a critical point of $f(x, y)$, so it is attained at $(-\frac{5}{3}, \frac{4}{3})$. For any point (x, y) outside of R, we have $f(x, y) > 15$ (by the very definition of R), so the global minimum of $f(x, y)$ is $\frac{25}{3}$ and is attained at $(-\frac{5}{3}, \frac{4}{3})$. It follows that the point on the plane $x + y + z = 4$ that is closest to the origin is $\boxed{(-\frac{5}{3}, \frac{4}{3}, \frac{13}{3})}$.

Problem 3.10: Find the point on the plane $2x + 3y + 6z - 10 = 0$ closest to the point $(-2, 5, 1)$ by using the method of Lagrange Multipliers. Can you justify that your answer is a global minimum and not just a local minimum?

Solution: We see that our constraint function is $g(x, y, z) = 2x+3y+6z-10$, and the function that we are trying to optimize is the distaince from a point (x, y, z) on the plane to the point $(-2, 5, 1)$, which is given by

$$\begin{aligned} h(x, y, z) &= \sqrt{(x - (-2))^2 + (y - 5)^2 + (z - 1)^2} \\ &= \sqrt{x^2 + 4x + 4 + y^2 - 10y + 25 + z^2 - 2z + 1}. \end{aligned} \tag{558}$$

Since $h(x, y, z)$ and $f(x, y, z) = (h(x, y, z))^2$ have their absolute minimum(s) occurring at the same location(s), we will optimize $f(x, y, z)$ subject to $g(x, y, z) = 0$ instead since the resulting calculations will be easier. Since our constraint function defines an open region (a plane) the method of Lagrange multipliers will give us all of the critical points in the open region, and we will compare the values of $f(x, y, z)$ at the critical points to the values of $f(x, y, z)$ as (x, y, z) approaches the boundary. Noting that

$$\nabla g(x, y, z) = \langle 2, 3, 6 \rangle \text{ and} \tag{559}$$

$$\nabla f(x, y, z) = \langle 2x + 4, 2y - 10, 2z - 2 \rangle, \tag{560}$$

the method of Lagrange multipliers gives us the system of equations

$$\begin{aligned} g(x, y, z) &= 0 \\ \vec{\nabla} f(x, y, z) &= \lambda \vec{\nabla} g(x, y, z) \end{aligned} \tag{561}$$

. . . .

$$\Leftrightarrow \begin{aligned} 2x + 3y + 6z - 10 &= 0 \\ \langle 2x + 4, 2y - 10, 2z - 2 \rangle &= \lambda \langle 2, 3, 6 \rangle \end{aligned} \tag{562}$$

. . . .

$$(563)\quad \Leftrightarrow \begin{array}{rcl} 2x+3y+6z-10 &=& 0 \\ 2x+4 &=& 2\lambda \\ 2y-10 &=& 3\lambda \\ 2z-2 &=& 6\lambda \end{array} \Leftrightarrow \begin{array}{rcl} 2x+3y+6z-10 &=& 0 \\ x &=& \lambda-2 \\ y &=& \frac{3}{2}\lambda+5 \\ z &=& 3\lambda+1 \end{array}$$

. . . .

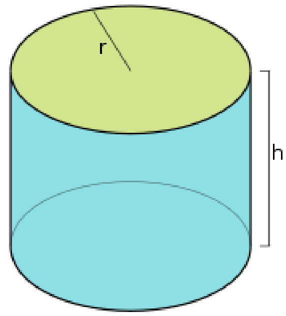
$$(564)\quad \rightarrow 0 = 2(\lambda-2)+3(\frac{3}{2}\lambda+5)+6(3\lambda+1)-10 = \frac{49}{2}\lambda+7 \rightarrow \lambda = -\frac{2}{7}$$

. . . .

$$(565)\quad \rightarrow \boxed{(x,y,z) = (-\frac{16}{7}, \frac{32}{7}, \frac{1}{7})}.$$

We see that a point (x, y, z) in the plane $2x + 3y + 6z - 10 = 0$ approaches the boundary of the plane (the 'outer edges' of the plane) if at least one of x, y, or z appraoches infinity. It follows that the square of the distance function $(f(x, y, z))$ approaches positive infinity as (x, y, z) approaches the boundary, so the absolute minimum exists and occurs at the critical point that we found.

Problem 3.11: Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of 32π in^3.



Solution: We recall that a cylinder of radius r and height h has a volume of $V = \pi r^2 h$ and a surface area (including the 2 circular ends) of $S = 2\pi r^2 + 2\pi rh$. It follows that we want to optimize the function $f(r,h) = 2\pi r^2 + 2\pi rh$ subject to the constraint $0 = g(r,h) = \pi r^2 h - 32\pi$. Since

$$(566)\quad \nabla f(r,h) = \langle 4\pi r + 2\pi h, 2\pi r\rangle \text{ and } \nabla g(r,h) = \langle 2\pi rh, \pi r^2\rangle, \text{ we obtain}$$

$$(567)\quad \begin{array}{rcl} 4\pi r + 2\pi h &=& 2\pi\lambda rh \\ 2\pi r &=& \pi\lambda r^2 \\ \pi r^2 h &=& 32\pi \end{array} \overset{r\neq 0}{\rightarrow} \begin{array}{rcl} 2r + h &=& \lambda rh \\ 2 &=& \lambda r \\ r^2 h &=& 32 \end{array} \rightarrow \begin{array}{rcl} 2r + h &=& 2h \\ 2 &=& \lambda r \\ r^2 h &=& 32 \end{array}$$

$$(568)\quad \rightarrow \begin{array}{rcl} 2r &=& h \\ 2 &=& \lambda r \\ r^2 h &=& 32 \end{array} \rightarrow \begin{array}{rcl} 2r &=& h \\ 2 &=& \lambda r \\ 2r^3 &=& 32 \end{array} \rightarrow r = \sqrt[3]{16} = 2\sqrt[3]{2} \rightarrow h = 4\sqrt[3]{2}.$$

Since the cylinder does not have a maximum surface area when subjected to the constraint $V = 32\pi$, we see that the critical point that we found has to correspond to a local minimum. The extreme/boundary cases occur when either $r \to \infty$ or $h \to \infty$, in which case we also have $S \to \infty$. It follows that $f(r,h)$ attains a minimum value of $24\pi\sqrt[3]{4}$ when $(r,h) = \boxed{(2\sqrt[3]{2}, 4\sqrt[3]{2})}$.

Problem 3.12: Economists model the output of manufacturing systems using production functions that have many of the same properties as utility functions. The family of Cobb-Douglas production functions has the form $P = f(K, L) = CK^aL^{1-a}$, where K represents capital, L represents labor, and C and a are positive real numbers with $0 < a < 1$. If the cost of capital is p dollars per unit, the cost of labor is q dollars per unit, and the total available budget is B, then the constraint takes the form $pK + qL = B$. Find the values of K and L that maximize the production function

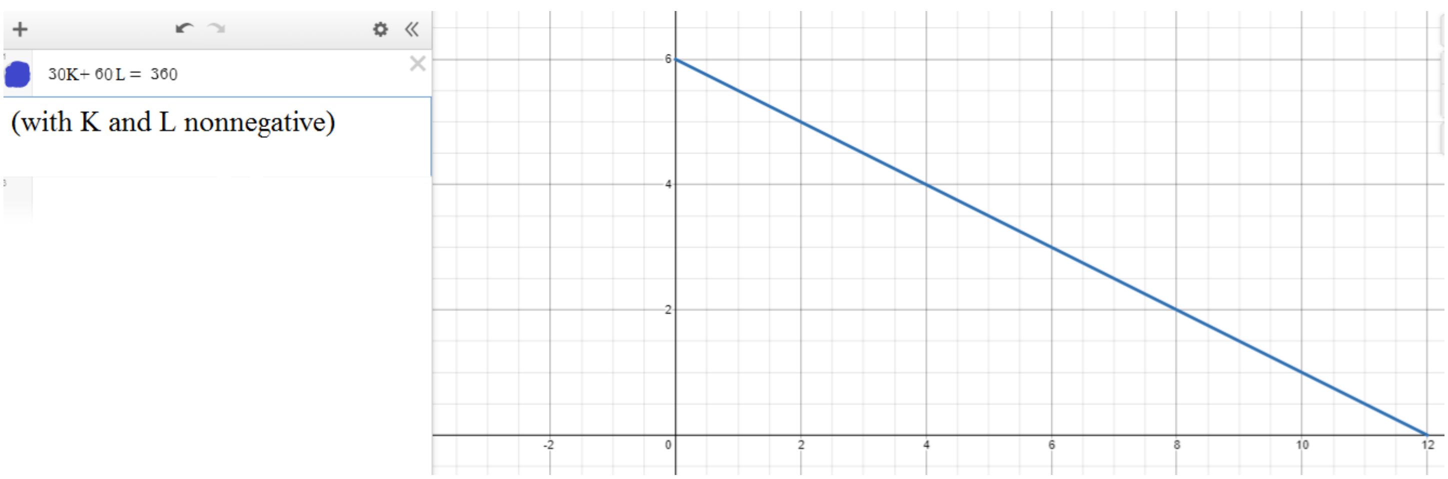
$$P = f(K, L) = 10K^{\frac{1}{3}}L^{\frac{2}{3}} \tag{569}$$

subject to

$$30K + 60L = 360, \tag{570}$$

assuming $K \geq 0$ and $L \geq 0$.

Solution: We see that the region defined by the constraint is the line segment from $(K, L) = (0, 6)$ to $(K, L) = (12, 0)$, which is a closed and bounded region with boundary.



The method of Lagrange multipliers will give us all of the critical points in the interior of the line segment, and we will then compare the values of f at the critical points with the values of f at the boundary (the 2 end points of the line segment) in order to find the absolute maximum and absolute minimum values. We begin by identifying our constraint function $g(K, L)$, its gradient field $\nabla g(K, L)$, and the gradient field $\nabla f(K, L)$ of our optimization function as

$$g(K,L) = 30K + 60L - 360, \nabla g(K,L) = \langle 30, 60\rangle, \text{ and} \tag{571}$$

$$\nabla f(K,L) = \langle \frac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}, \frac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}\rangle. \tag{572}$$

The method of Lagrange multipliers gives us the system of equations

$$\begin{aligned} g(K,L) &= 0 \\ \nabla f(K,L) &= \lambda \nabla g(K,L) \end{aligned} \tag{573}$$

....

$$\Leftrightarrow \begin{aligned} 30K + 60L - 360 &= 0 \\ \langle \tfrac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}, \tfrac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}\rangle &= \lambda\langle 30, 60\rangle \end{aligned} \tag{574}$$

....

$$\Leftrightarrow \begin{aligned} 30K + 60L - 360 &= 0 \\ \tfrac{10}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} &= 30\lambda \\ \tfrac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} &= 60\lambda \end{aligned} \tag{575}$$

....

$$\to \frac{20}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} = 60\lambda = \frac{20}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} \to K = L \tag{576}$$

....

$$\to 0 = 30K + 60L - 360 = 90L - 360 \to L = 4 \to \boxed{(K,L) = (4,4)}. \tag{577}$$

Since $(4,4)$ is the only critical point given to use by the method of Lagrange multipliers and

$$f(4,4) = 10 \cdot 4^{\frac{1}{3}}4^{\frac{2}{3}} = 10 \cdot 4 = 40 > 0 = f(12,0) = f(0,6), \tag{578}$$

we see that the production function attains its absolute maximum value (subject to the given constraint) of 40 at $(4,4)$.

Problem 3.13: Given the production function $P = f(K, L) = K^aL^{1-a}$ and the budget constraint $pK + qL = B$, where a, p, q, and B are given, show that P is maximized when $K = aB/p$ and $L = (1 - a)B/q$. (Recall that $p, q, K, L \geq 0$ and $0 < a < 1$ in order for the model to make sense in the real world and for the production function f to be well defined.)

Solution: We see that the region defined by the constraint is the line segment from $(K, L) = (0, \frac{B}{q})$ to $(K, L) = (\frac{B}{p}, 0)$, which is a closed and bounded region with boundary. The method of Lagrange multipliers will give us all of the critical points in the interior of the line segment, and we will then compare the values of f at the critical points with the values of f at the boundary (the 2 end points of the line segment) in order to find the absolute maximum and absolute minimum values. We begin by identifying our constraint function $g(K, L)$, its gradient field $\nabla g(K, L)$, and the gradient field $\nabla f(K, L)$ of our optimization function as

(579) $$g(K, L) = pK + qL - B, \nabla g(K, L) = \langle p, q\rangle, \text{ and}$$

(580) $$\nabla f(K, L) = \langle aK^{a-1}L1 - a, (1 - a)K^aL^{-a}\rangle.$$

The method of Langrange multipliers gives us the system of equations

(581) $$\begin{aligned} g(K, L) &= 0 \\ \nabla f(K, L) &= \lambda\nabla g(K, L) \end{aligned}$$

. . . .

(582) $$\Leftrightarrow \begin{aligned} pK + qL - B &= 0 \\ \langle aK^{a-1}L1 - a, (1 - a)K^aL^{-a}\rangle &= \lambda\langle p, q\rangle \end{aligned}$$

. . . .

(583) $$\Leftrightarrow \begin{aligned} pK + qL - B &= 0 \\ aK^{a-1}L1 - a &= p\lambda \\ (1 - a)K^aL^{-a} &= q\lambda \end{aligned}$$

. . . .

(584) $$\rightarrow qaK^{a-1}L^{1-a} = pq\lambda = p(1 - a)K^aL^{-a}$$

. . . .

$$\rightarrow qaL = p(1-a)K \rightarrow L = \frac{p(1-a)}{qa}K \tag{585}$$

. . . .

$$\rightarrow 0 = pK + qL - B = pK + \frac{p(1-a)}{a}K - B \rightarrow K = \frac{Ba}{p} \tag{586}$$

. . . .

$$\rightarrow L \overset{\text{(By (585))}}{=} \frac{B(1-a)}{q}, \text{ so} \tag{587}$$

. . . .

$$(K, L) = (\frac{Ba}{p}, \frac{B(1-a)}{q}) \tag{588}$$

is the only critical point obtained by the method of Lagrange multipliers. We see that $K, L > 0$ at this critical point, so

$$f(K, L) > 0 = f(0, \frac{B}{q}) = f(\frac{B}{p}, 0). \tag{589}$$

Since the value of f at the (only) critical point is larger than the values of f on the boundary (the end points) we see that f attains its absolute maximum value at the critical point as desired.

Problem 3.14: Find the absolute minimum and absolute maximum values of the function

$$f(x,y) = x^2 + 4y^2 + 1 \tag{590}$$

over the region

$$R = \{(x,y) : x^2 + 4y^2 \leq 1\}. \tag{591}$$

You should know how to solve this type of problem using lagrange multipliers, but you can avoid using lagrange multipliers (and even avoid parameterization of the boundary) in this particular problem if you think about it carefully.

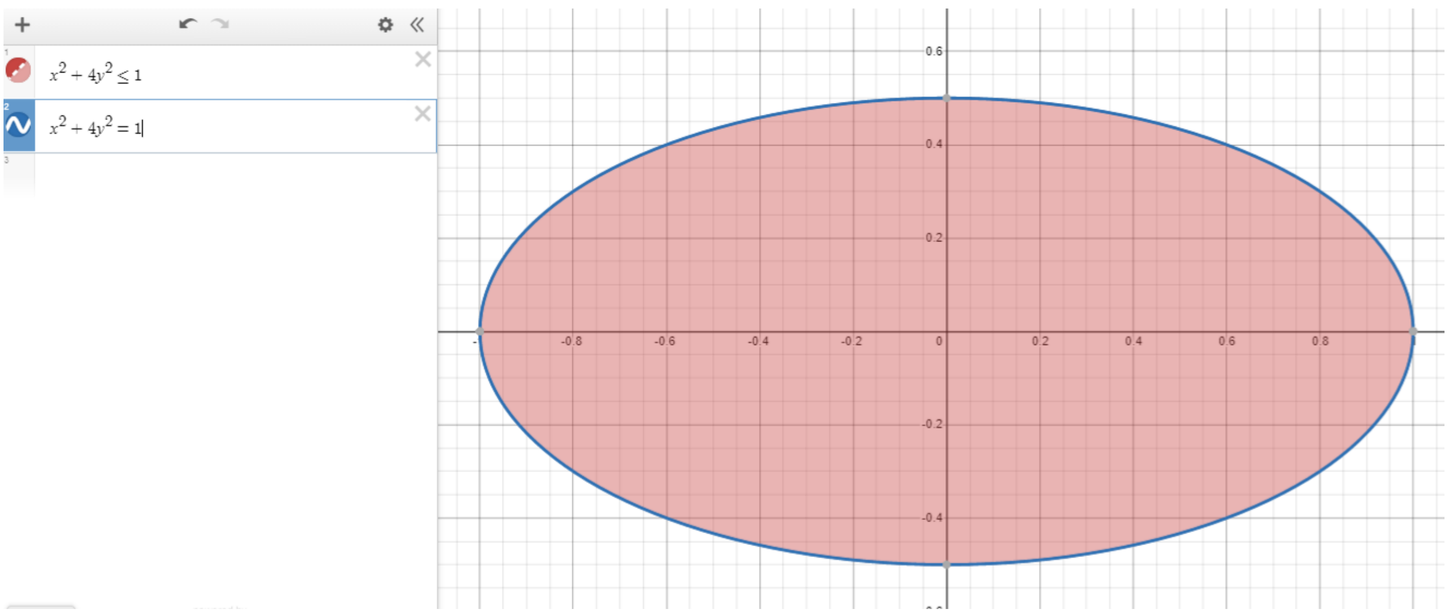


FIGURE 27. The interior of the R is shaded in red and the boundary of R is blue.

Solution: Since the region R is a closed and bounded region, and the function f is continuous, the extreme value theorem tells us that the absolute minimum and absolute maximum values of f must be achieved on the boundary of R or at a critical point in the interior of R. We first find all of the critical points of f. We see that

$$\begin{array}{rcl} f_x(x,y) & = & 0 \\ f_y(x,y) & = & 0 \end{array} \Leftrightarrow \begin{array}{rcl} 2x & = & 0 \\ 8y & = & 0 \end{array} \Leftrightarrow (x,y) = (0,0). \tag{592}$$

We see that $(0,0) \in R$ and that $f(0,0) = 1$. Next we will determine the absolute minimum and absolute maximum values of f on the boundary of R. Since the boundary of R is given by $x^2 + 4y^2 = 1$, we see that $f(x,y) = 2$ for every (x,y) on the boundary of R, so we immediately see that f achieves its absolute minimum value of 1 at $(0,0)$ and its absolute maximum value of 2 at any (x,y) on the boundary of R.

If we were not lucky enough to instatly notice that $f(x,y) = 2$ for every (x,y) on the boundary of R, then we would try to handle the boundary by using the method of Lagrange multipliers. More specifically, we would try to optimize the function $f(x,y) = 1 + x^2 + 4y^2$ subject to the constraint $g(x,y) = x^2 + 4y^2 - 1 = 0$. Noting that

$$\nabla g(x,y) = \langle 2x, 8y\rangle \text{ and } \nabla f(x,y) = \langle 2x, 8y\rangle \tag{593}$$

the method of Lagrange multipliers gives us the system of equations

$$\begin{array}{rcl} g(x,y) & = & 0 \\ \nabla f(x,y) & = & \lambda\nabla g(x,y) \end{array} \Leftrightarrow \begin{array}{rcl} g(x,y) & = & 0 \\ \langle 2x, 8y\rangle & = & \lambda\langle 2x, 8y\rangle \end{array} \tag{594}$$

$$\Leftrightarrow \begin{array}{rcl} g(x,y) & = & 0 \\ 2x & = & 2\lambda x \\ 8y & = & 8\lambda y \end{array}. \tag{595}$$

Letting $\lambda = 1$, we see that every point (x,y) on the boundary of R (which is the same as every point (x,y) satisfying the constraint $g(x,y) = 0$ also satisfies the system of equations given to us by the method of Lagrange multipliers. This seems bad at first since the boundary has infinitely many points, so it looks like the method of Lagrange multipliers did not help us in our search for the absolute minimum and absolute maximum values that occur on the boundary. However, it turns out that the only time every point on the boundary of our region R (assuming that R has a piecewise smooth boundary, which it always will in this class) is a critical point is when $f(x,y)$ is constant on the region R (as it was in this problem), so the problem turns out to be easier in these cases since you can determine the value of $f(x,y)$ on the boundary of R by checking the value at any random point (x,y) on the boundary of R.

Problem 3.15: Show that each of the following functions $f(x,y)$ have exactly 1 critical point that is a local extrema, but not a global extrema.[9]

(i) $f(x,y) = e^{3x} + y^3 - 3ye^x$.
(ii) $f(x,y) = x^2 + y^2(1+x)^3$.

Remark: A continuously differentiable single variable function $f(x)$ that has exactly 1 critical point that is a local extrema will also have that critical point be a global extrema. This problem shows that the same phenomena does not hold for functions of 2 or more variables.

Solution to (i): We begin by finding the critical points of $f(x,y)$.

$$\begin{aligned} 0 = & f_x(x,y) = 3e^{3x} - 3ye^x \to y = e^{2x} \to e^{4x} = e^x \to x = 0 \to y = 1 \\ 0 = & f_y(x,y) = 3y^2 - 3e^x \qquad y^2 = e^x \end{aligned} \tag{596}$$

We will now proceed to apply the second derivative test to the critical point $(0,1)$ to verify that it is a local extremum.

$$\begin{aligned} & f_{xx}(x,y) = 9e^{3x} - 3ye^x, \\ & f_{yy}(x,y) = 6y, \\ & f_{x,y}(x,y) = -3e^x, \\ \to & D(0,1) = f_{xx}(0,1)f_{yy}(0,1) - (f_{xy}(0,1))^2 = 6\cdot 6 - (-3)^2 = 27. \end{aligned} \tag{597}$$

Since $D(0,1) > 0$ and $f_{xx}(0,1) > 0$, we see that f has a local minimum at $(0,1)$. To see that f does not have a global minimum or global maximum, it suffices to observe that $f(0,y) = y^3 - 3y + 1$.

Solution to (ii): We begin by finding the critical points of $f(x,y)$.

$$\begin{aligned} 0 = & f_x(x,y) = 2x + 3y^2(1+x)^2 \to x = -\frac{3}{2}y^2(1+x)^2 \to x = 0 \\ 0 = & f_y(x,y) = 2y(1+x)^3 \qquad y = 0 \text{ or } x = -1 \qquad \to \quad y = 0 \end{aligned} \tag{598}$$

We will now proceed to apply the second derivative test to the critical point $(0,0)$ to verify that it is a local extremum.

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[9] I took item (i) from Tom Vogel of Texas A& M and item (ii) from Henry Wente of University of Toledo.

$$\begin{aligned}(599)\qquad f_{xx}(x,y) &= 2+6y^2(1+x),\\ f_{yy}(x,y) &= 2(1+x)^3,\\ f_{x,y}(x,y) &= 6y(1+x)^2,\\ \rightarrow D(0,0) &= f_{xx}(0,0)f_{yy}(0,0)-(f_{xy}(0,0))^2 = 2\cdot 2-0^2 = 4.\end{aligned}$$

Since $D(0,0) > 0$ and $f_{xx}(0,0) > 0$, we see that f has a local minimum at $(0,0)$. To see that f does not have a global minimum or global maximum, it suffices to observe that $f(x,1) = (1+x)^3 + x^2$.

Problem 4.1: Evaluate

(600) $$\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^1 yx\sin(x^2)dydx,$$

(601) $$\int_0^1\int_0^{\sqrt{\frac{\pi}{2}}} yx\sin(x^2)dxdy, \text{ and}$$

(602) $$\left(\int_0^{\sqrt{\frac{\pi}{2}}} x\sin(x^2)dx\right)\left(\int_0^1 ydy\right)$$

Note that all 3 integrals should result in the same value once evaluated. Please show your work for the calculations of each of the 3 integrals separately.

Solution: We see that

(603) $$\int_0^{\sqrt{\frac{\pi}{2}}}\int_0^1 yx\sin(x^2)dydx = \int_0^{\sqrt{\frac{\pi}{2}}}\left(\frac{y^2}{2}x\sin(x^2)\Big|_{y=0}^1\right)dx$$

(604) $$= \int_0^{\sqrt{\frac{\pi}{2}}}\frac{1}{2}x\sin(x^2)dx \overset{u=x^2}{=} \int_{x=0}^{\sqrt{\frac{\pi}{2}}}\frac{1}{4}\sin(u)du = -\frac{1}{4}\cos(u)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}$$

(605) $$= -\frac{1}{4}\cos(x^2)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}} = \boxed{\frac{1}{4}},$$

(606) $$\int_0^1\int_0^{\sqrt{\frac{\pi}{2}}} yx\sin(x^2)dxdy \overset{u=x^2}{=} \int_0^1\int_{x=0}^{\sqrt{\frac{\pi}{2}}}\frac{1}{2}y\sin(u)dudy$$

(607)
$$= \int_0^1\left(-\frac{1}{2}y\cos(u)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)dy = \int_0^1\left(-\frac{1}{2}y\cos(x^2)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)dy = \int_0^1\frac{1}{2}ydy$$

. . . .

$$= \frac{1}{4}y^2\Big|_0^1 = \boxed{\frac{1}{4}}, \text{ and} \tag{608}$$

. . . .

$$\left(\int_0^{\sqrt{\frac{\pi}{2}}} x\sin(x^2)dx\right)\left(\int_0^1 ydy\right) \overset{u=x^2}{=} \left(\int_{x=0}^{\sqrt{\frac{\pi}{2}}} \frac{1}{2}\sin(u)du\right)\left(\frac{1}{2}y^2\Big|_0^1\right) \tag{609}$$

. . . .

$$= \left(-\frac{1}{2}\cos(u)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)\cdot\frac{1}{2} = \left(-\frac{1}{2}\cos(x^2)\Big|_{x=0}^{\sqrt{\frac{\pi}{2}}}\right)\cdot\frac{1}{2} = \frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{4}}. \tag{610}$$

Problem 4.2: Suppose that the second partial derivative of f are continuous on $R = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}$. Show that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y) dA = f(a, b) - f(a, 0) - f(0, b) + f(0, 0). \tag{611}$$

Hint: Think about the fundamental theorem of calculus.

Solution: We see that

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y) dA = \int_0^b \int_0^a \frac{\partial^2 f}{\partial x \partial y}(x, y) dx dy = \int_0^b \frac{\partial f}{\partial y}(x, y)\Big|_{x=0}^{a} dy \tag{612}$$

$$= \int_0^b \left(\frac{\partial f}{\partial y}(a, y) - \frac{\partial f}{\partial y}(0, y) \right) dy = (f(a, y) - f(0, y))\Big|_0^b. \tag{613}$$

$$= f(a, b) - f(0, b) - f(a, 0) + f(0, 0). \tag{614}$$

Alternatively, since the second partial derivatives of f are continuous on R, we can use Clairaut's Theorem to perform the calculations in the following fashion.

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y) dA = \int_0^a \int_0^b \frac{\partial^2 f}{\partial y \partial x}(x, y) dy dx = \int_0^a \frac{\partial f}{\partial x}(x, y)\Big|_{y=0}^{b} dx \tag{615}$$

$$= \int_0^a \left(\frac{\partial f}{\partial x}(x, b) - \frac{\partial f}{\partial x}(x, 0) \right) dx = (f(x, b) - f(x, 0))\Big|_0^a. \tag{616}$$

$$= f(a, b) - f(a, 0) - f(0, b) + f(0, 0). \tag{617}$$

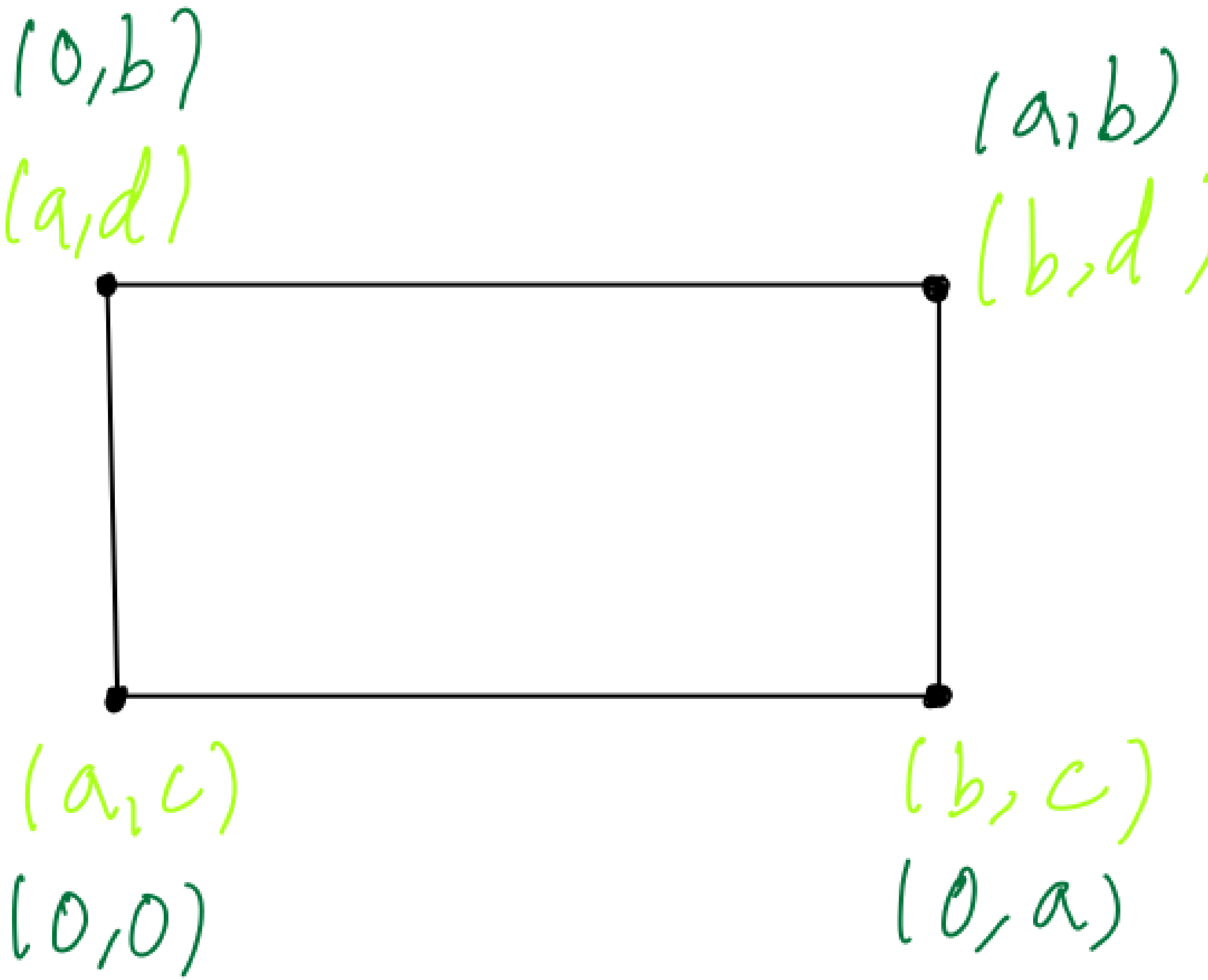
Remark: A similar method can show that if $R = \{(x, y) : a \le x \le b, c \le y \le d\}$, then

$$\iint_R \frac{\partial^2 f}{\partial x \partial y}(x, y) dA = f(b, d) - f(a, d) - f(b, c) + f(a, c). \tag{618}$$

The Fundamental Theorem of Calculus told us that

$$\int_a^b \frac{df}{dx}(x)dx = f(b) - f(a). \tag{619}$$

Comparing equations (619) and (618), we see that instead taking the difference at the 2 endpoints of a line segment, we are adding 2 opposite corners of the rectangular region R ($f(b,d)$ and $f(a,c)$, or $f(a,b)$ and $f(0,0)$ from the original problem) and subtracting from that the sum of the other 2 opposite corners ($f(a,d)$ and $f(b,c)$, or $f(a,0)$ and $f(0,b)$ from the original problem).



Problem 4.3: Let $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.
(a) Evaluate $\iint_R \cos(x\sqrt{y})dA$.
(b) Evaluate $\iint_R x^3 y\cos(x^2y^2)dA$.

Hint: Choose a convenient order of integration.

Solution to a: Noting that $\int \cos(cx)dx$ is easily computable, but $\int \cos(c\sqrt{y})dy$ is not easily computable, we decide to use the order of integration given by $dA = dxdy$. It follows that

(620) $$\iint_R \cos(x\sqrt{y})dA = \int_0^1\int_0^1 \cos(x\sqrt{y})dxdy$$

(621) $$\overset{u=x\sqrt{y}}{=} \int_0^1\int_0^1 \frac{\cos(x\sqrt{y})}{\sqrt{y}}\sqrt{y}dxdy \overset{u=x\sqrt{y}}{=} \int_0^1\int_{x=0}^1 \frac{\cos(u)}{\sqrt{y}}dudy$$

(622) $$= \int_0^1 \left(\frac{\sin(u)}{\sqrt{y}}\bigg|_{x=0}^1\right)dy = \int_0^1 \left(\frac{\sin(x\sqrt{y})}{\sqrt{y}}\bigg|_{x=0}^1\right)dy = \int_0^1 \frac{\sin(\sqrt{y})}{\sqrt{y}}dy$$

(623) $$\overset{u=\sqrt{y}}{=} \int_0^1 2\sin(\sqrt{y})\frac{dy}{2\sqrt{y}} \overset{u=\sqrt{y}}{=} \int_{y=0}^1 2\sin(u)du = -2\cos(u)\bigg|_{y=0}^1$$

(624) $$= -2\cos(\sqrt{y})\bigg|_{y=0}^1 = \boxed{2-2\cos(1)}.$$

Solution to b: Noting that $\int c_1x^3\cos(c_2x^2)dx$ is not easily computable, but $\int c_1y\cos(c_2y^2)dy$ is easily computable, we decide to use the order of integration given by $dA = dydx$. It follows that

(625) $$\iint_R x^3y\cos(x^2y^2)dA = \int_0^1\int_0^1 x^3y\cos(x^2y^2)dydx$$

$$\overset{u=y^2}{=} \int_0^1 \int_0^1 \frac{x^3}{2}\cos(x^2y^2)2ydydx \overset{u=y^2}{=} \int_0^1 \int_{y=0}^1 \frac{x^3}{2}\cos(x^2u)dudx \tag{626}$$

. . . .

$$\overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2}\cos(x^2u)x^2dudx \overset{v=x^2u}{=} \int_0^1 \int_{y=0}^1 \frac{x}{2}\cos(v)dvdx \tag{627}$$

. . . .

$$= \int_0^1 \left(\frac{x}{2}\sin(v)\Big|_{y=0}^1 \right) dx = \int_0^1 \left(\frac{x}{2}\sin(x^2u)\Big|_{y=0}^1 \right) dx \tag{628}$$

. . . .

(629)

$$= \int_0^1 \left(\frac{x}{2}\sin(x^2y^2)\Big|_{y=0}^1 \right) dx = \int_0^1 \frac{x}{2}\sin(x^2)dx \overset{u=x^2}{=} \int_0^1 \frac{1}{4}\sin(x^2)2xdx$$

. . . .

$$\overset{u=x^2}{=} \int_{x=0}^1 \frac{1}{4}\sin(u)du = -\frac{1}{4}\cos(u)\Big|_{x=0}^1 \tag{630}$$

. . . .

$$= -\frac{1}{4}\cos(x^2)\Big|_{x=0}^1 = \boxed{\frac{1}{4} - \frac{1}{4}\cos(1)}. \tag{631}$$

Problem 4.4: Let $R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. Let F be an antiderivative of f satisfying $F(0) = 0$, and let G be an antiderivative of F. Show that if f and F are integrable, and $r, s \ge 1$ are real numbers, then

$$\iint_R x^{2r-1}y^{s-1}f(x^r y^s)dA = \frac{G(1) - G(0)}{rs}. \tag{632}$$

Hint: Pick a convenient order of integration, then apply u-substitution. It also helps if you do problem 14.1.60 before doing this problem.

Solution: We note that Problem 14.1.60b was a special instance of this problem in which $r = s = 2$ and $f(t) = \cos(t)$. Therefore we will proceed in a similar fashion, but we will slightly simplify our solution by merging the first 2 u-substitutions that were performed in the solution to Problem 14.1.60b into a single u-substitution. We now see that

$$\iint_R x^{2r-1}y^{s-1}f(x^r y^s)dA = \int_0^1 \int_0^1 x^{2r-1}y^{s-1}f(x^r y^s)dydx \tag{633}$$

$$\overset{u=x^r y^s}{=} \int_0^1 \int_0^1 x^{r-1}f(x^r y^s)x^r y^{s-1}dydx \overset{u=x^r y^s}{=} \int_0^1 \int_{y=0}^1 \frac{x^{r-1}}{s}f(u)dudx \tag{634}$$

$$= \int_0^1 \left(\frac{x^{r-1}}{s}F(u)\Big|_{y=0}^1\right)dx = \int_0^1 \left(\frac{x^{r-1}}{s}F(x^r y^s)\Big|_{y=0}^1\right)dx \tag{635}$$

$$= \int_0^1 \left(\frac{x^{r-1}}{s}F(x^r) - \frac{x^{r-1}}{s}\underbrace{F(0)}_{=0}\right)dx = \int_0^1 \frac{x^{r-1}}{s}F(x^r)dx \tag{636}$$

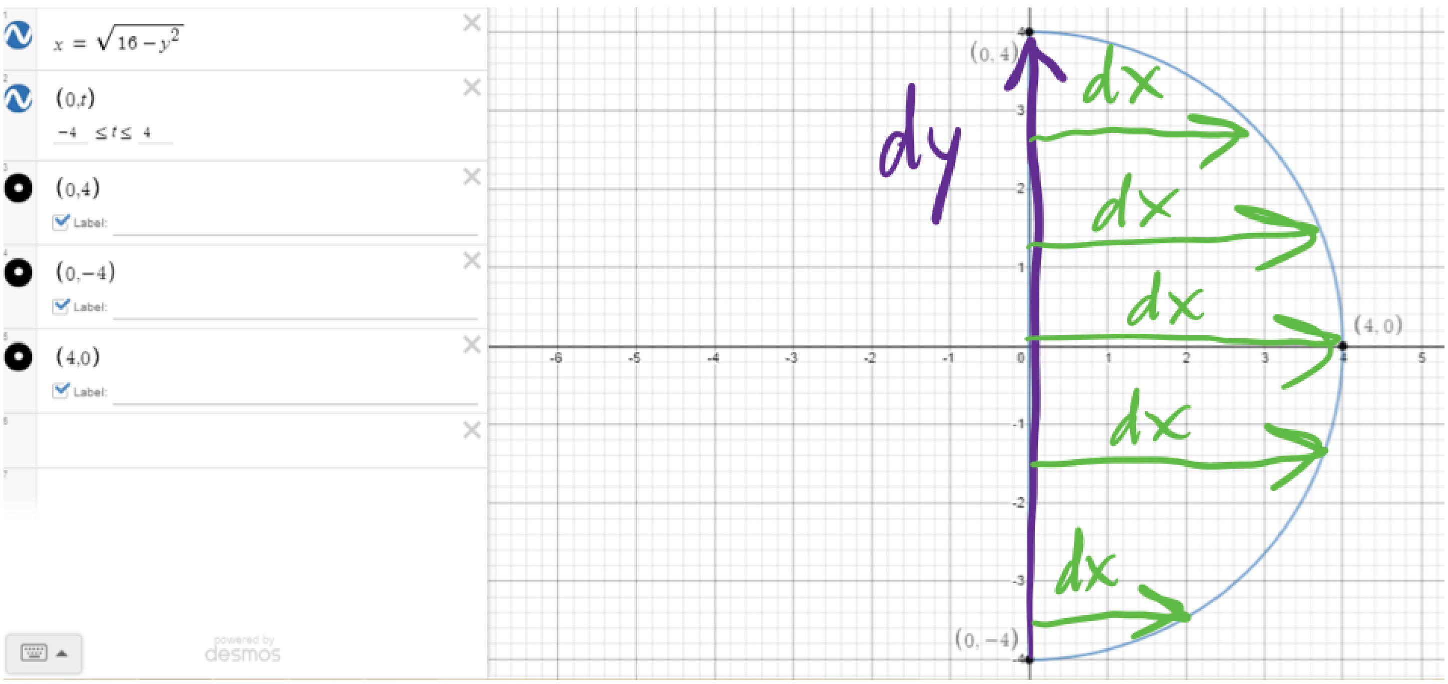
$$\overset{u=x^r}{=} \int_0^1 \frac{1}{rs}F(x^r)rx^{r-1}dx \overset{u=x^r}{=} \int_{x=0}^1 \frac{1}{rs}F(u)du \tag{637}$$

$$= \frac{1}{rs}G(u)\Big|_{x=0}^{1} = \frac{1}{rs}G(x^r)\Big|_{x=0}^{1} = \frac{G(1) - G(0)}{rs}. \tag{638}$$

Problem 4.5: Let R be the region in quadrants 1 and 4 bounded by the semicircle of radius 4 centered at $(0,0)$. Sketch a picture of R, then evaluate

$$\iint_R x^2 y dA. \tag{639}$$

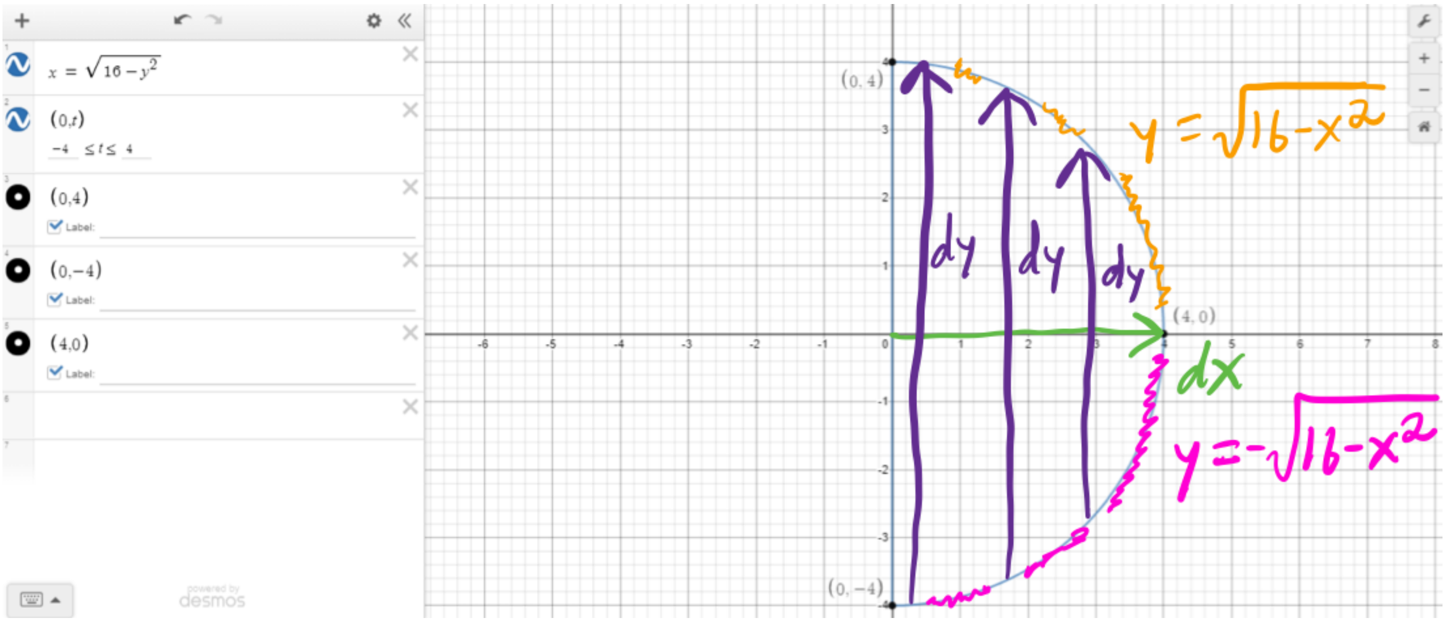
Solution 1: If we decide to integrate using the order $dA = dxdy$, then we obtain the picture and calculations show below.



$$\iint_R x^2 y dA = \int_{-4}^{4} \int_{0}^{\sqrt{16-y^2}} x^2 y dx dy = \int_{-4}^{4} \left(\frac{x^3}{3} y \Big|_{x=0}^{\sqrt{16-y^2}} \right) dy \tag{640}$$

$$= \int_{-4}^{4} \frac{1}{3} y (16 - y^2)^{\frac{3}{2}} dy = -\frac{1}{15}(16 - y^2)^{\frac{5}{2}} \Big|_{-4}^{4} = \boxed{0}. \tag{641}$$

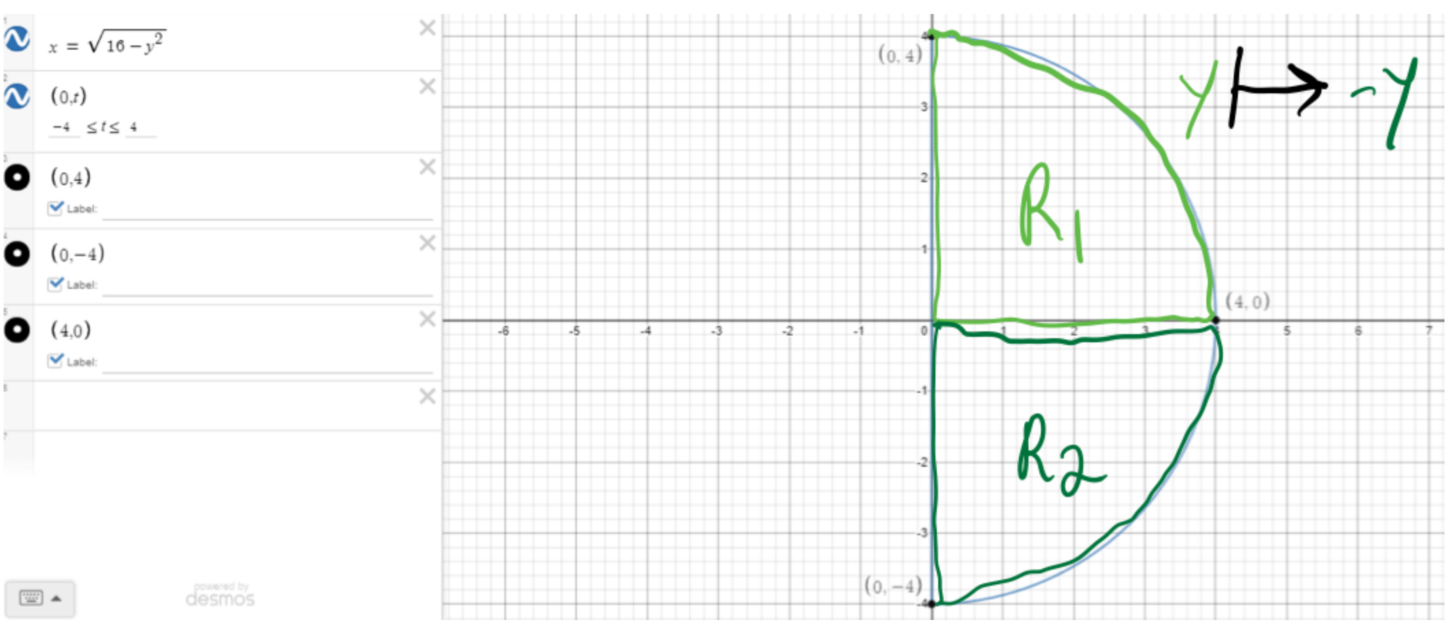
Solution 2: If we decide to integrate using the order $dA = dydx$, then we obtain the picture and calculations show below.



$$\iint_R x^2 y\,dA = \int_0^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} x^2 y\,dy\,dx = \int_0^4 \frac{1}{2}\left(x^2y^2 \Big|_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \right) dx \tag{642}$$

$$= \int_0^4 0\,dx = \boxed{0}. \tag{643}$$

Solution 3: Using the symmetry shown below,



we see that

$$\iint_R x^2 y dA = \iint_{R_1} x^2 y dA + \iint_{R_2} x^2 y dA \tag{644}$$

$$= -\iint_{R_2} x^2 y dA + \iint_{R_2} x^2 y dA = \boxed{0}. \tag{645}$$

To see the details of the above calculation worked out in more detail, we proceed as we did in solution 2.

$$\iint_{R_1} x^2 y dA = \int_0^4 \int_0^{\sqrt{16-x^2}} x^2 y dy dx = \int_0^4 \int_0^{\sqrt{16-x^2}} x^2(-y)(-dy) dx \tag{646}$$

$$\overset{y=-y}{=} \int_0^4 \int_0^{-\sqrt{16-x^2}} x^2 y dy dx = -\int_0^4 \int_{-\sqrt{16-x^2}}^0 x^2 y dy dx \tag{647}$$

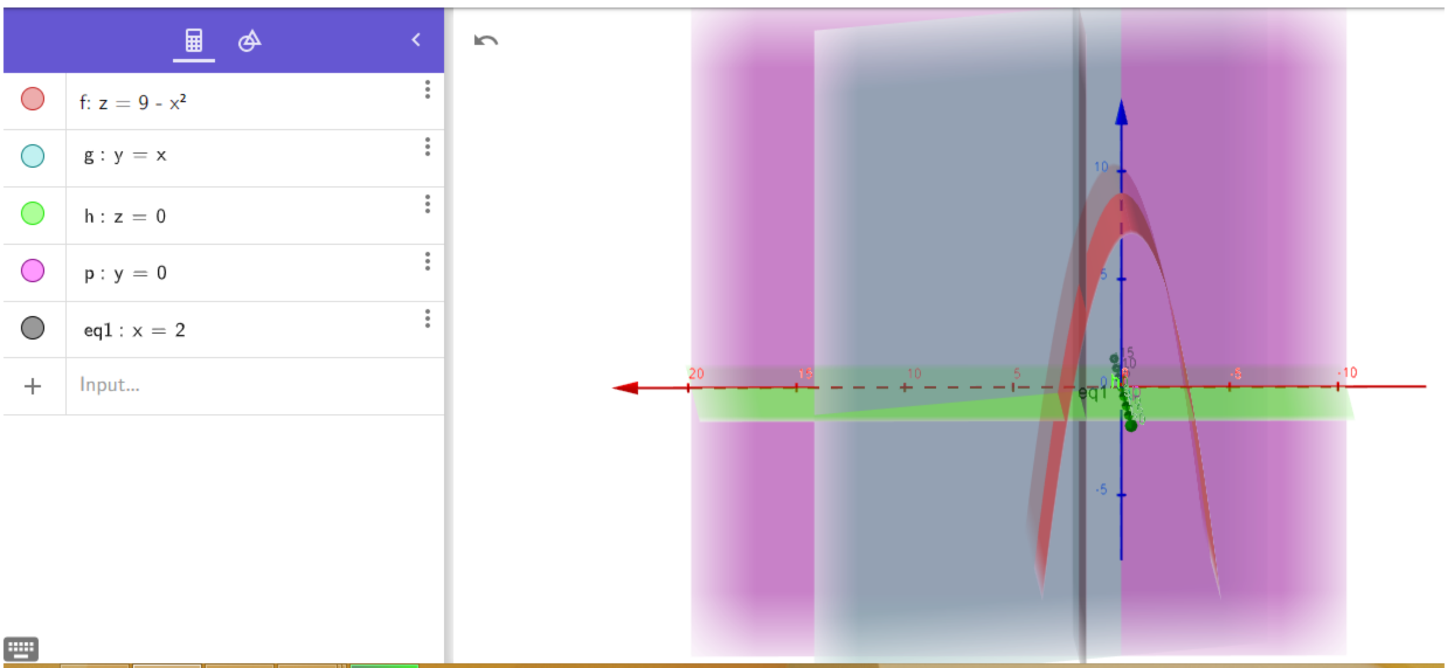
$$= -\iint_{R_2} x^2 y dA. \tag{648}$$

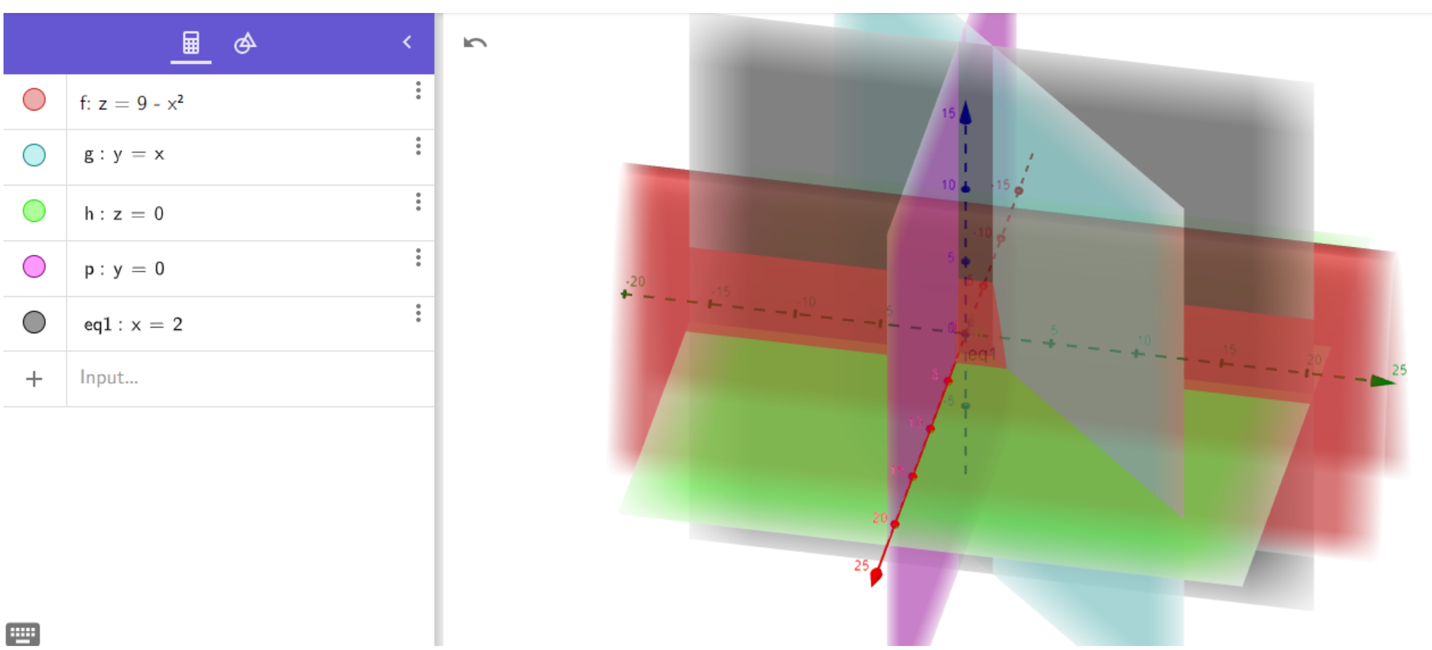
Problem 5.9: Rewrite the the triple integral

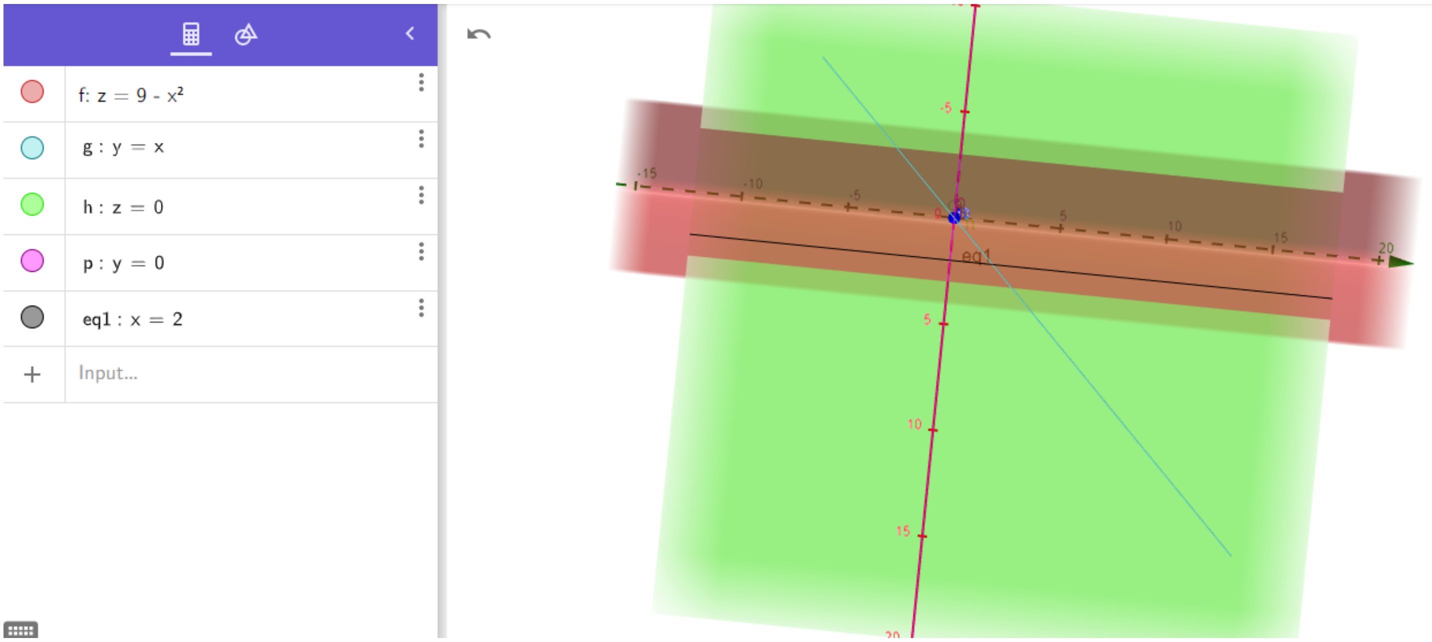
$$\int_0^2 \int_0^{9-x^2} \int_0^x f(x,y,z)dydzdx \tag{649}$$

using the order $dzdxdy$.

First Solution: We envision the 3-dimensional solid that is described by the bounds of the triple integral in the currect order of $dydzdx$, and then we traverse the solid using the new order of $dzdxdy$.

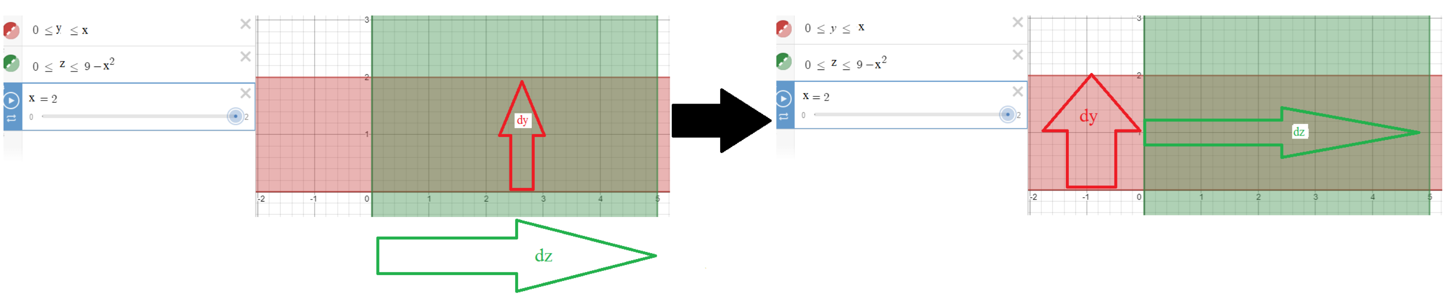


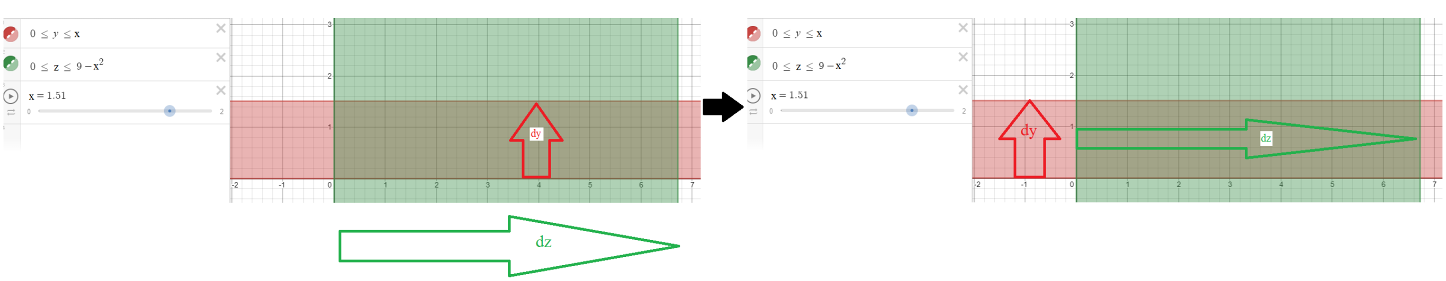




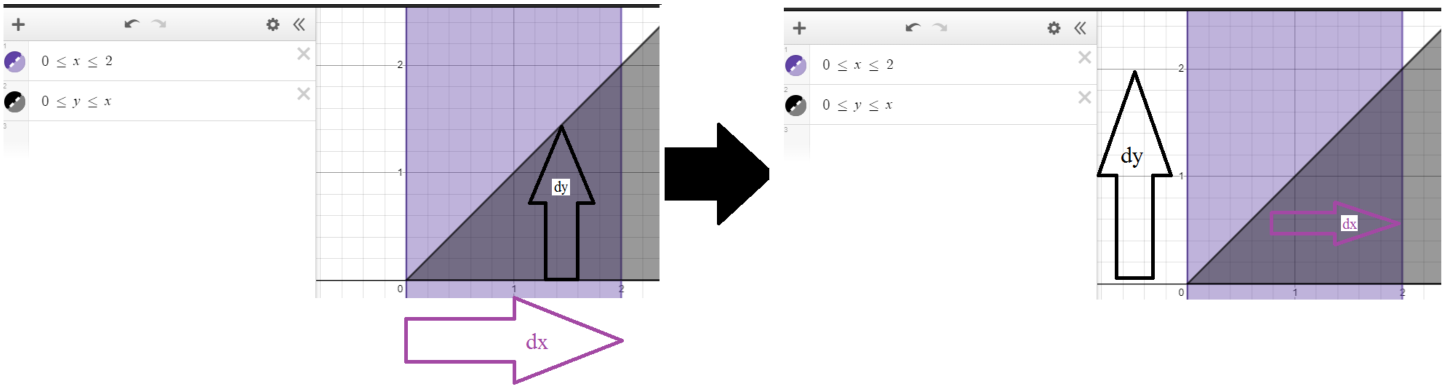
$$\boxed{\int_0^2\int_y^2\int_0^{9-x^2} f(x,y,z)dzdxdy}. \tag{650}$$

Second Solution: In order to avoid drawing and thinking about 3-dimensional regions, we will perform 2 separate changes of order. We will first change the order from $dydzdx$ to $dzdydx$, and then we will change the order from $dzdydx$ to $dzdxdy$.



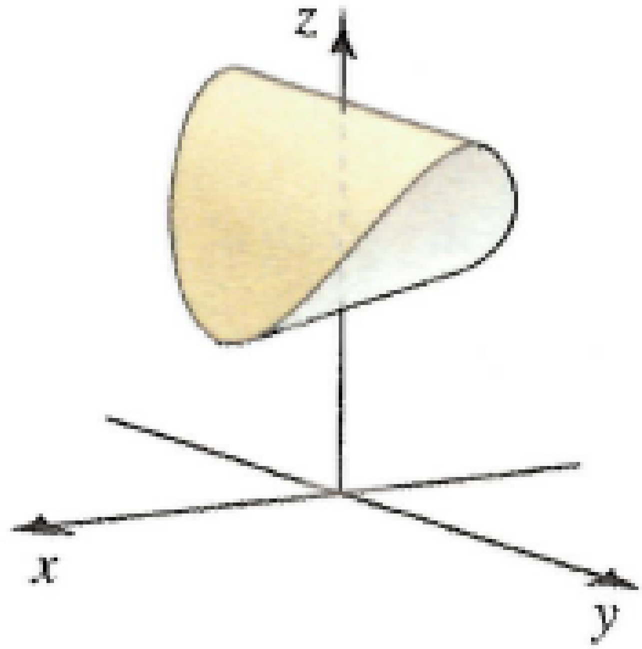


$$(651) \qquad \int_0^2 \int_0^{9-x^2} \int_0^x f(x,y,z)\,dydzdx = \int_0^2 \int_0^x \int_0^{9-x^2} f(x,y,z)\,dzdydx$$

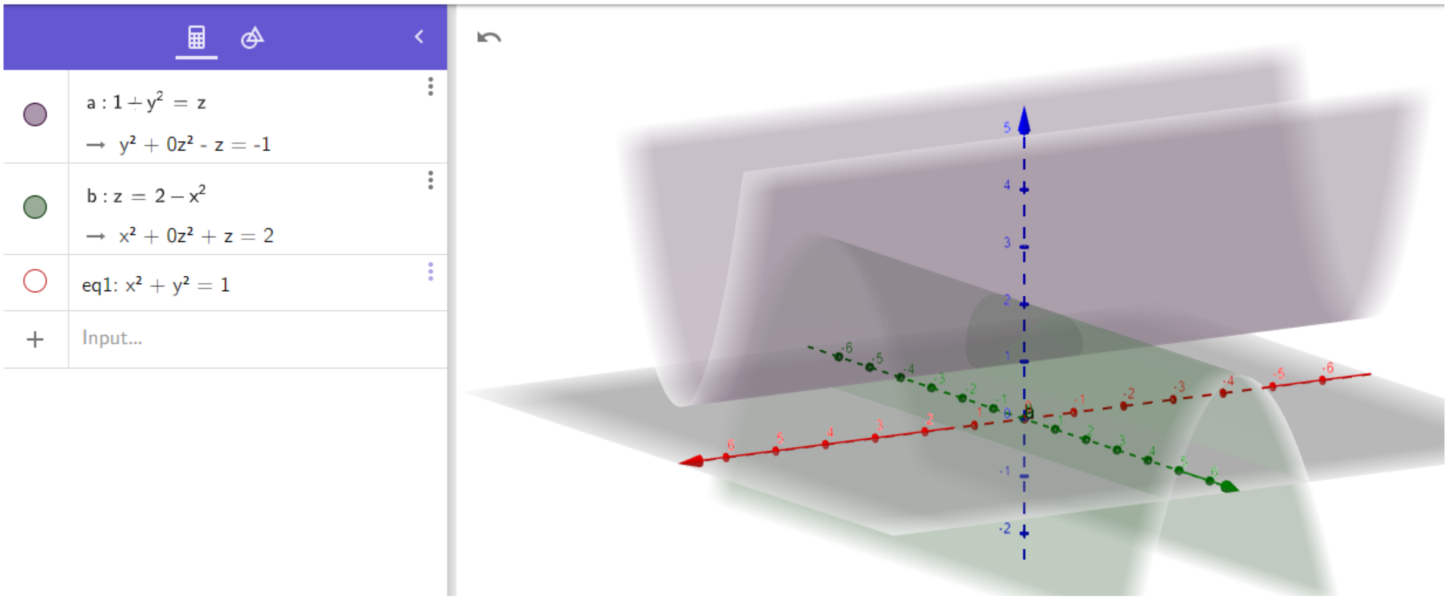


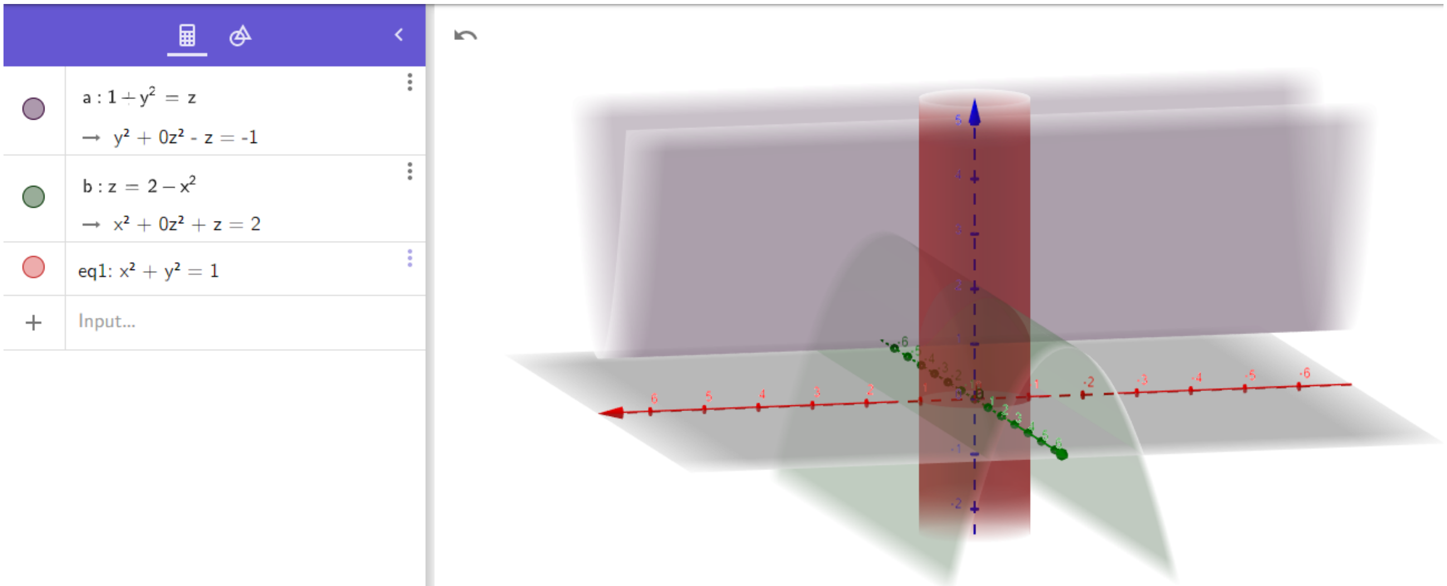
$$(652) \qquad \int_0^2 \int_0^x \int_0^{9-x^2} f(x,y,z)\,dzdydx = \boxed{\int_0^2 \int_y^2 \int_0^{9-x^2} f(x,y,z)\,dzdxdy}.$$

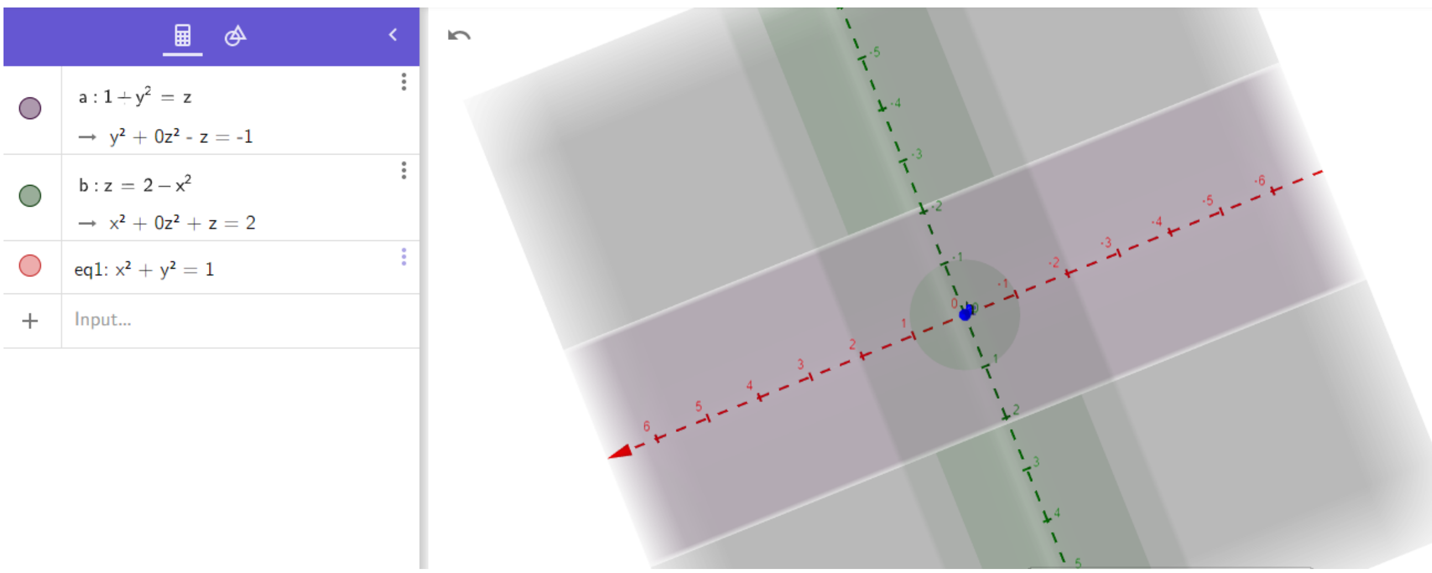
Problem 5.10: Find the volume of the solid S that is bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$.



Solution: S is a 3 dimensional solid that is defined as the region inbetween 2 surfaces. First, we find the intersection I of $z = y^2 + 1$ and $z = 2 - x^2$ to satisfy $y^2 + 1 = 2 - x^2$ or $x^2 + y^2 = 1$.







It follows that the (x, y)-coordinates of I are the circle of radius 1 centered at the origin. Note that the intersection I is not itself a circle since the z-coordinate is not constant on the intersection. NThankfully, for the purposes of calculating the volume of S, we only need to know the projection R of I onto the xy-plane (along with the interior of the projection), which is the same as knowing the the (x, y)-coordinates of I.

$$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bottom}})dA \tag{653}$$

$$= \int_0^{2\pi} \int_0^1 \left((2 - (r\cos(\theta))^2) - ((r\sin(\theta))^2 + 1)\right) r dr d\theta \tag{654}$$

$$= \int_0^{2\pi}\int_0^1 \left(1 - r^2\cos^2(\theta) - r^2\sin^2(\theta)\right) r dr d\theta \tag{655}$$

$$= \int_0^1\int_0^{2\pi} \left(r - r^3\right) d\theta dr = \int_0^{\sqrt{3}} \left(r\theta - r^3\theta\right)\Big|_{\theta=0}^{2\pi} dr \tag{656}$$

$$= \int_0^1 2\pi\left(r - r^3\right) dr = 2\pi\left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_0^1 = \boxed{\frac{\pi}{2}}. \tag{657}$$

Remark: We could have also calculated the volume by using a triple integral in cylindrical coordinates as follows.

$$\text{Volume}(S) = \iiint_S 1 dV = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r^2\sin^2(\theta)+1}^{2-r^2\cos^2(\theta)} r dz dr d\theta = \boxed{\pi}. \tag{658}$$

Problem 7.1: Use a scalar line integral to find the length of the curve

(659) $$\vec{r}(t) = \langle 20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2}\rangle, \text{ for } 0 \le t \le 2.$$

Solution: Firstly, we note that

(660) $$\vec{r}\,'(t) = \langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle.$$

We now see that

(661)
$$\text{Arclength(C)} = \int_C 1ds = \int_0^2 |\vec{r}\,'(t)|dt = \int_0^2 |\langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle|dt$$

(662) $$= \int_0^2 \sqrt{\left(5\cos(\frac{t}{4})\right)^2 + \left(-5\sin(\frac{t}{4})\right)^2 + \left(\frac{1}{2}\right)^2}dt$$

(663) $$= \int_0^2 \sqrt{25\cos^2(\frac{t}{4}) + 25\sin^2(\frac{t}{4}) + \frac{1}{4}}dt = \int_0^2 \sqrt{25\frac{1}{4}}dt$$

(664) $$= \sqrt{25\frac{1}{4}}t\Big|_0^2 = 2\sqrt{25\frac{1}{4}} = \boxed{\sqrt{101}}.$$

Problem 7.2: Find the work required to move an object along the line segment from $(1,1,1)$ to $(8,4,2)$ through the forcefield $\vec{F}$ given by

$$\vec{F} = \frac{\langle x,y,z\rangle}{x^2+y^2+z^2}. \tag{665}$$

Solution 1: Firstly, we recall that one method of parameterizing the line segment that starts at $\vec{p}$ and ends at $\vec{q}$ is to use the parameterization

$$\vec{r}(t) = (1-t)\vec{p} + t\vec{q} = \vec{p} + t(\vec{q}-\vec{p}), \quad 0 \le t \le 1. \tag{666}$$

It follows that

$$\vec{r}(t) = \langle 1,1,1\rangle + t\left(\langle 8,4,2\rangle - \langle 1,1,1\rangle\right) = \langle 1+7t, 1+3t, 1+t\rangle, \quad 0 \le t \le 1, \tag{667}$$

is a parameterization of the line segment from $(1,1,1)$ to $(8,4,2)$. We now see that

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)dt \tag{668}$$

$$= \int_0^1 \underbrace{\frac{\langle 1+7t, 1+3t, 1+t\rangle}{(1+7t)^2+(1+3t)^2+(1+t)^2}}_{\vec{F}(\vec{r}(t))}\cdot\underbrace{\langle 7,3,1\rangle dt}_{d\vec{r}} \tag{669}$$

$$= \int_0^1 \frac{(1+7t)\cdot 7 + (1+3t)\cdot 3 + (1+t)\cdot 1}{1+14t+49t^2+1+6t+9t^2+1+2t+t^2}dt \tag{670}$$

$$= \int_0^1 \frac{11+59t}{3+22t+59t^2}dt = \int_0^1 \frac{t+\frac{11}{59}}{t^2+\frac{22}{59}t+\frac{3}{59}}dt = \int_0^1 \frac{t+\frac{11}{59}}{(t+\frac{11}{59})^2+\frac{56}{3481}}dt \tag{671}$$

$$= \frac{1}{2}\ln\left((t+\frac{11}{59})^2+\frac{56}{3481}\right)\Big|_0^1 = \boxed{\frac{1}{2}\ln(28)}. \tag{672}$$

Solution 2: We note that for $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$ we have $\nabla\varphi = \vec{F}$, so the Fundamental Theorem for Line Integrals (section 3.3) allows us to simplify the calculations from equations (668)-(672) as follows.

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi\cdot d\vec{r} = \varphi\left((8,4,2)\right) - \varphi\left((1,1,1)\right) \tag{673}$$

$$= \frac{1}{2}\ln(8^2+4^2+2^2) - \frac{1}{2}\ln(1^2+1^2+1^2) = \frac{1}{2}\ln(84) - \frac{1}{2}\ln(3) = \boxed{\frac{1}{2}\ln(28)}. \tag{674}$$

Problem 7.3: Determine whether the vector field $\vec{F}$ given by

$$\vec{F} = \langle y - e^{x+y}, x - e^{x+y} + 1, \frac{1}{z}\rangle \tag{675}$$

is a conservative vector field. If $\vec{F}$ is conservative, find a potential function φ.

Solution: We see that

$$\begin{aligned} &\vec{F} = \langle m, n, p\rangle, \quad \text{with} \\ &m(x,y,z) = y - e^{x+y}, \quad n(x,y,z) = x - e^{x+y} + 1, \quad p(x,y,z) = \frac{1}{z}. \\ &\text{Since } \frac{\partial m}{\partial y} = 1 - e^{x+y} = \frac{\partial n}{\partial x}, \quad \frac{\partial n}{\partial z} = 0 = \frac{\partial p}{\partial y}, \quad \frac{\partial m}{\partial z} = 0 = \frac{\partial p}{\partial x}, \end{aligned} \tag{676}$$

we see that $\vec{F}$ is a conservative vector field. We will now find the potential function φ for $\vec{F}$. We recall that

$$\langle m, n, p\rangle = \vec{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle. \tag{677}$$

We will now handle the 3 scalar differential equations that arise from (677) in order to find φ (but only up to a constant).

$$\begin{aligned} &\varphi_x(x,y,z) = m(x,y,z) = y - e^{x+y} \\ \to &\varphi(x,y,z) = \int (y - e^{x+y})dx = xy - e^{x+y} + h(y,z) \\ &x - e^{x+y} + 1 = n(x,y,z) = \varphi_y(x,y,z) = x - e^{x+y} + h_y(y,z) \\ \to &h_y(y,z) = 1 \to h(y,z) = \int 1dy = y + g(z) \\ \to &\varphi(x,y,z) = xy - e^{x+y} + y + g(z) \\ &\frac{1}{z} = p(x,y,z) = \varphi_z(x,y,z) = g_z(z) = g'(z) \\ \to &g(z) = \int \frac{1}{z}dz = \ln|z| + C \\ \to &\boxed{\varphi(x,y,z) = xy - e^{x+y} + y + \ln|z| + C}. \end{aligned} \tag{678}$$

Problem 7.4: Evaluate

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 5 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r}, \tag{679}$$

where C is the curve that is shown in the picture below.

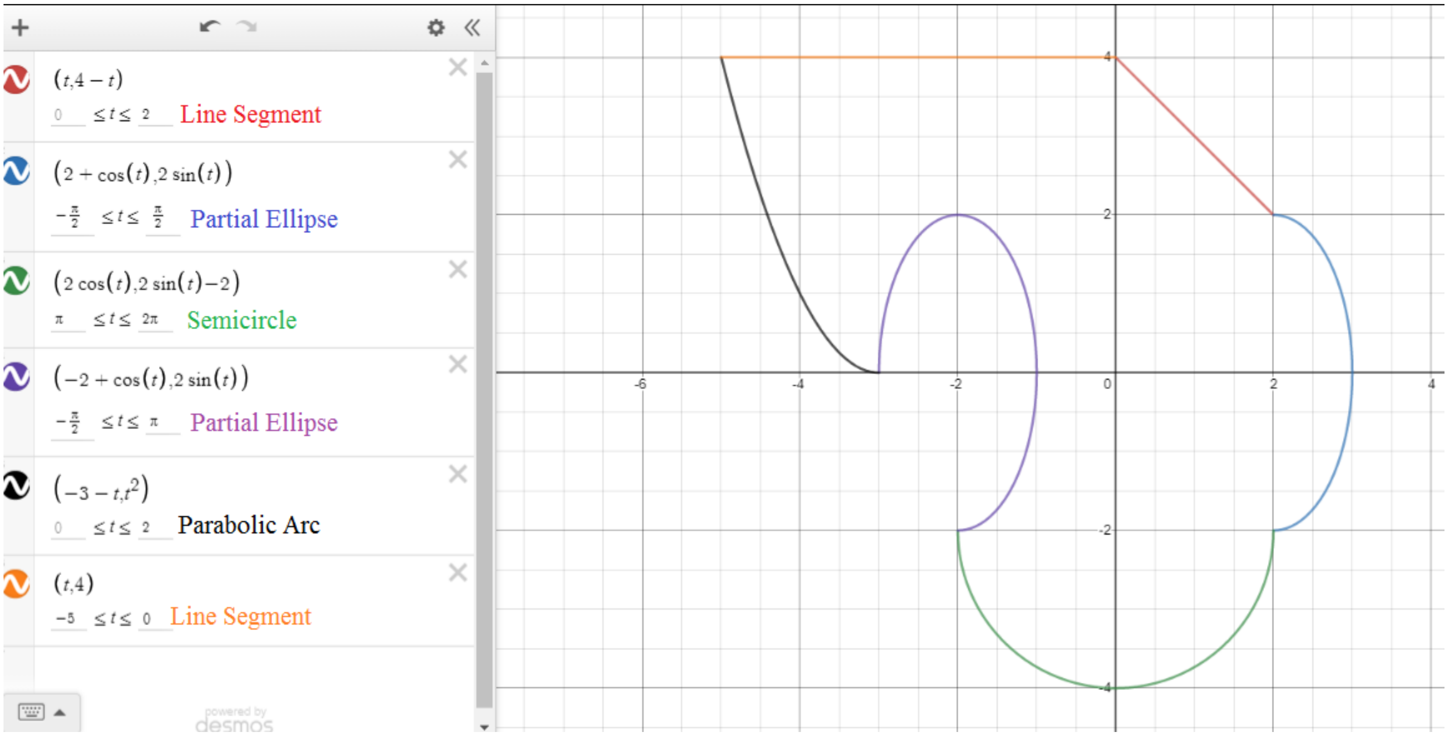


FIGURE 28. The curve C.

Solution: Letting

$$m(x,y,z) = \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 5 + x))) - 1, \text{ and} \tag{680}$$

$$n(x,y,z) = y^3 + 2 + e^{y^2}, \text{ we see that} \tag{681}$$

$$\vec{F} := \langle m, n \rangle, \text{ satisfies} \tag{682}$$

$$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \tag{683}$$

so $\vec{F}$ is a conservative vector field. We also see that

(684) $$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 5 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}.$$

Since $\vec{F}$ is conservative and C is a (simple, piecewise smooth, oriented) closed curve, and $\vec{F}$ is continuous on C and its interior, we see that

(685) $$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}.$$

Challenge for the brave: Letting C once again denote the curve in figure 28, evaluate

(686) $$\int_C \langle y, 0 \rangle \cdot d\vec{r}.$$

Problem 8.1: An idealized two-dimensional ocean is modeled by the square region $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. with boundary $\mathcal{C}$. Consider the stream function $\Psi(x, y) = 4\cos(x)\cos(y)$ defined on R. Some of the level curves of Ψ are shown in the figure below.

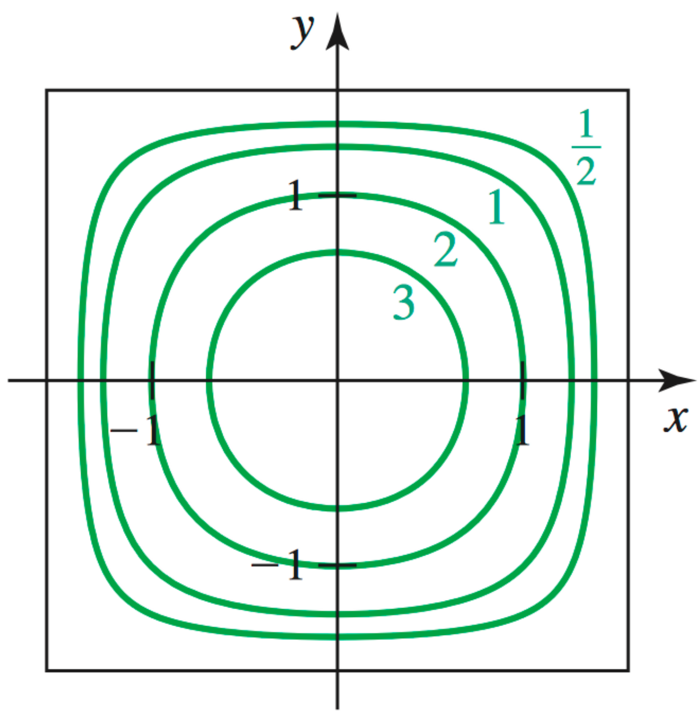


FIGURE 29. Some level curves of the stream function $\Psi(x, y)$.

(a) The horizontal (east-west) component of the velocity is $u = \Psi_y$ and the vertical (north-south) component of the velocity is $v = -\Psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.
(b) Is the velocity field source free? Explain.
(c) Is the velocity field irrotational? Explain.
(d) Find the total outward flux across $\mathcal{C}$.
(e) Find the circulation on $\mathcal{C}$ assuming counterclockwise orientation.

Solution to part (a): We see that

$$u(x, y) = \Psi_y(x, y) = -4\cos(x)\sin(y), \text{ and} \tag{687}$$

$$v(x, y) = -\Psi_x(x, y) = 4\sin(x)\cos(y), \tag{688}$$

so the velocity field $\vec{F} = \vec{F}(x, y)$ is given by

$$\vec{F}(x, y) = \langle u(x, y), v(x, y)\rangle = \langle -4\cos(x)\sin(y), 4\sin(x)\cos(y)\rangle. \tag{689}$$

Solution to part (b): We see that the divergence of $\vec{F}$ is given by

$$\text{Div}(\vec{F}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4\sin(x)\sin(y) - 4\sin(x)\sin(y) = 0, \tag{690}$$

so the $\boxed{\text{velocity field } \vec{F} \text{ is source free.}}$ In fact, we can show that any vector field $\vec{F} = \langle f, g\rangle = \langle \Psi_y, -\Psi_x\rangle$ that arises from a stream function Ψ is source free. It suffices to observe that

$$\text{Div}(\vec{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = (\Psi_y)_x + (-\Psi_x)_y = \Psi_{yx} - \Psi_{xy} = 0. \tag{691}$$

This should be compared to the fact that any vector field $\vec{F} = \langle \varphi_x, \varphi_y\rangle$ coming from a potential function φ is conservative/irrotational.

Solution to part (c): We see that the curl of $\vec{F}$ is given by

$$\begin{aligned}\text{Curl}(F) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 4\cos(x)\cos(y) - (-4\cos(x)\cos(y)) \\ &= 8\cos(x)\cos(y) \not\equiv 0,\end{aligned} \tag{692}$$

so the $\boxed{\text{velocity field } \vec{F} \text{ is not irrotational.}}$

Solution to part (d): Using the flux form of Green's Theorem we see that

$$\int_{\mathcal{C}} \vec{F}\cdot\hat{n}ds = \iint_R \text{Div}(\vec{F})dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0dxdy = \boxed{0}. \tag{693}$$

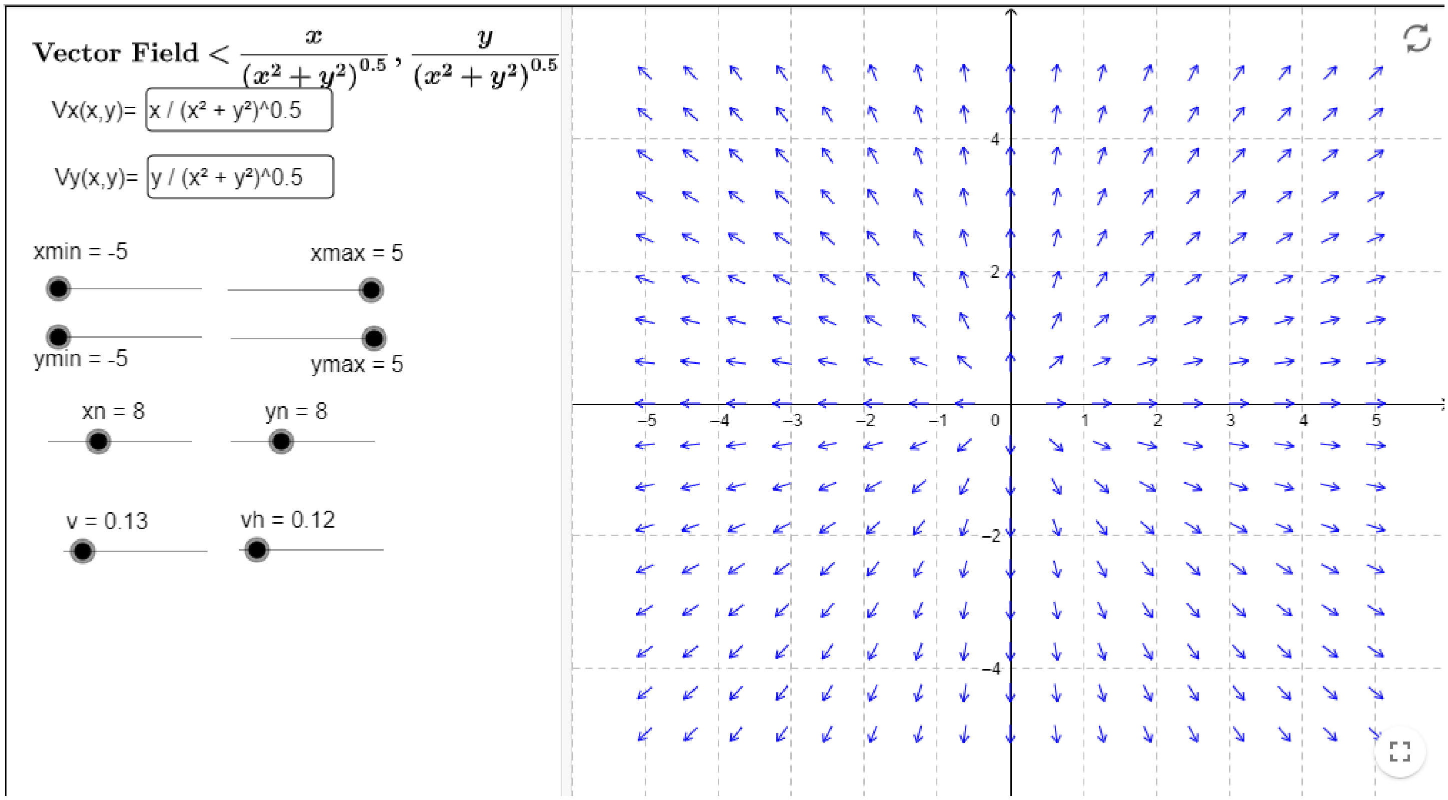
Solution to part (e): Using the circulation form of Green's Theorem we see that

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \iint_R \text{Curl}(\vec{F})dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos(x)\cos(y)dxdy \tag{694}$$

$$= 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y)dy\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx\right) = 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)dx\right)^2 \tag{695}$$

$$= 8\left(\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)^2 = 8 \cdot 2^2 = \boxed{32}. \tag{696}$$

Problem 8.2: Consider the radial field $\vec{F}(x,y) = \frac{\langle x,y\rangle}{\sqrt{x^2+y^2}} = \frac{\vec{r}}{|\vec{r}|}$ shown below.



(a) Explain why the conditions of Green's Theorem do not apply to $\vec{F}$ on a region R containing the origin.

(b) Let R be the unit disk centered at the origin and compute

(697) $$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA.$$

(c) Evaluate the line integral in the flux form of Green's Theorem applied to the region R and the vector field $\vec{F}$.

(d) Do the results of parts (b) and (c) agree? Explain.

Solution to part (a): We see that for

(698) $$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \text{ and } g(x,y) = \frac{y}{\sqrt{x^2+y^2}},$$

we have $\vec{F} = \langle f, g\rangle$. One of the conditions of Green's Theorem (flux form and circulation form) is that f and g have continuous first partial derivatives in R. Since neither of f and g are continuous at $(0,0)$, their first partial derivatives don't even exist at $(0,0)$, so they are not continuous. It follows that the conditions of Green's Theorem are not satisfied if $(0,0) \in R$.

Solution to part (b): We see that

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} + x\left(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} \cdot 2x\right) = \frac{y^2}{\sqrt{x^2+y^2}^3}, \text{ and} \tag{699}$$

$$\frac{\partial g}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} + y\left(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}} \cdot 2y\right) = \frac{x^2}{\sqrt{x^2+y^2}^3}. \tag{700}$$

It follows that

$$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA = \iint_R \left(\frac{y^2}{\sqrt{x^2+y^2}^3} + \frac{x^2}{\sqrt{x^2+y^2}^3}\right) dA \tag{701}$$

$$= \iint_R \frac{1}{\sqrt{x^2+y^2}} dA = \int_0^{2\pi}\int_0^1 \frac{1}{r} r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 dr\,d\theta = \boxed{2\pi}. \tag{702}$$

Solution to part (c): We recall that

$$\vec{r}(t) = \langle \cos(t), \sin(t)\rangle, 0 \le t \le 2\pi \tag{703}$$

is the parameterization by arclength of the unit circle. In this case we may naturally identify $\vec{r}(t)$ with the radial vector $\vec{r}$, so we will do so by abuse of notation. Furthermore, recalling that $\hat{n}$ is the *outward* unit normal vector, we see that

$$\hat{n}(t) = \langle \cos(t), \sin(t)\rangle = \vec{r}(t), 0 \le t \le 2\pi. \tag{704}$$

It follows that

$$\int_{\mathcal{C}} \vec{F} \cdot \hat{n} ds = \int_{\mathcal{C}} \frac{\vec{r}}{|\vec{r}|} \cdot (\vec{r}(t)) ds = \int_0^{2\pi} \frac{|\vec{r}(t)|^2}{|\vec{r}(t)|} dt = \int_0^{2\pi} \frac{1^2}{1} dt = \boxed{2\pi}. \tag{705}$$

Solution to part (d): Even though the conditions of Green's Theorem do not apply, the answers to parts (b) and (c) are the same. This shows that the conditions of Green's Theorem are sufficient conditions but not necessary conditions to attain the result of Green's Theorem.

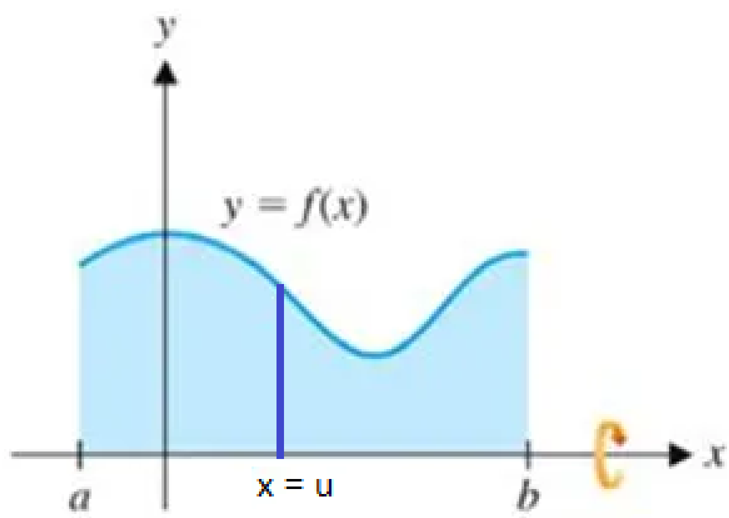
Problem 8.3: Suppose $y = f(x)$ is a continuous and positive function on $[a, b]$. Let $\mathcal{S}$ be the surface generated when the graph of $f(x)$ is revolved about the x-axis.

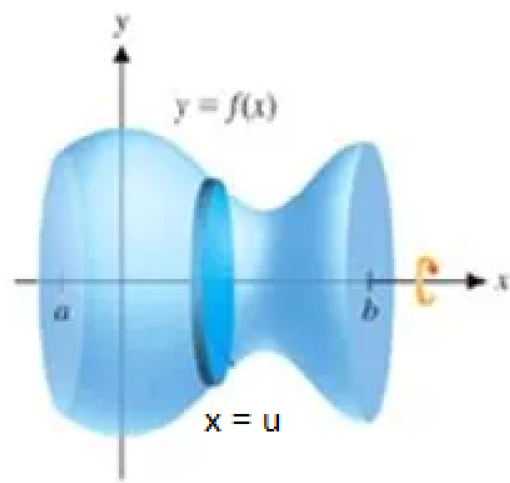
(a) Show that $\mathcal{S}$ is described parametrically by $\vec{r}(u, v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$, for $a \leq u \leq b$, $0 \leq v \leq 2\pi$.

(b) Find an integral that gives ths surface area of $\mathcal{S}$.

(c) Apply the result of part (b) to the surface $\mathcal{S}_1$ generated with $f(x) = x^3$, for $1 \leq x \leq 2$.

Solution to (a): We see that for each value of u inbetween a and b, if we rotate the point $(u, f(u))$ about the x-axis then we generate a circle C of radius $f(u)$ in the plane $x = u$ as shown in the picture below.





We see that the x-coordinate at every point of the circle C is u. It now suffices to recall that the parametrization of a circle of radius $f(u)$ in the xy-plane is $\langle f(u)\cos(v), f(u)\sin(v)\rangle$ for $0 \leq v \leq 2\pi$, but we have a circle in the plane $x = u$ (which is parallel to the yz-plane), so we obtain the parametrization $\vec{r}(u, v) = \langle u, f(u)\cos(v), f(u)\sin(v)\rangle$ for $a \leq u \leq b$ and $0 \leq v \leq 2\pi$ as desired.

Solution to (b): We begin by calculating

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(u)\cos(v) & f'(u)\sin(v) \\ 0 & -f(u)\sin(v) & f(u)\cos(v) \end{vmatrix} \tag{706}$$

$$= \hat{i}(f(u)f'(u)\cos^2(v) + f(u)f'(u)\sin^2(v)) - \hat{j}(f(u)\cos(v)) + \hat{k}(-f(u)\sin(v))$$

$$= f(u)f'(u)\hat{i} - f(u)\cos(v)\hat{j} - f(u)\sin(v)\hat{k}, \text{ hence}$$

$$\left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| = \sqrt{(f(u)f'(u))^2 + (-f(u)\cos(v))^2 + (-f(u)\sin(v))^2} \tag{707}$$

$$= f(u)\sqrt{(f'(u))^2 + 1}.$$

We now see that

$$\text{Surface Area}(\mathcal{S}) = \iint_{\mathcal{S}} 1dS = \int_a^b \int_0^{2\pi} f(u)\sqrt{(f'(u))^2 + 1}dvdu \tag{708}$$

$$= \boxed{2\pi \int_a^b f(u)\sqrt{(f'(u))^2 + 1}du}$$

Solution to (c): From part (b) we see that

$$\text{Surface Area}(\mathcal{S}_1) = 2\pi \int_1^2 u^3\sqrt{(3u^2)^2 + 1}du = 2\pi \int_1^2 u^3\sqrt{9u^4 + 1}du \tag{709}$$

$$\overset{w=9u^4+1}{=} 2\pi \int_{u=1}^2 \sqrt{w}\frac{dw}{36} = \frac{\pi}{18} \cdot \frac{2}{3}w^{\frac{3}{2}}\Big|_{u=1}^2$$

$$= \frac{\pi}{27}(9u^4 + 1)^{\frac{3}{2}}\Big|_1^2 = \boxed{\frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right)}$$

Problem 8.4: Given a sphere of radius R and a length $0 < L \leq 2R$, show that the surface area of the strips of length L on the sphere depend only on L and not on the location of the strip.

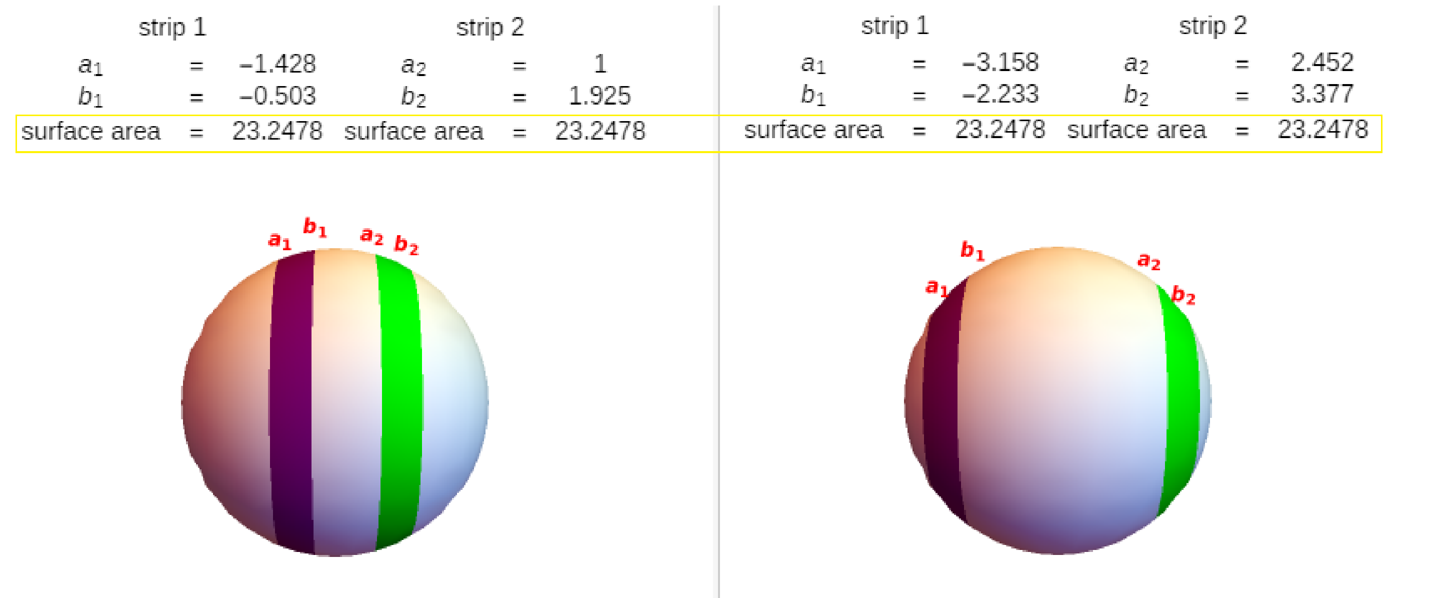


FIGURE 30. An example of Problem 8.4 with $L = 0.925$ and $R = 4$.

Hint: Problem 8.3 can help.

Solution: We begin by recalling that the graph of $f(x) = \sqrt{R^2 - x^2}$ for $-R \leq x \leq R$ is the upper half of the circle of radius R centered at the origin of the xy-plane. We may now use Problem 8.3(b) to see that for any $-R \leq a \leq R-L$ the surface area obtained by revolving $f(x)$ for $a \leq x \leq a+L$ is

$$\begin{aligned}(710) \qquad & 2\pi\int_a^{a+L}\sqrt{R^2-u^2}\sqrt{(\frac{-u}{\sqrt{R^2-u^2}})^2+1}du \\ =&2\pi\int_a^{a+L}\sqrt{R^2-u^2}\sqrt{\frac{R^2}{R^2-u^2}}du = 2\pi\int_a^{a+L}Rdu = \boxed{2\pi RL}\end{aligned}$$

Problem 8.5(Rain on roofs)**:** Let $z = s(x,y)$ define the surface $\mathcal{S}$ over a region R in the xy-plane, where $z \geq 0$ on R. Show that the downward flux of the vertical vector field $\vec{F} = \langle 0,0,-1\rangle$ across $\mathcal{S}$ equals the area of R. Interpret the result physically.

Solution: We see that the surface $\mathcal{S}$ can be parametrized by $\vec{r}(x,y) = \langle x, y, s(x,y)\rangle$ for $(x,y) \in R$. We now proceed to calculate $\hat{n}dS$, the vector normal to the surface whose length is proportional to the differential area at each point.

$$\begin{aligned}\hat{n}dS &= \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & s_x(x,y) \\ 0 & 1 & s_y(x,y)\end{vmatrix} \\ &= \hat{i}(-s_x(x,y)) - \hat{j}s_y(x,y) + \hat{k}(1) \\ &= -s_x(x,y)\hat{i} - s_y(x,y)\hat{j} + \hat{k}.\end{aligned} \tag{711}$$

We now see that the downward flux of the vector field $\vec{F}$ is given by

$$\begin{aligned}\iint_{\mathcal{S}} \vec{F}\cdot\hat{n}dS &= \iint_R \langle 0,0,-1\rangle \cdot \langle -s_x(x,y), -s_y(x,y), 1\rangle dA \\ &= \iint_R -1dA = \boxed{-\text{Area}(R)}.\end{aligned} \tag{712}$$

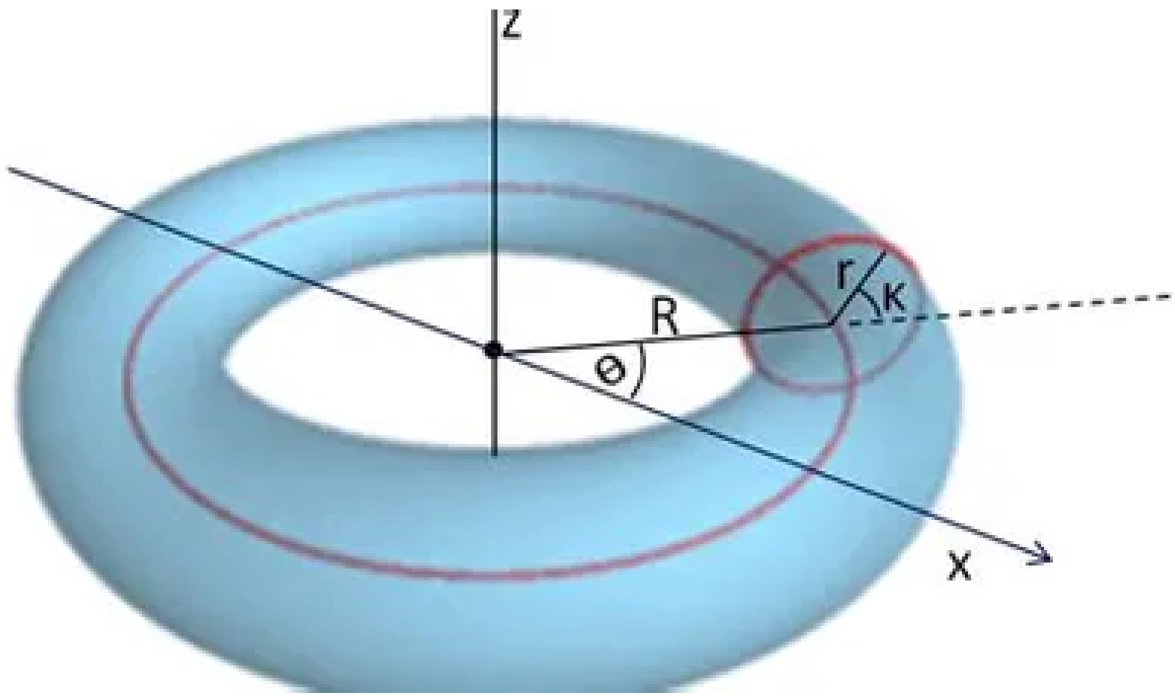
One way in which to physically interpret this result is the following. If $\mathcal{S}$ is modeling the roof of a house built over the region R, and $\vec{F}$ represents the force of rain drops that are falling straight down, then the downward flux of the rain on the roof (the force imparted by the rain onto the roof) of the house depends only on the area of the base of the house, not the shape of the roof.

Problem 8.6(Surface Area and Volume of a Torus)**:**

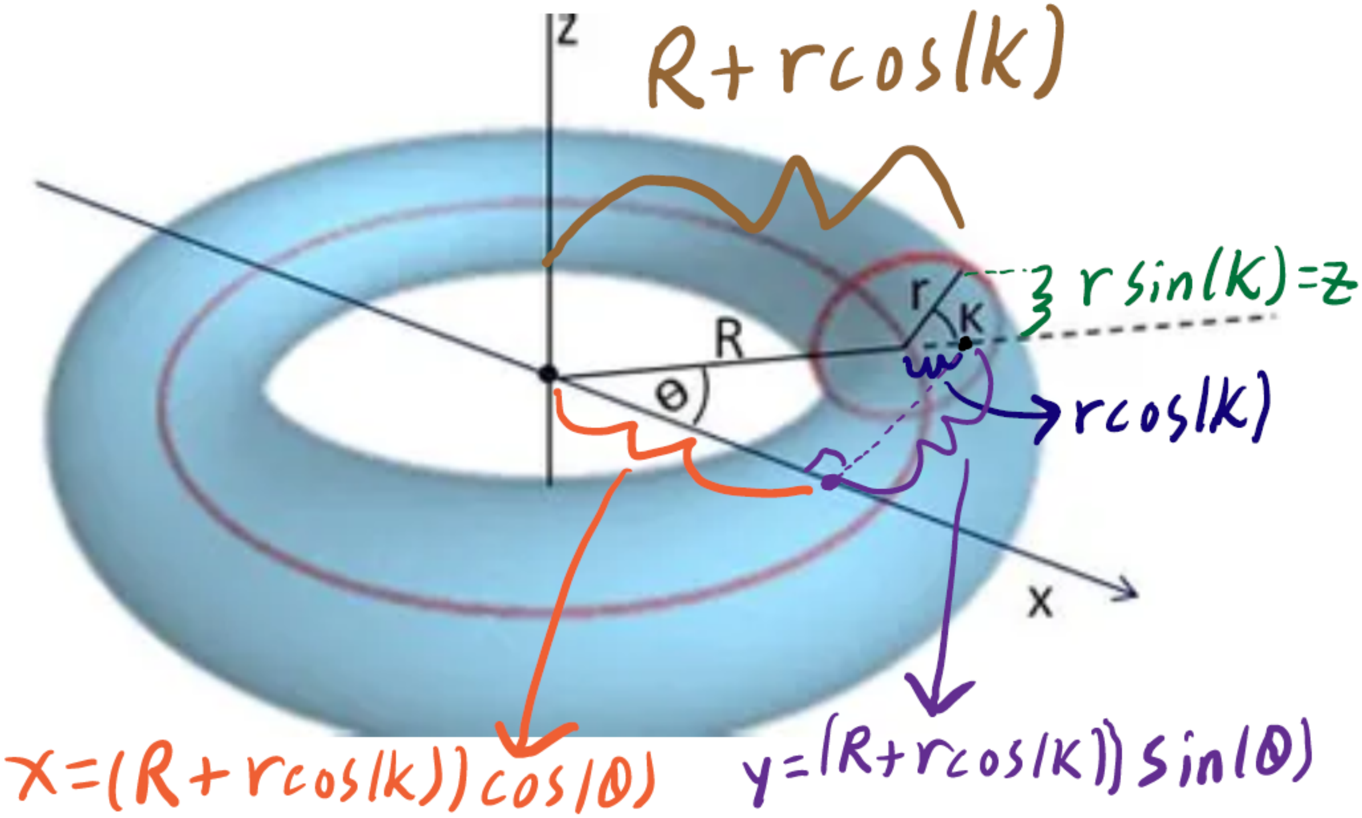
(a) Show that a torus T with radii $R > r$ (See figure) may be described parametrically by $\vec{r}(K,\theta) = \langle (R+r\cos(K))\cos(\theta), (R+r\cos(K))\sin(\theta), r\sin(K)\rangle$, for $0 \le K \le 2\pi$, $0 \le \theta \le 2\pi$.

(b) Show that the surface area of the torus T is $4\pi^2 Rr$.
Interestingly, the arclength of the small circle is $2\pi r$ and the arclength of the large circle inside the torus is $2\pi R$, so the surface area of the torus happens to be the product of the arclengths of the 2 circles from which it is created.

(c) Use part (a) to find a parametrization $\vec{s}(K,\theta,r)$ for the solid torus $\mathcal{T}$ (T from part (a) as well as its interior), then use $\vec{s}$ and a change of variables to show that the volume of $\mathcal{T}$ is $\pi r^2 R$.



Solution to (a): The justification that $\vec{r}(K,\theta)$ is indeed a parametrization for T is given by the diagram below.



Solution to (b): We begin by calculating

$$(713)\quad \frac{\partial \vec{r}}{\partial K}\times\frac{\partial \vec{r}}{\partial \theta}$$
$$=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -r\sin(K)\cos(\theta) & -r\sin(K)\sin(\theta) & r\cos(K) \\ -(R+r\cos(K))\sin(\theta) & (R+r\cos(K))\cos(\theta) & 0 \end{vmatrix}$$

$$=\hat{i}(-r\cos(K)(R+r\cos(K))\cos(\theta)$$
$$-\hat{j}(-r\cos(K)(R+r\cos(K))\sin(\theta))$$
$$\hat{k}\Big(-r\sin(K)\cos(\theta)(R+r\cos(K))\cos(\theta)$$
$$-r\sin(K)\sin(\theta)(R+r\cos(K))\sin(\theta)\Big)$$

$$=-r\cos(K)\cos(\theta)(R+r\cos(K))\hat{i}$$
$$+r\cos(K)\sin(\theta)(R+r\cos(K))\hat{j}$$
$$-r\sin(K)(R+r\cos(K))\hat{k},\text{ hence}$$

$$(714)\quad \left|\frac{\partial \vec{r}}{\partial K}\times\frac{\partial \vec{r}}{\partial \theta}\right| = (R+r\cos(K))\sqrt{(r\cos(K)\cos(\theta))^2+(r\cos(K)\sin(\theta))^2+(-r\sin(K))^2}$$
$$=(R+r\cos(K))\sqrt{r^2\cos^2(K)+r^2\sin^2(K)}$$
$$=r(R+r\cos(K))$$

We now see that

$$(715)\quad \text{Surface Area}(T) = \iint_T 1dS = \int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial \vec{r}}{\partial K}\times\frac{\partial \vec{r}}{\partial \theta}\right| dKd\theta$$
$$=\int_0^{2\pi}\int_0^{2\pi}(rR+r^2\cos(K))dKd\theta$$
$$=\int_0^{2\pi}\left(rRK+r^2\sin(K)\Big|_{K=0}^{2\pi}\right)d\theta$$
$$=\int_0^{2\pi}2\pi rRd\theta=\boxed{4\pi^2rR}.$$

Solution to (c): We only need to replace the radius r with a new radius $0 \leq \rho \leq r$ in order to get toroidal shells within the original torus, so we obtain the parametrization

$$\vec{s}(K,\theta,\rho) = \langle (R+\rho\cos(K))\cos(\theta), (R+\rho\cos(K))\sin(\theta), \rho\sin(K)\rangle \quad \text{for } 0 \leq K \leq 2\pi, 0 \leq \theta \leq 2\pi, \text{ and } 0 \leq \rho \leq r. \tag{716}$$

Now that we have found $\vec{s}$, we can we can calculate the Jacobian of the transformation $(x,y,z) = \vec{s}(K,\theta,r)$. We see that

$$J(K,\theta,\rho) = \begin{vmatrix} \frac{\partial x}{\partial K} & \frac{\partial y}{\partial K} & \frac{\partial z}{\partial K} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \end{vmatrix} \tag{717}$$

$$= \begin{vmatrix} -\rho\sin(K)\cos(\theta) & -\rho\sin(K)\sin(\theta) & \rho\cos(K) \\ -(R+\rho\cos(K))\sin(\theta) & (R+\rho\cos(K))\cos(\theta) & 0 \\ \cos(K)\cos(\theta) & \cos(K)\sin(\theta) & \sin(K) \end{vmatrix}$$

$$\begin{aligned} &= -\rho\sin(K)\cos(\theta)(R+\rho\cos(K))\cos(\theta)\sin(K) \\ &\quad - (-\rho\sin(K)\sin(\theta))(-(R+\rho\cos(K))\sin(\theta))\sin(K) \\ &\quad + \rho\cos(K)\Big(-(R+\rho\cos(K))\sin(\theta)\cos(K)\sin(\theta) \\ &\qquad - \cos(K)\cos(\theta)(R+\rho\cos(K))\cos(\theta)\Big) \\ &= (R+\rho\cos(K))\Big(-\rho\sin^2(K)\cos^2(\theta) - \rho\sin^2(K)\sin^2(\theta) \\ &\qquad + \rho\cos^2(K)(-\sin^2(\theta) - \cos^2(\theta))\Big) \\ &= (R+\rho\cos(K))(-\rho\sin^2(K) - \rho\cos^2(K)) = -\rho(R+\rho\cos(K)). \end{aligned}$$

Recalling that $dV = dxdydz = |J(K,\theta,\rho)|dKd\theta d\rho$, we see that

$$
\begin{aligned}
(718) \qquad \text{Volume}(\mathcal{T}) &= \iiint_T 1dV = \int_0^{2\pi}\int_0^{2\pi}\int_0^r |J(K,\theta,\rho)|dKd\theta d\rho \\
&= \int_0^r\int_0^{2\pi}\int_0^{2\pi} \rho(R+\rho\cos(K))dKd\theta d\rho \\
&= \int_0^{2\pi}\int_0^{2\pi}\left(\rho R+\rho^2\sin(K)\Big|_{K=0}^{2\pi}\right)d\theta d\rho \\
&= \int_0^r\int_0^{2\pi}\rho Rd\theta d\rho = 2\pi\int_0^r \rho Rd\rho \\
&= 2\pi(\frac{1}{2}\rho^2R\Big|_{\rho=0}^{r}) = \boxed{\pi r^2 R}.
\end{aligned}
$$

Problem 8.7: Let $\mathcal{S}$ be the upper half of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$ and let $\vec{F}=\langle z,x,y\rangle$. Use Stoke's theorem to evaluate

$$\iint_{\mathcal{S}}(\nabla\times\vec{F})\cdot\hat{n}dS. \tag{719}$$

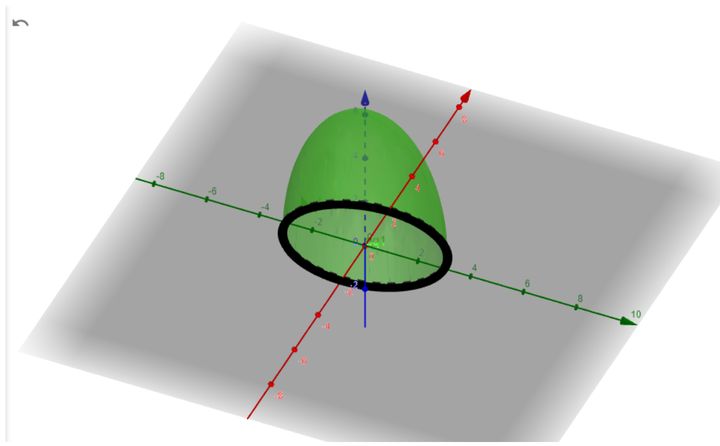


FIGURE 31. A view of $\mathcal{S}$ and $\partial\mathcal{S}$.

Solution: We see that the boundary $\partial\mathcal{S}$ of $\mathcal{S}$ is obtained when $z=0$, so it is given by the equation $\frac{x^2}{4}+\frac{y^2}{9}=1$. Since $\partial\mathcal{S}$ is an ellipse in the $xy-plane$, we see that it can be parametrized by $\vec{r}(t)=\langle 2\cos(t),3\sin(t),0\rangle$ for $0\le t\le 2\pi$. We observe that

$$\vec{F}(\vec{r}(t))=\langle 0,2\cos(t),3\sin(t)\rangle \text{ and } \vec{r}\,'(t)=\langle -2\sin(t),3\cos(t),0\rangle. \tag{720}$$

We now use Stoke's theorem to see that

(721)
$$\begin{aligned}\iint_{\mathcal{S}}(\nabla\times\vec{F})\cdot\hat{n}dS &= \int_{\partial\mathcal{S}}\vec{F}\cdot d\vec{r}=\int_0^{2\pi}\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt\\ &=\int_0^{2\pi}\langle 0,2\cos(t),3\sin(t)\rangle\cdot\langle -2\sin(t),3\cos(t),0\rangle dt\\ &=\int_0^{2\pi}(0+6\cos^2(t)+0)dt\ \left(\cos(2t)=2\cos^2(t)-1\right)\\ &=\int_0^{2\pi}(3\cos(2t)+3)dt=\frac{3}{2}\sin(2t)+3t\Big|_0^{2\pi}=\boxed{6\pi}.\end{aligned}$$

Problem 8.8: Let C be the circle $x^2+y^2=12$ in the plane $z=0$ (as a subset of $\mathbb{R}^3$) and let $\vec{F}=\langle(x+4)^x, y\ln(y+4), e^{z^2+\sqrt{z}}\rangle$. Use Stoke's theorem to evaluate

$$\oint_C \vec{F}\cdot d\vec{r}. \tag{722}$$

Solution: It is clear that the line integral in equation (722) is very difficult to evaluate directly, and the formulation of the problem suggests that the surfaces integral arising from Stoke's theorem will be easier to evaluate. To this end, we begin by verifying that $\nabla\times\vec{F}=\vec{0}$ whenever $\vec{F}$ is of the form $\vec{F}=\langle f_1(x), f_2(y), f_3(z)\rangle$.

$$\begin{aligned}\nabla\times\vec{F} &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1(x) & f_2(y) & f_3(z)\end{vmatrix}\\ &= \hat{i}\left(\frac{\partial f_3(z)}{\partial y}-\frac{\partial f_2(y)}{\partial z}\right)-\hat{j}\left(\frac{\partial f_3(z)}{\partial x}-\frac{\partial f_1(x)}{\partial z}\right)\\ &\quad+\hat{k}\left(\frac{\partial f_2(y)}{\partial x}-\frac{\partial f_1(x)}{\partial y}\right)\\ &= 0\hat{i}+0\hat{j}+0\hat{k}=\vec{0}.\end{aligned} \tag{723}$$

We may now view C as the boundary ∂S of the upper half of the sphere of radius $2\sqrt{3}$ $\mathcal{S}$ so that we may apply Stoke's Theorem.

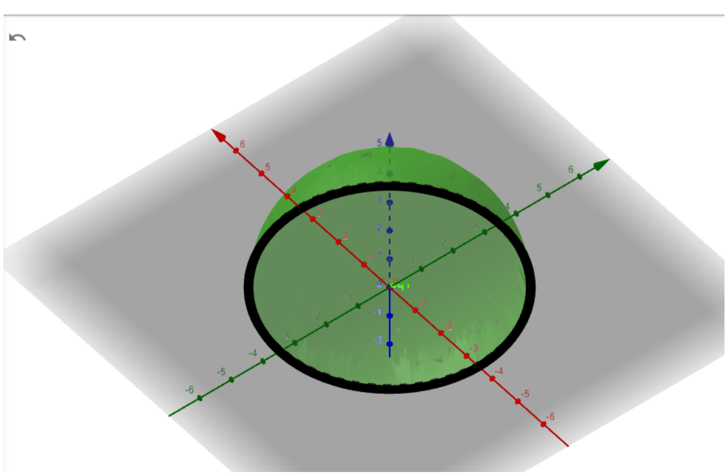


FIGURE 32. A view of C and $\mathcal{S}$.

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{\mathcal{S}} (\nabla \times \vec{F}) dS = \iint_{\mathcal{S}} 0 dS = \boxed{0}. \tag{724}$$

Remark: We could have applied the same procedure for the vector field $\vec{F} = \langle (x+4)^x, y\ln(y+4)\rangle$ by identifying it with the vector field $\vec{F} = \langle (x+4)^x, y\ln(y+4), 0\rangle$. In particular, a 2-dimensional circulation integral may become easier by viewing it as a circulation integral in 3-dimensions and using Stoke's theorem.

Problem 8.9: Let $\mathcal{S}$ be the surface of the cube cut from the first octant by the planes $x = 1, y = 1$, and $z = 1$. Let $\vec{F} = \langle x^2, 2xz, y^2 \rangle$. Use the Divergence theorem to evaluate the net outward flux of $\vec{F}$ across $\mathcal{S}$.

Solution: We begin by observing that

$$\text{Div}(\vec{F}) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}(y^2) = 2x + 0 + 0 = 2x. \tag{725}$$

We may now apply the Diverence theorem to see that

$$\begin{aligned} \text{Flux}(\vec{F}, \mathcal{S}) &= \iint_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = \iiint_{\text{int}(\mathcal{S})} \text{Div}(\vec{F}) dV \\ &= \int_0^1 \int_0^1 \int_0^1 2x dx dy dz = \int_0^1 \int_0^1 (x^2 \Big|_{x=0}^1) dy dz \\ &= \int_0^1 \int_0^1 1 dy dz = \boxed{1}. \end{aligned} \tag{726}$$

Remark: One of the benefits to calculating the divergence in this problem with the divergence theorem rather than by direct calculation is that it is easier to evaluate 1 triple integral than a sum of 6 surface integrals.

Problem 8.10: Let $\mathcal{S}$ be the boundary of the ellipsoid $\frac{x^2}{4} + y^2 + z^2 = 1$ and let $\vec{F} = \langle x^2 e^y \cos(z), -4xe^y \cos(z), 2xe^y \sin(z)\rangle$. Evaluate the outward flux of $\vec{F}$ across $\mathcal{S}$.

Solution: We begin by observing that

$$\begin{aligned}\text{Div}(\vec{F}) &= \frac{\partial}{\partial x}(x^2 e^y \cos(z)) + \frac{\partial}{\partial y}(-4xe^y \cos(z)) + \frac{\partial}{\partial z}(2xe^y \sin(z)) \\ &= 2xe^y \cos(z) - 4xe^y \cos(z) + 2xe^y \cos(z) = 0.\end{aligned} \tag{727}$$

We may now apply the Diverence theorem to see that

$$\begin{aligned}\text{Flux}(\vec{F}, \mathcal{S}) &= \iint_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = \iiint_{\text{int}(\mathcal{S})} \text{Div}(\vec{F}) dV \\ &= \iiint_{\text{int}(\mathcal{S})} 0 dV = \boxed{0}.\end{aligned} \tag{728}$$

Problem 9.1: Three people play a game in which there are always 2 winners and 1 loser. They have the understanding that the loser always gives each winner an amount equal to what the winner already has. After 3 games, each has lost once and each has $24. With how much money did each begin?

Solution: Let us assume that player 1 begins with $x, player 2 begins with $y, and player 3 begins with $z. We may further assume without loss of generality that player 1 loses round 1, player 2 loses round 2, and player 3 loses round 3. We then obtain the following table.

	Player 1	Player 2	Player 3
Money at the Start	x	y	z
Money at the end of round 1	x-y-z	2y	2z
Money at the end of round 2	2x-2y-2z	-x+3y-z	4z
Money at the end of round 3	4x-4y-4z	-2x+6y-2z	-x-y+7z

We now obtain and solve the following system of equations.

$$\begin{array}{rcrcrcr} 4x & - & 4y & - & 4z & = & 24 \\ -2x & + & 6y & - & 2z & = & 24 \\ -x & - & y & + & 7z & = & 24 \end{array} \to \left[\begin{array}{ccc|c} 4 & -4 & -4 & 24 \\ -2 & 6 & -2 & 24 \\ -1 & -1 & 7 & 24 \end{array}\right] \tag{729}$$

$$\overset{\substack{R_1+4R_3 \\ R_2-2R_3}}{\to} \left[\begin{array}{ccc|c} 0 & -8 & 24 & 120 \\ 0 & 8 & -16 & -24 \\ -1 & -1 & 7 & 24 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\to} \left[\begin{array}{ccc|c} -1 & -1 & 7 & 24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right] \tag{730}$$

(731)

$$\overset{-R_1}{\to} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right] \overset{R_3+R_2}{\to} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & 0 & 8 & 96 \end{array}\right] \overset{\substack{\frac{1}{8}R_2 \\ \frac{1}{8}R_3}}{\to} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -21 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right] \xrightarrow{\substack{R_1 + 5R_3 \\ R_2 + 2R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 39 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right] \tag{732}$$

$$\rightarrow (x, y, z) = \boxed{(39, 21, 12)}. \tag{733}$$

Problem 9.2: For the following problems, determine all possiblities for the solution set (from among infinitely many solutions, a unique solution, or no solution) of the system of linear equations described. After determining the possibilities for the solution set create concrete examples of systems corresponding to each possibility.

(1) A homogeneous system of 4 equations in 5 unknowns.
(2) A system of 4 equations in 3 unknowns.
(3) A system of 3 equations in 4 unknowns that has $x_1 = -1$, $x_2 = 0$, $x_3 = 2$, $x_4 = -3$ as a solution.
(4) A homogeneous system of 3 equations in 3 unknowns.
(5) A homogeneous system of 3 equations in 3 unknowns that has solution $x_1 = 1$, $x_2 = 3$, $x_3 = -1$.
(6) A system of 2 equations in 3 unknowns.

You are free to make use of the following facts.

(1) Any homogeneous system of equations is consistent.
   - This is seen by the fact that the trivial solution (the solution in which all variables are equal to 0) is always a solution to a homogeneous system of equations.

(2) If a consistent system of equations (a system of equations with at least 1 solution) has more than 1 solution, then it has infinitely many solutions.
(3) If a consistent system of equations has more variables than equations, then it has infinitely many solutions.

Solution:

(1) By facts (1) and (3) we see that there are infinitely many solutions.

$$\begin{array}{ccccccccccc} x_1 & & & & & & & & & = & 0 \\ & & x_2 & & & & & & & = & 0 \\ & & & & x_3 & & & & & = & 0 \\ & & & & & & x_4 & + & x_5 & = & 0 \end{array} \tag{734}$$

has infinitely many solutions.

(2) Anything is possible. The system could be inconsistent, it could have a unique solution, or it could have infinitely many solutions.

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \\ 2x_3 & = & 2 \end{array} \text{ has no solutions.} \tag{735}$$

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \\ 2x_3 & = & 0 \end{array} \text{ has a unique solution.} \tag{736}$$

$$\begin{array}{rcl} x_1 + x_2 & = & 0 \\ 2x_1 + 2x_2 & = & 0 \\ x_3 & = & 0 \\ 2x_3 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{737}$$

(3) By facts (1) and (3) we see that there are infinitely many solutions.

$$\begin{array}{rcl} x_1 - x_4 & = & 2 \\ x_2 & = & 0 \\ x_3 + 2x_4 & = & -4 \end{array} \text{ has infinitely many solutions.} \tag{738}$$

(4) The system has to be consistent by fact (1). The system could have a unique solution, or it could have infinitely many solutions.

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \\ x_3 & = & 0 \end{array} \text{ has a unique solution.} \tag{739}$$

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 + x_3 & = & 0 \\ 2x_2 + 2x_3 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{740}$$

(5) The system is consistent by fact (1). Since we are given a solution other than the trivial solution, fact (2) tells us that there are infinitely many solutions.

$$\begin{array}{rcl} x_1 + x_2 + 4x_3 & = & 0 \\ x_2 + 3x_3 & = & 0 \\ x_1 + x_3 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{741}$$

(6) It is possible that the system is inconsistent and has no solutions. By fact (1), the only possible alternative is an infinite number of solutions.

(742) $$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array} \text{ has no solutions.}$$

(743) $$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \end{array} \text{ has infinitely many solutions.}$$

Problem 9.3: For what value(s) of a does the following system have nontrivial solutions?

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -x_1 & + & ax_2 & + & x_3 & = & 0 \\ 3x_1 & + & 4x_2 & - & x_3 & = & 0 \end{array}. \tag{744}$$

Solution: Let us first represent the system of equations as an augmented matrix that we will reduce into echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & a & 1 & 0 \\ 3 & 4 & -1 & 0 \end{array}\right] \overset{\substack{R_2+R_1 \\ R_3-3R_1}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & a+2 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \tag{745}$$

In order to continue the row reduction, we would like to use the row operation $R_3 + \frac{2}{a+2}R_2$, but this is only valid if $a + 2 \neq 0$, which occurs if and only if $a = -2$. So let us assume that $a \neq -2$ for now and we will handle $a = -2$ as a separate case.

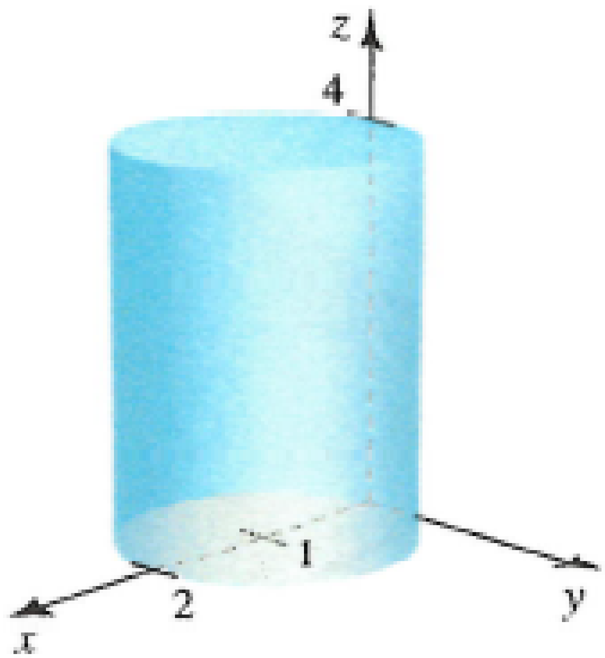
$$\overset{R_3+\frac{2}{a+2}R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & a+2 & 2 & 0 \\ 0 & 0 & \frac{4}{a+2}-4 & 0 \end{array}\right] \tag{746}$$

If $\frac{4}{a+2} - 4 \neq 0$, then equation (744) will only have the trivial solution. Since we are searching for the value(s) of a that result in nontrivial solutions to equation (744), we solve $\frac{4}{a+2} - 4 = 0$ and see that $\boxed{a = -1}$. The only other possible value of a is $a = -2$ which we will now consider as a separate case. Plugging $a = -2$ back into (745) we obtain

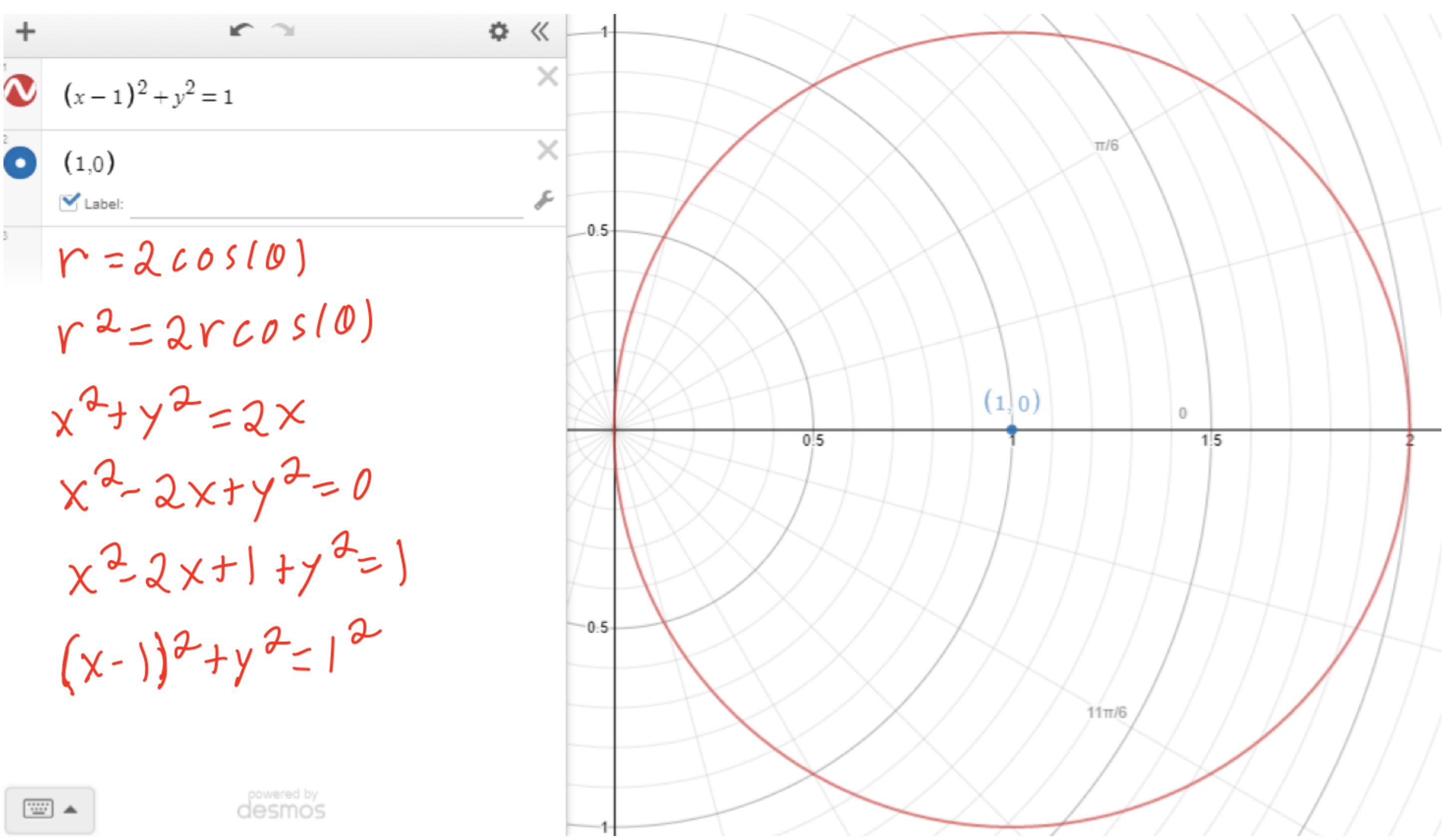
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \overset{R_2\leftrightarrow R_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \overset{\substack{-\frac{1}{2}R_2 \\ \frac{1}{2}R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]. \tag{747}$$

Since the system represented in equation (747) only has the trivial solution, we see that -2 is not one of the desired values of a. In conclusion, the only value of a that results in nontrivial solutions for equation (744) is $\boxed{a = -1}$.

Problem 5.11: Find the volume of the solid cylinder E whose height is 4 and whose base is the disk $\{(r,\theta) : 0 \leq r \leq 2\cos(\theta)\}$.



Solution: We first look at the cross section of E in the xy−plane to help us determine our bounds.



$$\text{Volume}(E) = \iiint_E 1dV = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} rdrd\theta dz \tag{748}$$

. . . .

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_0^{2\cos(\theta)} d\theta dz = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2(\theta)d\theta dz \tag{749}$$

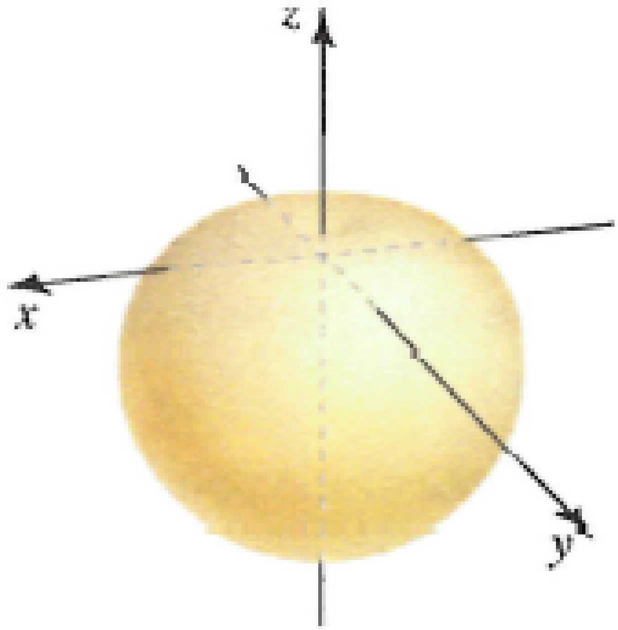
. . . .

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2\theta)+1)d\theta dz = \int_0^4 (\frac{1}{2}\sin(2\theta)+\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \tag{750}$$

. . . .

$$= \int_0^4 \pi dz = \boxed{4\pi}. \tag{751}$$

Problem 5.12: Find the volume of the solid cardiod of revolution $D = \{(\rho, \varphi, \theta) : 0 \le \rho \le \frac{1}{2}(1 - \cos(\varphi)), 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$.



Solution: In this problem, the description of the region is just a reordering of the description that we need to write down our triple integral in spherical coordinates to find the volume. We see that

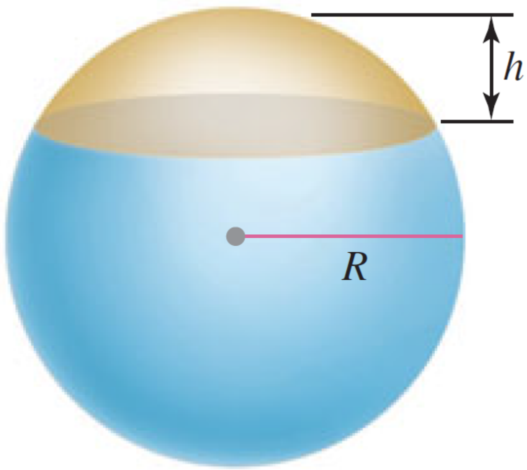
$$\text{Volume}(D) = \iiint_D 1dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{1}{2}(1-\cos(\varphi))} \rho^2 \sin(\varphi) d\rho d\varphi d\theta \tag{752}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3 \sin(\varphi)\Big|_0^{\frac{1}{2}(1-\cos(\varphi))} d\varphi d\theta \tag{753}$$

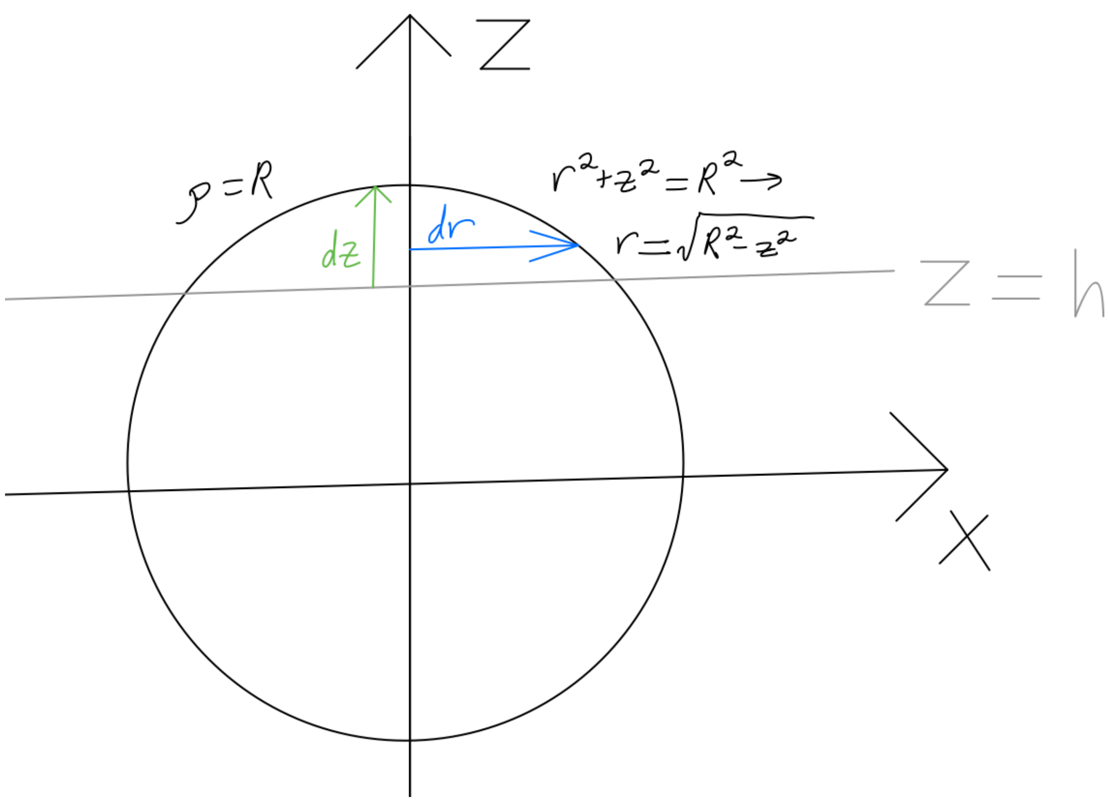
$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\Bigg(\underbrace{\frac{1}{2}(1-\cos(\varphi))}_{u}\Bigg)^3 \underbrace{\sin(\varphi)d\varphi}_{2du} d\theta = \int_0^{2\pi} \frac{1}{6}u^4\Big|_{\varphi=0}^{\pi} d\theta \tag{754}$$

$$= \int_0^{2\pi} \frac{1}{6}\left(\frac{1}{2}(1-\cos(\varphi))\right)^4\Big|_0^{\pi} d\theta = \int_0^{2\pi}\frac{1}{6}d\theta = \boxed{\frac{\pi}{3}}. \tag{755}$$

Problem 5.13: Find the volume of S, the cap of a sphere of radius R with thickness h.



Solution 1: We will first solve this problem using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane, which corresponds to the $\theta = 0$ and $\theta = \pi$ cross sections combined. Since we are working in cylindrical coordinates, the cross section will be handled in coordinates similar to Cartesian coordinates.



(756) $$\text{Vol}(S) = \int_0^{2\pi}\int_{R-h}^{R}\int_0^{\sqrt{R^2-z^2}} r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_{R-h}^{R} \frac{1}{2}r^2\Big|_{r=0}^{\sqrt{R^2-z^2}} dz\,d\theta$$

. . . .

(757) $$= \frac{1}{2}\int_0^{2\pi}\int_{R-h}^{R}(R^2 - z^2)\,dz\,d\theta = \frac{1}{2}\int_0^{2\pi}(R^2 z - \frac{1}{3}z^3\Big|_{z=R-h}^{R})\,d\theta$$

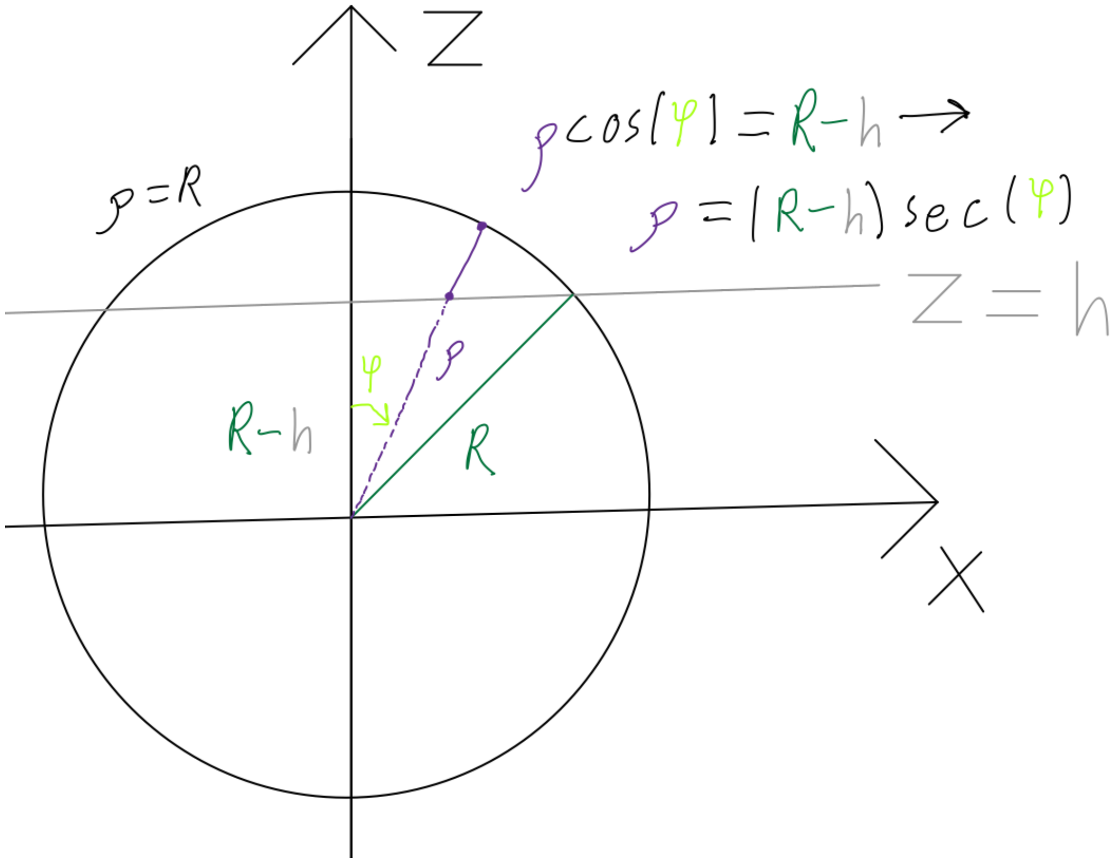
. . . .

(758) $$= \frac{1}{2}\int_0^{2\pi}(R^3 - \frac{1}{3}R^3 - (R^2(R-h) - \frac{1}{3}(R-h)^3))\,d\theta = \frac{1}{2}\int_0^{2\pi}(Rh^2 - \frac{1}{3}h^3)\,d\theta$$

. . . .

(759) $$= \pi(Rh^2 - \frac{1}{3}h^3) = \boxed{\frac{\pi}{3}h^2(3R-h)}.$$

Solution 2: We will now solve this problem using spherical coordinates. Due to the symmetry of our solid with respect to θ we once again begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be handled in coordinates similar to polar coordinates.



(760) $$\text{Vol}(S) = \int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})}\int_{(R-h)\sec\phi}^{R} \rho^2\sin(\varphi)d\rho d\varphi d\theta$$

. . . .

(761) $$= \int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=(R-h)\sec\phi}^{R} d\varphi d\theta$$

. . . .

(762) $$= \frac{1}{3}\int_0^{2\pi}\int_0^{\cos^{-1}(\frac{R-h}{R})}(R^3\sin(\varphi) - (R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi d\theta$$

. . . .

(763) $$= \frac{2\pi}{3}\int_0^{\cos^{-1}(\frac{R-h}{R})}(R^3\sin(\varphi) - (R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi$$

. . . .

(764)
$$= \frac{2\pi}{3}\left(\int_0^{\cos^{-1}(\frac{R-h}{R})} R^3\sin(\varphi)d\varphi - \int_0^{\cos^{-1}(\frac{R-h}{R})}(R-h)^3\sin(\varphi)\sec^3(\varphi))d\varphi\right)$$

(765)
$$= \frac{2\pi}{3}\left(-R^3\cos(\varphi)\Big|_0^{\cos^{-1}(\frac{R-h}{R})} - \int_0^{\cos^{-1}(\frac{R-h}{R})}(R-h)^3\tan(\varphi)\sec^2(\varphi))d\varphi\right)$$

(766)
$$\overset{u=\tan(\varphi)}{=} \frac{2\pi}{3}\left(-R^3(\frac{R-h}{R}) - (-R^3\cdot 1) - \frac{1}{2}(R-h)^3\tan^2(\varphi)\Big|_0^{\cos^{-1}(\frac{R-h}{R})}\right)$$

(767) $$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{1-\cos^2(\varphi)}{\cos^2(\varphi)}\Big|_0^{\cos^{-1}(\frac{R-h}{R})}\right)$$

(768) $$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{1-(\frac{R-h}{R})^2}{(\frac{R-h}{R})^2}\right)$$

(769) $$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)^3\frac{R^2-(R-h)^2}{(R-h)^2}\right)$$

(770) $$= \frac{2\pi}{3}\left(R^2h - \frac{1}{2}(R-h)(2Rh-h^2)\right)$$

(771) $$= \frac{\pi}{3}(2R^2h - 2R^2h + 2Rh^2 + Rh^2 - h^3) = \boxed{\frac{\pi}{3}h^2(3R-h)}.$$

Remark: In both solutions we can easily check our final answer by noting that $h = 0$ results in a volume of 0, $h = R$ results in a volume of $\frac{2\pi}{3}R^3$ which is indeed the volume of a hemisphere of radius R, and $h = -R$ results in a volume of $\frac{4}{3}R^3$ which is indeed the volume of a sphere of radius R.

Problem 6.1: Let R be the region bounded by the lines $y - x = 0, y - x = 2, y + x = 0, y + x = 2$. Use a change of variables to evaluate

$$\iint_R \sqrt{y^2 - x^2}\, dA. \tag{772}$$

Solution: We use the substitution $u = y - x$ and $v = y + x$ as suggested by the defining equations of the boundary curves. We also see in the picture below that this substitution results in a simple tesselation of our region R, which shows us that the new region of integration in the uv-plane is just $R' = \{(u, v) \mid 0 \le u \le 2, 0 \le v \le 2\}$.

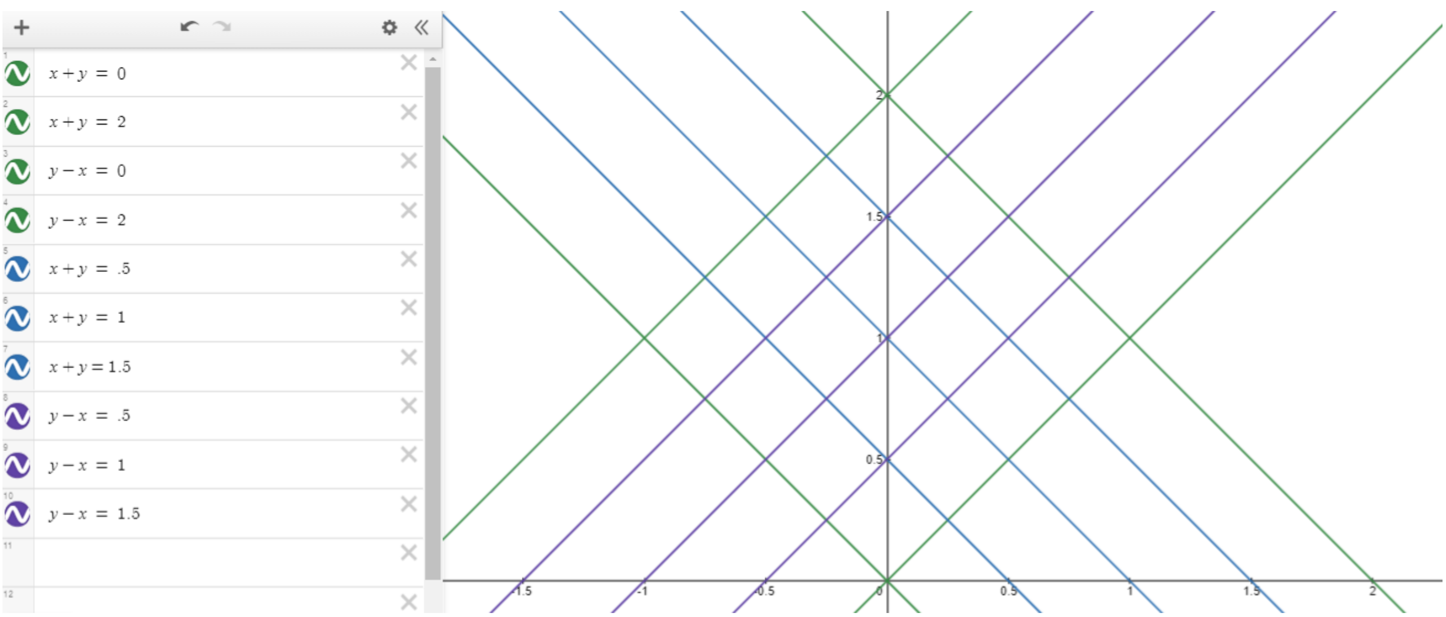


FIGURE 33. A picture of the region R and the tesselation that results from our given change of variables.

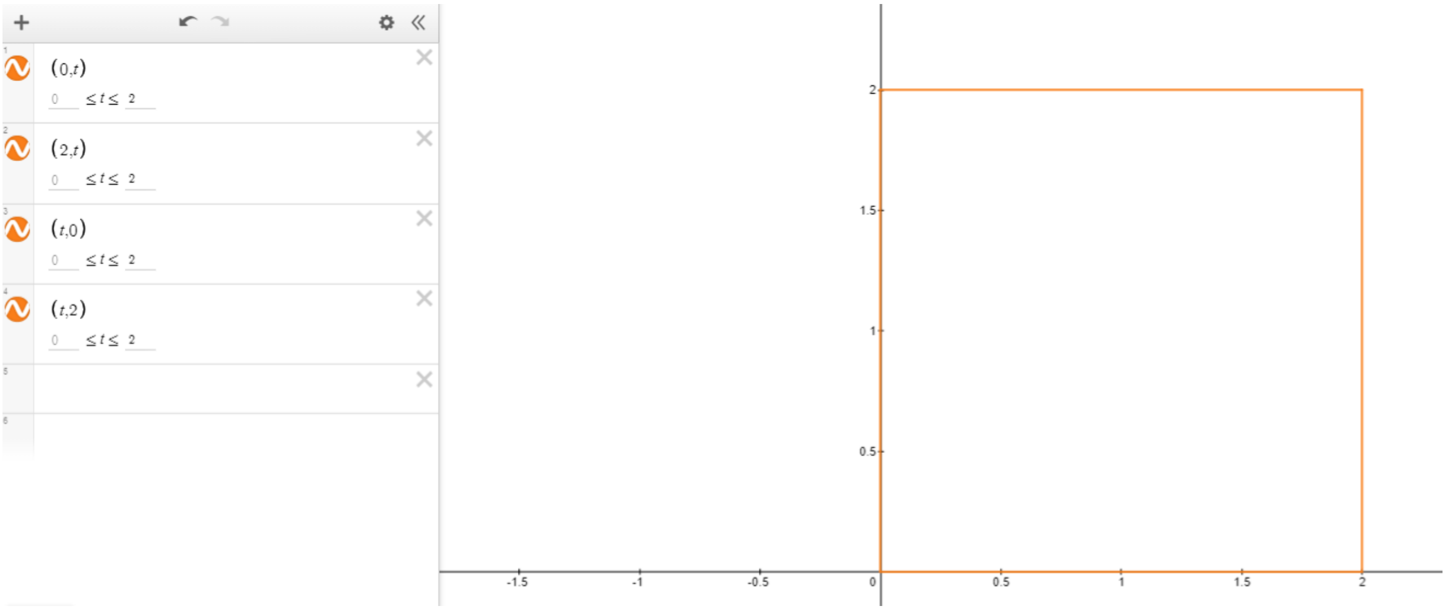


FIGURE 34. A picture of the region of integration in the uv-plane R'.

In order to calculate the Jacobian $J(u, v)$, we need to solve for x and y in terms of u and v. To this end, we see that

(773) $$\begin{array}{rcl} u & = & y - x \\ v & = & y + x \end{array} \rightarrow \begin{array}{rcl} x & = & \frac{1}{2}\Big((y+x) - (y-x)\Big) & = & \frac{1}{2}(v-u) \\ y & = & \frac{1}{2}\Big((y+x) + (y-x)\Big) & = & \frac{1}{2}(v+u) \end{array}.$$

We now see that

(774) $$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = (-\frac{1}{2}) \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2}.$$

It follows that $|J(u, v)| = |-\frac{1}{2}| = \frac{1}{2}$. We now see that

(775) $$\text{Area}(R) = \iint_R \sqrt{y^2 - x^2} dA = \iint_{R'} \sqrt{(y-x)(y+x)} \cdot |J(u, v)| dA$$

(776) $$= \int_0^2 \int_0^2 \sqrt{uv}\frac{1}{2} dudv = \frac{1}{2}\int_0^2 \int_0^2 u^{\frac{1}{2}} v^{\frac{1}{2}} dudv = \frac{1}{2}\int_0^2 \frac{2}{3} u^{\frac{3}{2}} v^{\frac{1}{2}} \Big|_{u=0}^2 dv$$

(777) $$= \frac{1}{2}\int_0^2 \frac{2}{3} 2^{\frac{3}{2}} v^{\frac{1}{2}} dv = \frac{2\sqrt{2}}{3}\int_0^2 v^{\frac{1}{2}} dv = \frac{4\sqrt{2}}{9} v^{\frac{3}{2}} \Big|_0^2 = \boxed{\frac{16}{9}}.$$

Problem 6.3: Find the volume of the solid D that is bounded by the planes $y - 2x = 0, y - 2x = 1, z - 3y = 0, z - 3y = 1, z - 4x = 0$, and $z - 4x = 3$.

Solution: We use the substitution $u = y - 2x, v = z - 3y$, and $w = z - 4x$ as suggested by the defining equations of the boundary curves. We also see in the pictures below that this substitution results in a simple tesselation of our region R, which shows us that the new region of integration in the uvw-space is just $R' = \{(u, v, w) \mid 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 3\}$.

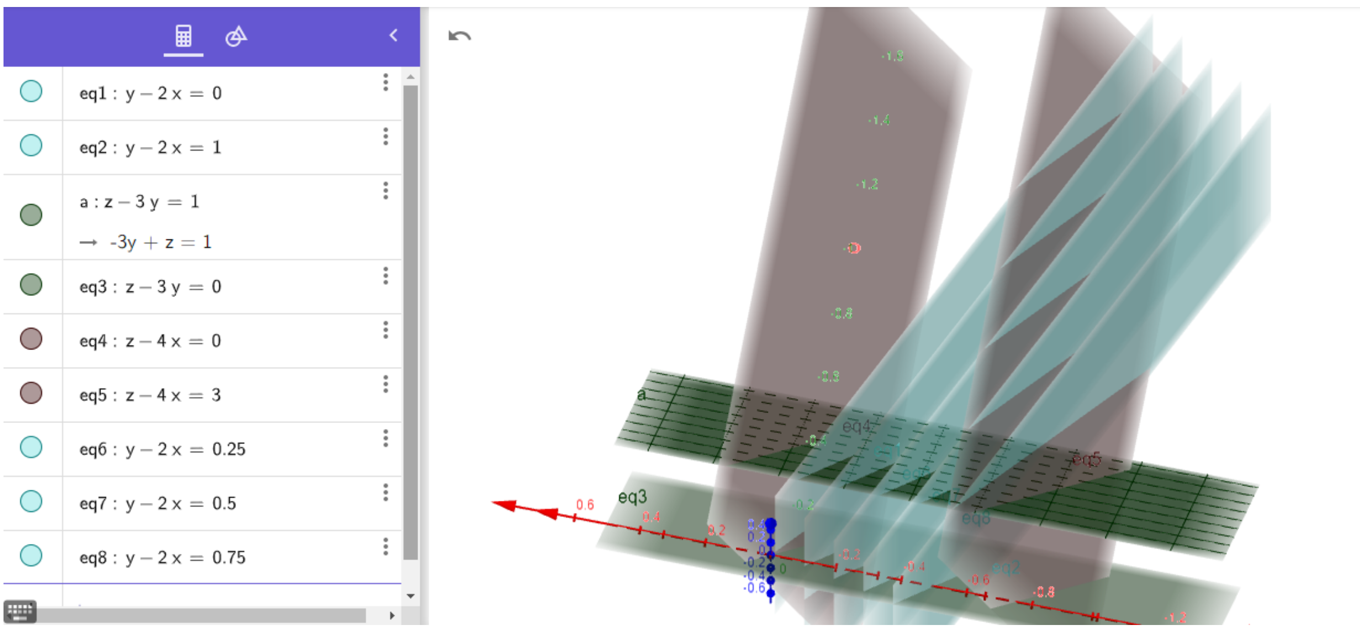


FIGURE 35. A visualization of the impact of changing the value of $y - 2x$.

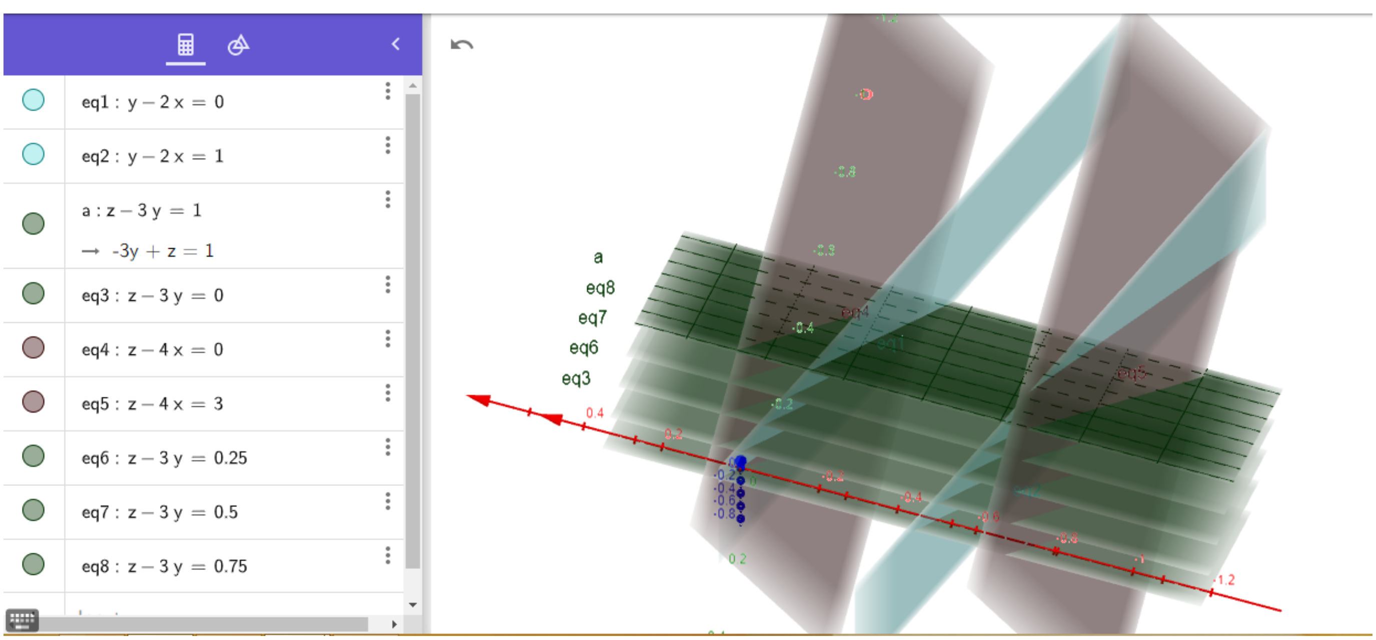


FIGURE 36. A visualization of the impact of changing the value of $z - 3y$.

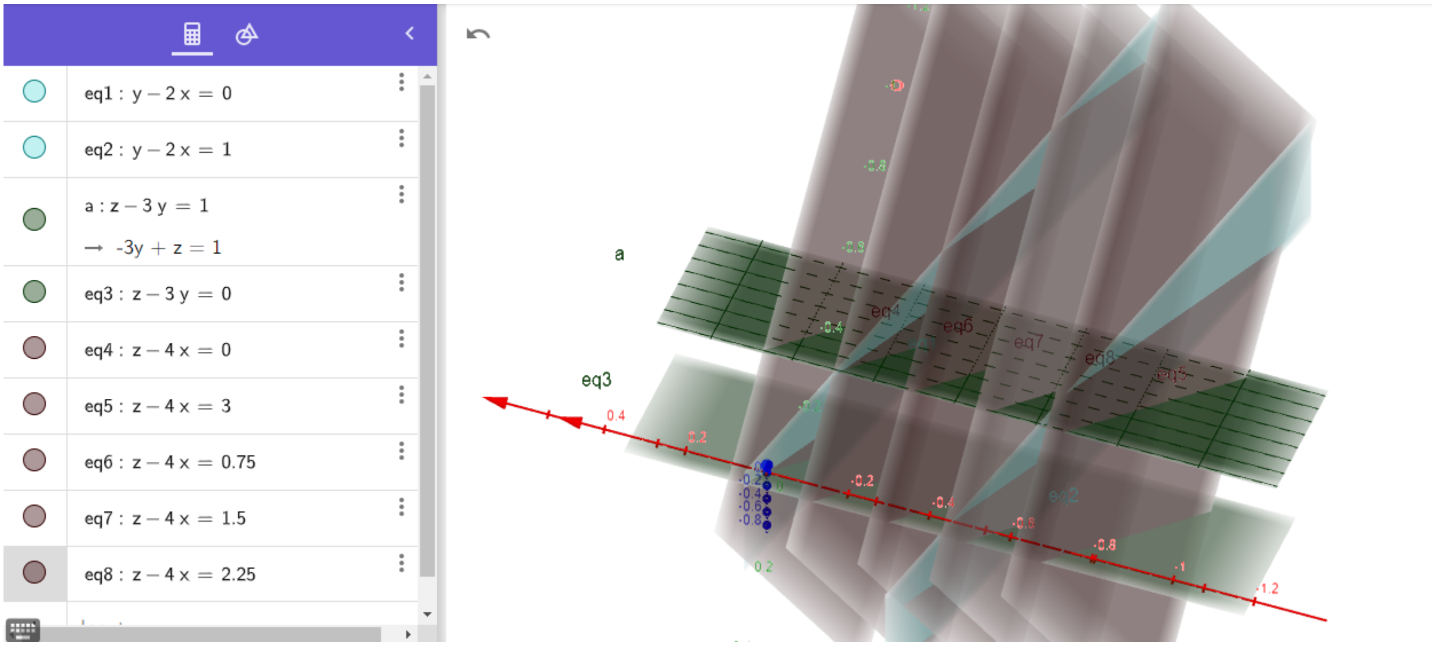


FIGURE 37. A visualization of the impact of changing the value of $z - 4x$.

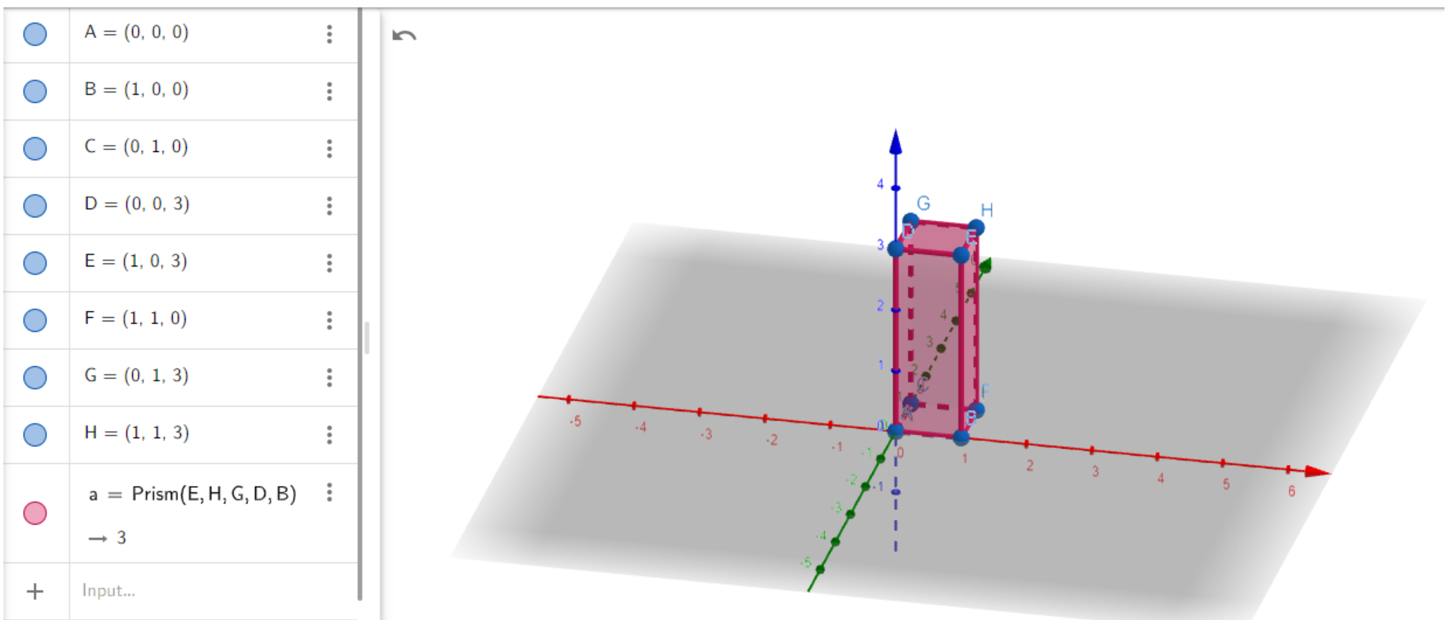


FIGURE 38. The region of integration R' in the uvw-space.

In order to calculate the Jacobian $J(u, v, w)$, we need to solve for x, y, and z in terms of u, v, and w. To this end, we see that

$$\begin{aligned} u &= y - 2x \\ v &= z - 3y \quad \to v - w = 4x - 3y \\ w &= z - 4x \end{aligned} \tag{778}$$

$$\to (v - w) + 3u = -2x \to x = \frac{1}{2}(-3u - v + w) \tag{779}$$

$$(780) \quad \rightarrow \begin{array}{lllll} y & = & u+2x & = & u-3u-v+w & = & -2u-v+w \\ z & = & w+4x & = & w-6u-2v+2w & = & -6u-2v+3w \end{array}.$$

We now see that

$$(781) \quad J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ -2 & -1 & 1 \\ -6 & -2 & 3 \end{vmatrix}$$

$$(782) \quad = -\frac{3}{2}\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} - (-\frac{1}{2})\begin{vmatrix} -2 & 1 \\ -6 & 3 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -2 & -1 \\ -6 & -2 \end{vmatrix}$$

$$(783) \quad = -\frac{3}{2}(-3+2) + \frac{1}{2}(-6+6) + \frac{1}{2}(4-6) = \frac{1}{2}.$$

It follows that $|J(u,v,w)| = \frac{1}{2}$. We now see that

$$(784) \quad \text{Volume}(D) = \iiint_D 1dV = \iiint_{D'} 1 \cdot |J(u,v,w)|dV$$

$$(785) \quad = \int_0^1 \int_0^1 \int_0^3 \frac{1}{2}dudvdw = (1-0)(1-0)(3-0)\frac{1}{2} = \boxed{\frac{3}{2}}.$$

Problem 6.4 (Parabolic coordinates): This problem has parts **a.-g.** spread out across the following pages. Your solutions to parts **a, b,** and **f** should include (hand drawn or computer generated) pictures.

Consider the Transformation T from the uv-plane to the xy-plane given by $T(u, v) = (u^2 - v^2, 2uv)$.

a. Show that the lines $u = a$ in the uv-plane map to parabolas in the xy-plane that open in the negative x-direction with vertices[10] on the positive x-axis.[11] Compare the images of the lines $u = a$ and $u = -a$ under T.

b. Show that the lines $v = b$ in the uv-plane map to parabolas in the xy-plane that open in the positive x-direction with vertices on the negative x-axis.[12] Compare the images of the lines $v = b$ and $v = -b$ under T.

c. Evaluate $J(u, v)$.

Solution to part a: We see that $T(a, v) = (a^2 - v^2, 2av)$. Setting $x = a^2 - v^2$ and $y = 2av$, we see that $v = \frac{1}{2a}y$, so $x = a^2 - (\frac{1}{2a}y)^2 = a^2 - \frac{1}{4a^2}y^2$, or equivalently, $x - a^2 = -\frac{1}{4a^2}y^2$. Since $-a^2 < 0$ and $-\frac{1}{4a^2} < 0$ when $a \neq 0$, we see (as mentioned in the footnote) that $T(a, v)$ is the parameterization of a parabola that opens in the negative x-direction and has its vertex on the positive x-axis. We see that $T(-a, v) = (a^2 - v^2, -2av) = (a^2 - (-v)^2, 2a(-v) = T(a, -v)$, so $T(a, v)$ and $T(-a, v)$ parameterize the same parabola in the xy-plane, but the parameterizations are in opposite directions (if $a \neq 0$). We also see that $T(0, v) = (-v^2, 0)$, which is a parameterization (with repitition) of the negative x-axis, which can be viewed as a degenerate parabola that opens in the negative x-direction and has its vertex at $(0, 0)$.

---

10 The vertex of the parabola $y = x^2$ is the point $(0, 0)$ and the vertex of the parabola $x = y^2$ is also $(0, 0)$.

11 You have to show that the curve $\vec{r_1}(v) = (a^2 - v^2, 2av)$ represents the same curve as $x + c = dy^2$ for some negative numbers c and d.

12 You have to show that the curve $\vec{r_2}(u) = (u^2 - b^2, 2ub)$ represents the same curve as $x + c = dy^2$ for some positive numbers c and d.

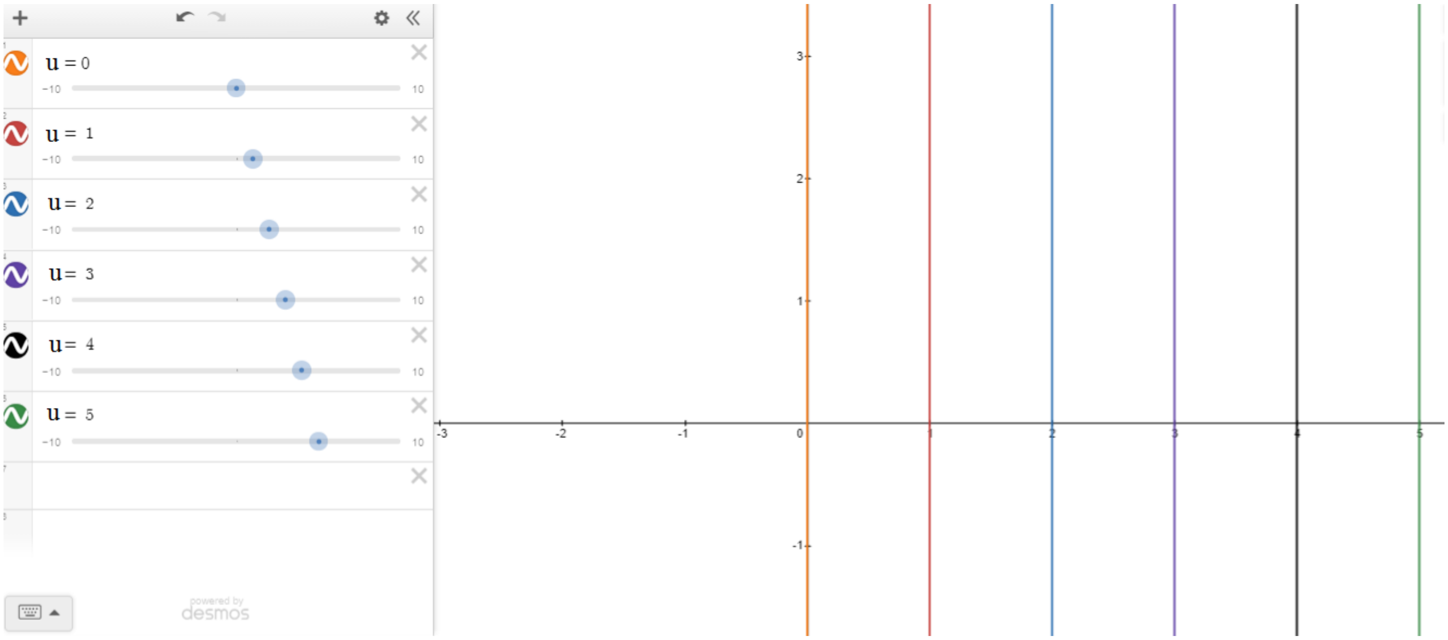


FIGURE 39. Vertical lines in the uv-plane.

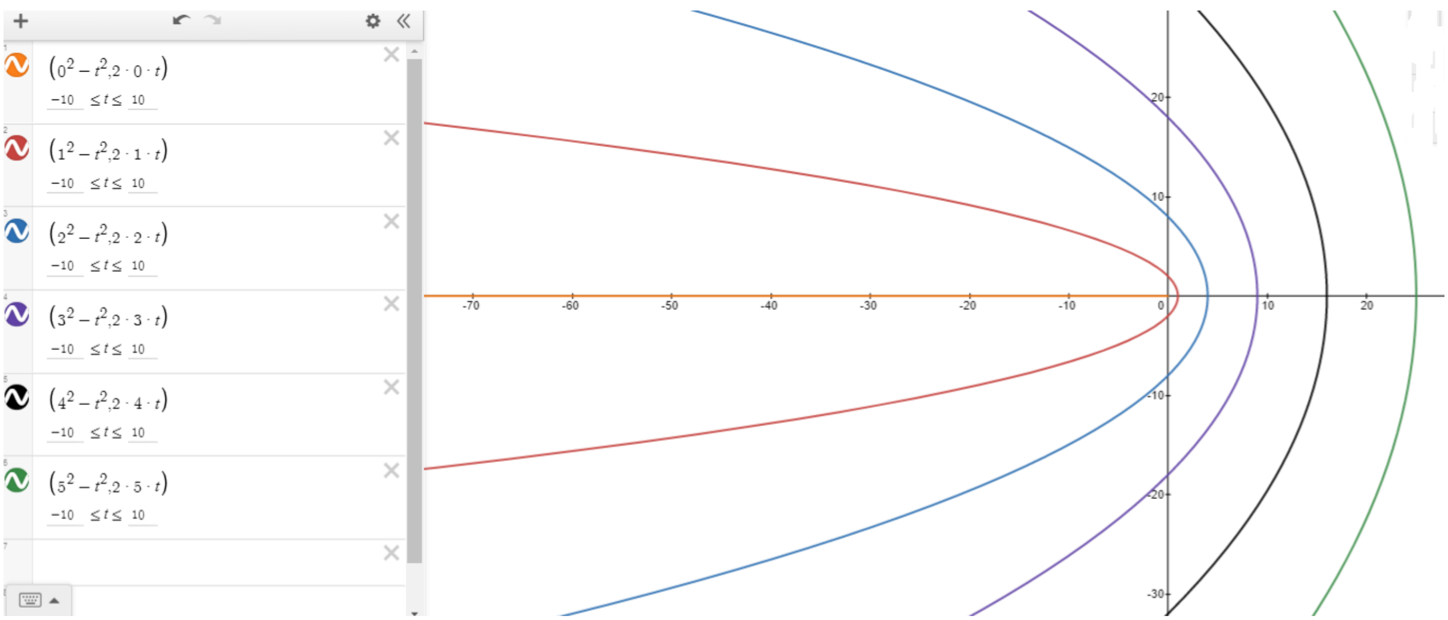


FIGURE 40. The parabolas in the xy-plane corresponding to vertical lines in the uv-plane under the transformation T.

Solution to part b: We see that $T(u,b) = (u^2 - b^2, 2ub)$. Setting $x = u^2 - b^2$ and $y = 2ub$, we see that $u = \frac{1}{2b}y$, so $x = (\frac{1}{2b}y)^2 - b^2 = \frac{1}{4b^2}y^2 - b^2$, or equivalently, $x + b^2 = \frac{1}{4b^2}y^2$. Since $b^2 > 0$ and $\frac{1}{4b^2} > 0$ when $b \neq 0$, we see (as mentioned in the footnote) that $T(u,b)$ is the parameterization of a parabola that opens in the postive x-direction and has its vertex on the negative x-axis. We see that $T(u,-b) = (u^2 - b^2, -2ub) = ((-u)^2 - b^2, 2(-u)b) = T(-u,b)$, so $T(u,b)$ and $T(u,-b)$ parameterize the same parabola in the xy-plane, but the parameterizations are in opposite directions (if $b \neq 0$). We also see that $T(u,0) = (u^2, 0)$, which is a parameterization (with repitition) of the positive x-axis, which can be viewed as a degenerate parabola that opens in the positive x-direction and has its vertex at $(0,0)$.

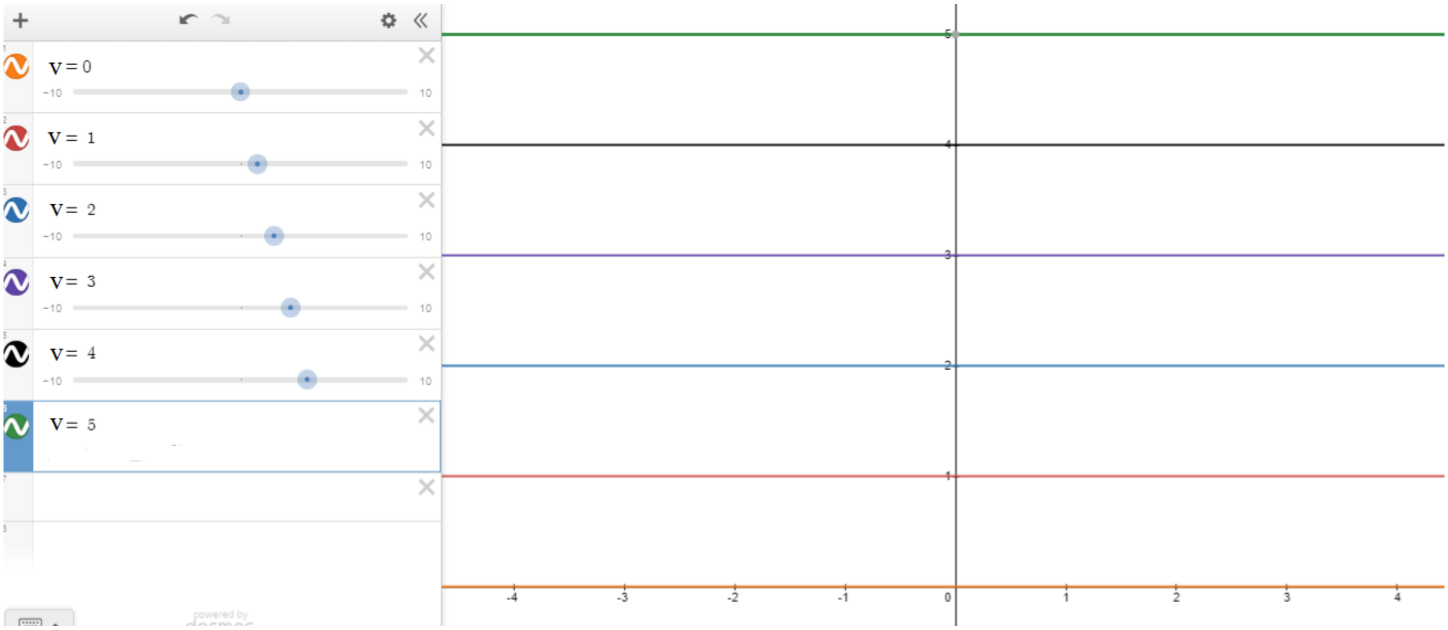


FIGURE 41. Horizontal lines in the uv-plane.

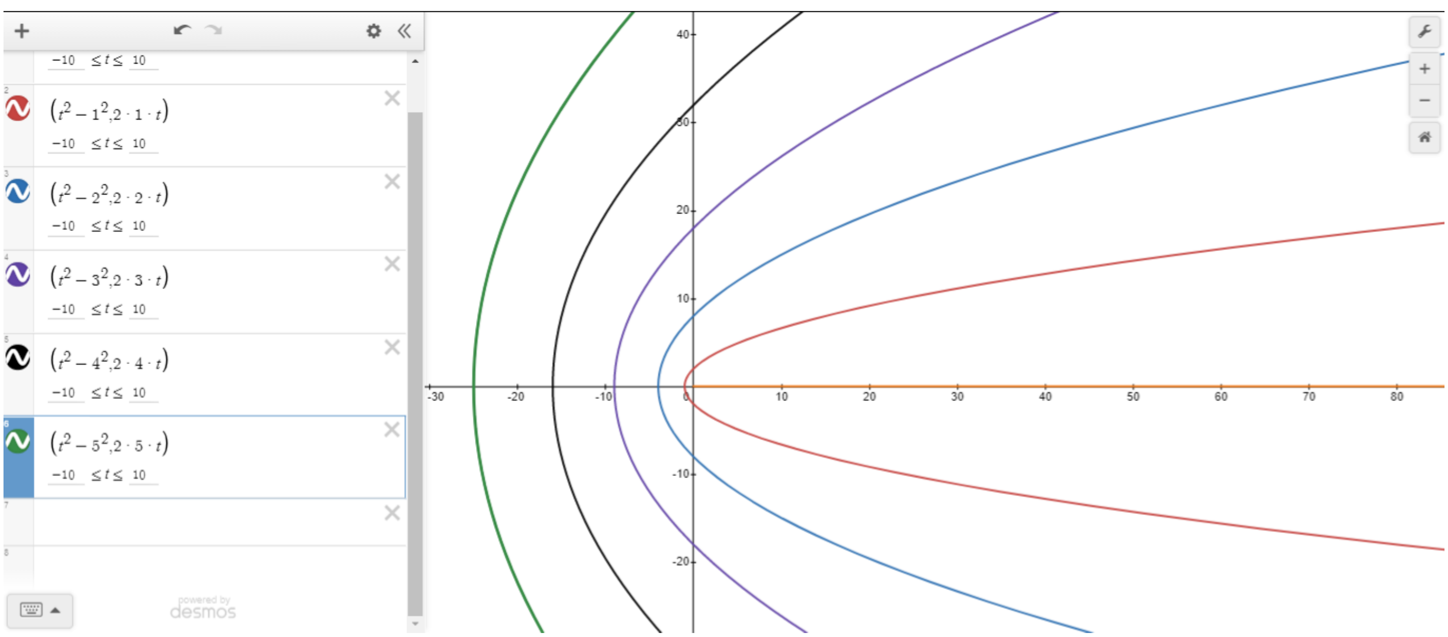


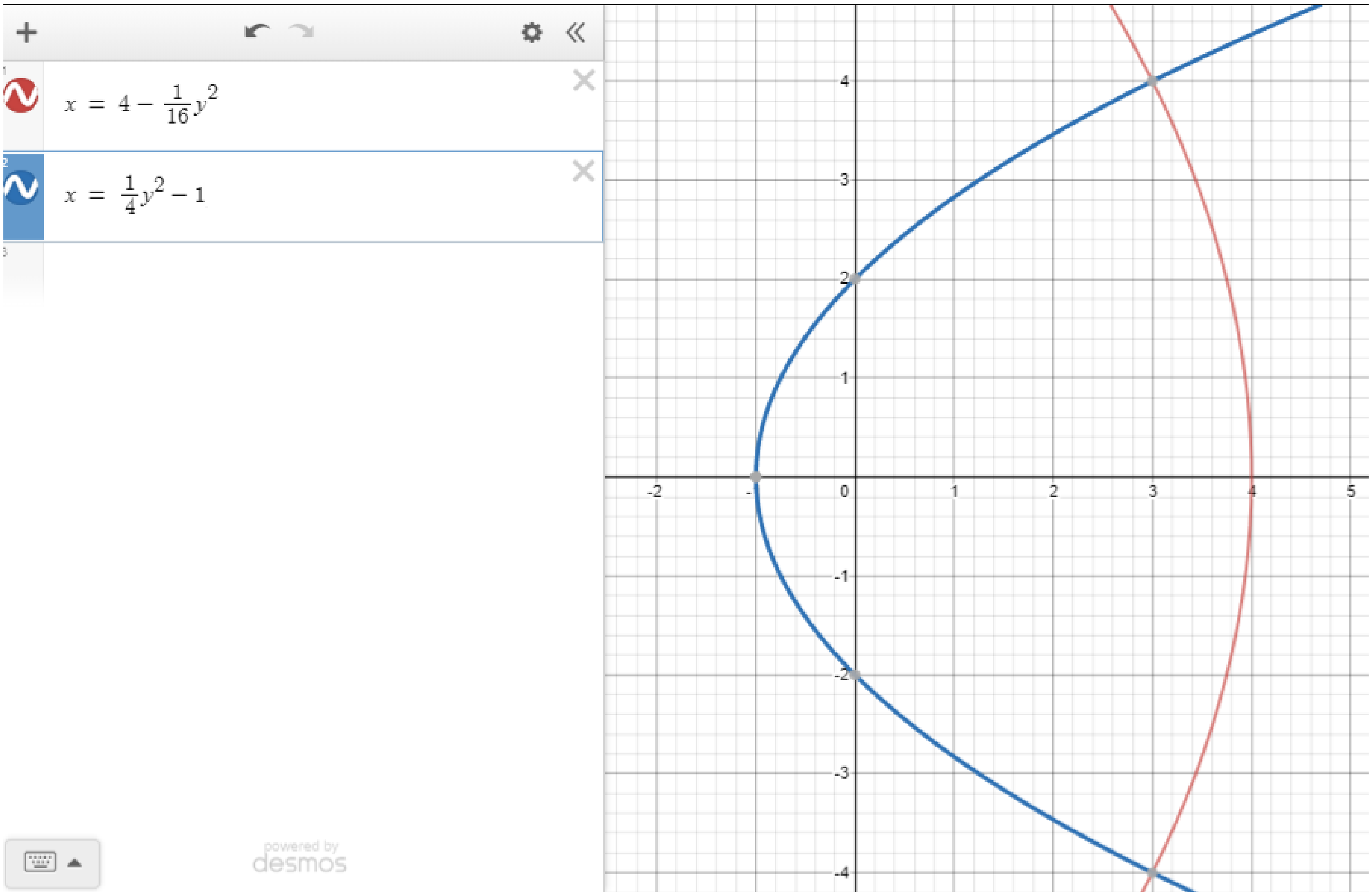
FIGURE 42. The parabolas in the xy-plane corresponding to horizontal lines in the uv-plane under the transformation T.

Solution to part c: Since $x = u^2 - v^2$ and $y = 2uv$, we see that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 2u \cdot 2u - (-2v) \cdot 2v = \boxed{4u^2 + 4v^2}. \tag{786}$$

We also observe that $|J(u,v)| = J(u,v) = 4u^2 + 4v^2$ since squares are always bigger than or equal to 0.

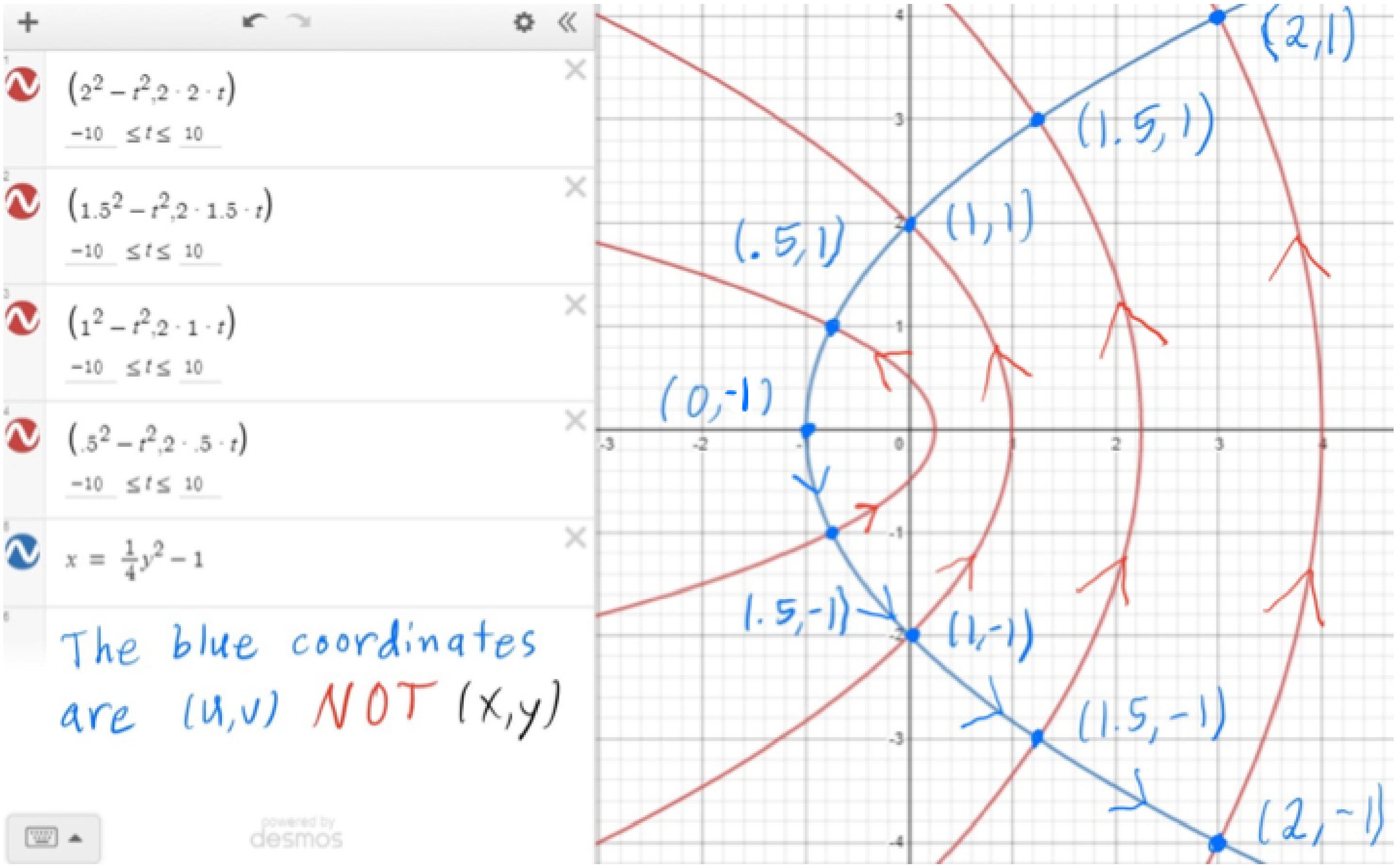
d. Use a change of variables into parabolic coordinates to find the area of the region R in the xy-plane bounded by the curves $x = 4 - \frac{1}{16}y^2$ and $x = \frac{1}{4}y^2 - 1$. Sketch a picture of the new region of integration as well.



Solution to part d: We begin by using parts **a** and **b** to see that the parabola $x = 4 - \frac{1}{16}y^2 = 2^2 - \frac{1}{4 \cdot 2^2}y^2$ in the xy-plane is the image under T of the line $u = 2$ (or $u = -2$) in the uv-plane and the parabola $x = \frac{1}{4}y^2 - 1 = \frac{1}{4 \cdot 1^2}y^2 - 1^2$ in the xy-plane is the image under T of the line $v = 1$ ($v = 1$) in the uv-plane. Note that for a postive number p, $T(2, p)$ is on the upper half of the parabola $x = 4 - \frac{1}{16}y^2$ and $T(2, -p)$ is on the lower half. Similarly, for $T(p, 1)$ is on the upper half of the parabola $x = \frac{1}{4}y^2 - 1$ and $T(-p, 1) = T(p, -1)$ is on the lower half. We also recall that T (almost) bijects the closed right (or left, or upper, or lower) half of the uv-plane to the xy-plane.[13] The picture below puts together all of the previous discussion to show that the region R in the xy-plane is the

[13] The map T from the uv-plane to the xy-plane is a one-to-one map if you restrict yourself to an open half of the uv-plane and an appropriate closed half of an axis (such as the open left half of the plane and the closed upper half of the y-axis), but T is not one-to-one on the entire uv-plane since $T(a, b) = T(-a, -b)$.

image under T of the region rectangle $R' = \{(u,v) \mid 0 \le u \le 2, -1 \le v \le 1\}$ in the uv-plane.

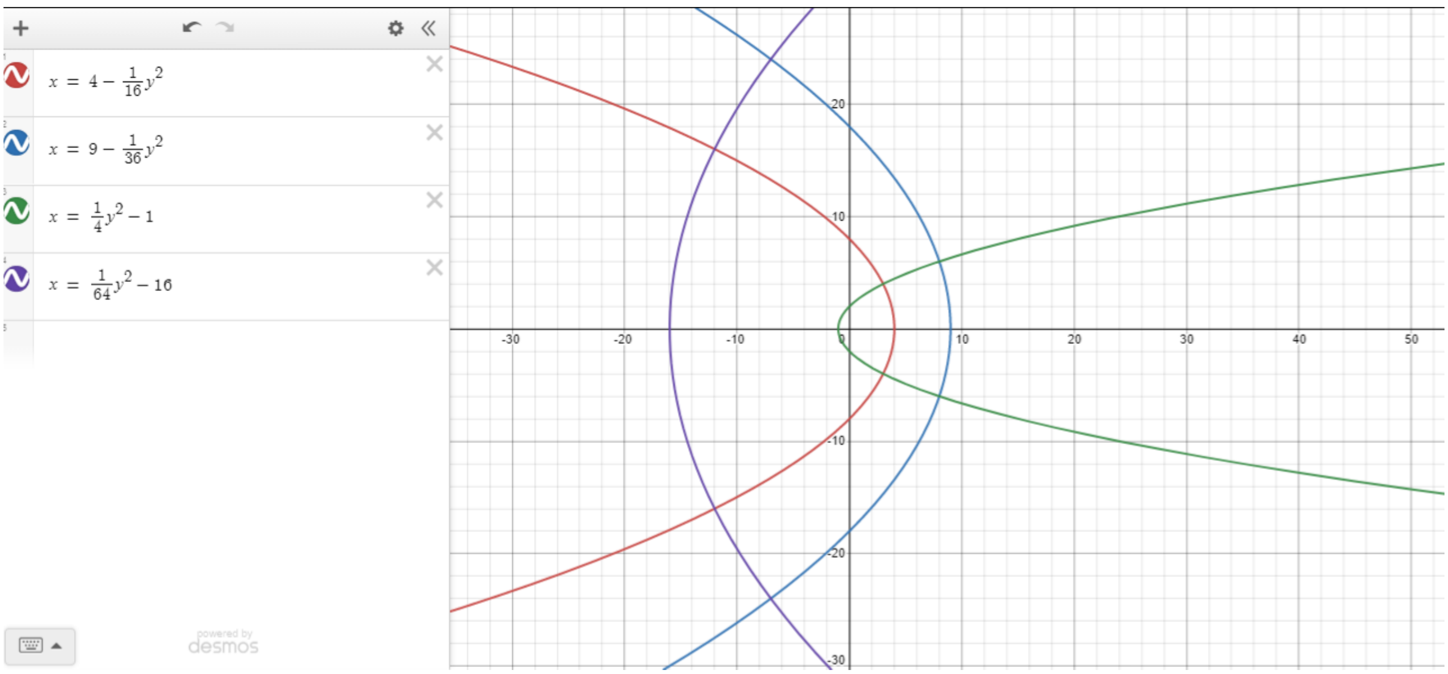


$$\text{(787)} \quad \text{Area}(R) = \iint_R 1dA = \iint_{R'} 1 \cdot |J(u,v)|dA = \int_0^2 \int_{-1}^1 |J(u,v)|dvdu$$

$$\text{(788)} \quad = \int_0^2 \int_{-1}^1 (4u^2+4v^2)dvdu = \int_0^2 (4u^2v+\frac{4}{3}v^3\Big|_{v=-1}^1)du = \int_0^2 (8u^2+\frac{8}{3})du$$

$$\text{(789)} \quad = \frac{8}{3}u^3 + \frac{8}{3}u\Big|_{u=0}^2 = \boxed{\frac{80}{3}}.$$

e. Use a change of variables into parabolic coordinates to find the area of the curved rectangle R above the x-axis bounded by $x = 4 - \frac{1}{16}y^2, x = 9 - \frac{1}{36}y^2, x = \frac{1}{4}y^2 - 1$, and $x = \frac{1}{64}y^2 - 16$. Sketch a picture of the new region of integration as well.



Solution to part e: We proceed as we did in part **d**. We note that $x = 4 - \frac{1}{16}y^2$ corresponds to $u = 2, -2$, $x = 9 - \frac{1}{36}y^2$ corresponds to $u = 3, -3$, $x = \frac{1}{4}y^2 - 1$ corresponds to $v = 1, -1$, and $x = \frac{1}{64}y^2 - 16$ corresponds to $v = 4, -4$. Since $y = 2uv$ is positive when u and v are both positive (or both negative), we obtain the parabolic rectangle above the x-axis as the image of the region $R' = \{(u, v) \mid 2 \leq u \leq 3, 1 \leq v \leq 4\}$ under T.

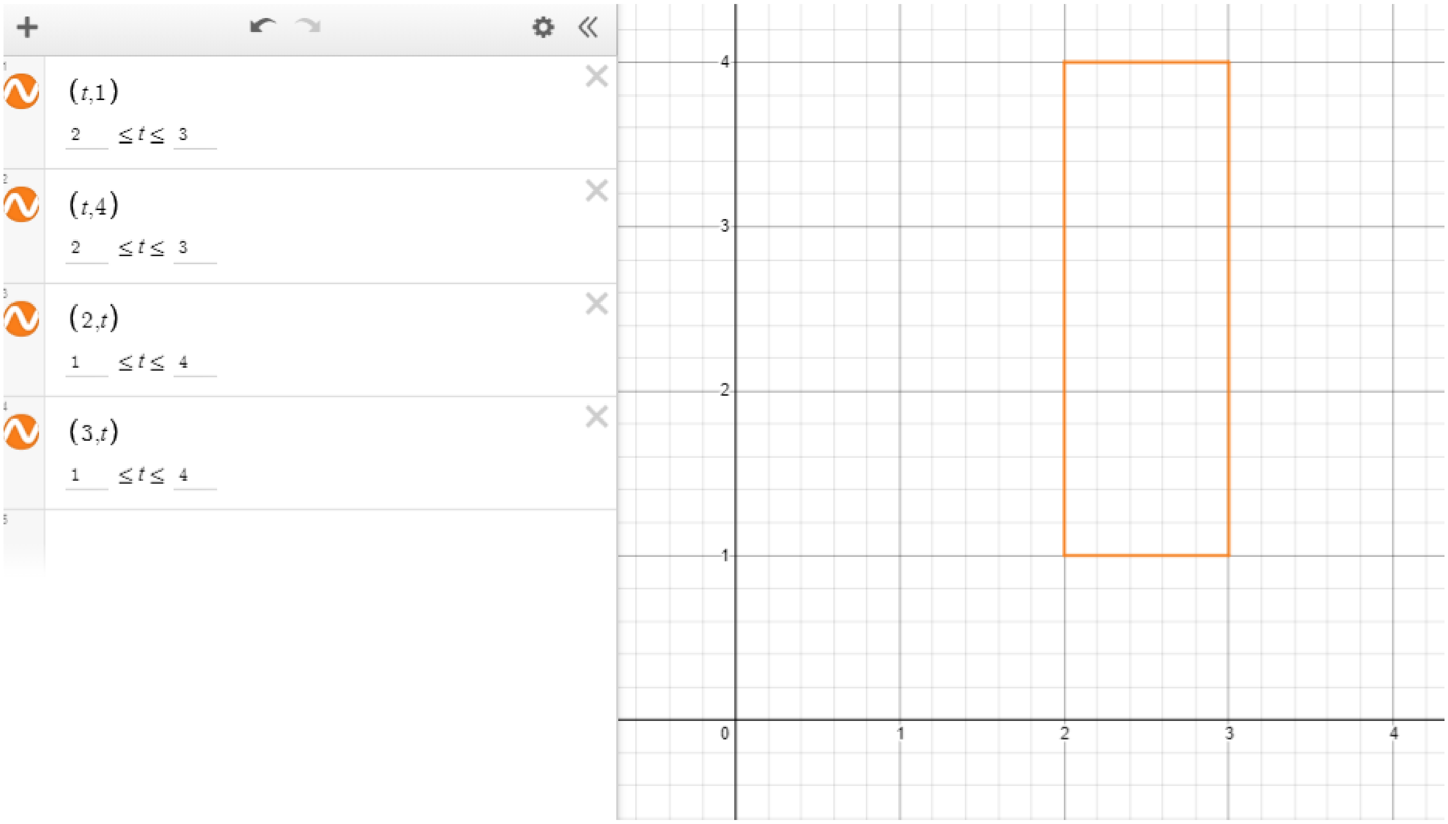


FIGURE 43. The new region of integration R' in the uv-plane.

We now see that

$$\text{Area}(R) = \iint_R 1dA = \iint_{R'} 1 \cdot |J(u,v)|dA = \int_2^3 \int_1^4 (4u^2 + 4v^2)dvdu \tag{790}$$

$$= \int_2^3 (4u^2v + \frac{4}{3}v^3\Big|_{v=1}^4)du = \int_2^3 (12u^2 + 84)du = 4u^3 + 84u\Big|_2^3 = \boxed{160}. \tag{791}$$

f. Describe the effect of the transformation $(u, v) \mapsto (2uv, u^2 - v^2)$ on horizontal and vertical lines in the uv-plane.[14]

Solution to part f: Let $S(u, v) = (2uv, u^2 - v^2)$. If P is a parabola that opens in the negative x-direction and has its vertex on the positive x-axis, then upon reflection over the line $x = y$, we obtain a parabola P' that opens in the negative y-direction and has its vertex on the positive y-axis. It follows that the image of the vertical line $u = a$ in the uv-plane under the transformation S gives a parabola in the xy-plane that opens in the negative y-direction and has its vertex on the positive y-axis. Similarly, if P is a parabola that opens in the positive x-direction and has its vertex on the negative x-axis, then upon reflection over the line $x = y$, we obtain a parabola P' that opens in the positive y-direction and has its vertex on the negative y-axis. It follows that the image of the horizontal line $v = b$ in the uv-plane under the transformation S gives a parabola in the xy-plane that opens in the positive y-direction and has its vertex on the negative y-axis.

[14] Remember that the transformation $(x, y) \mapsto (y, x)$ reflects points in the xy-plane across the line $y = x$. It will also help to use the results of parts **a.** and **b.** of this problem.

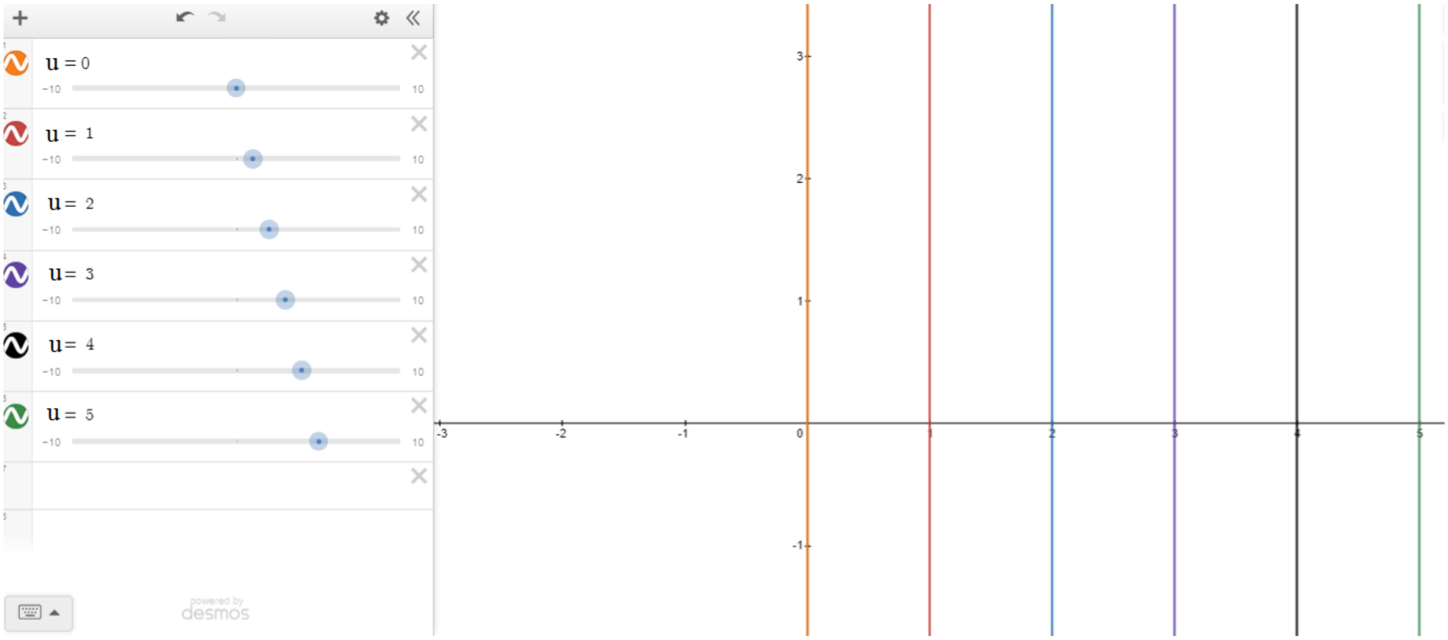


FIGURE 44. Vertical lines in the uv-plane.

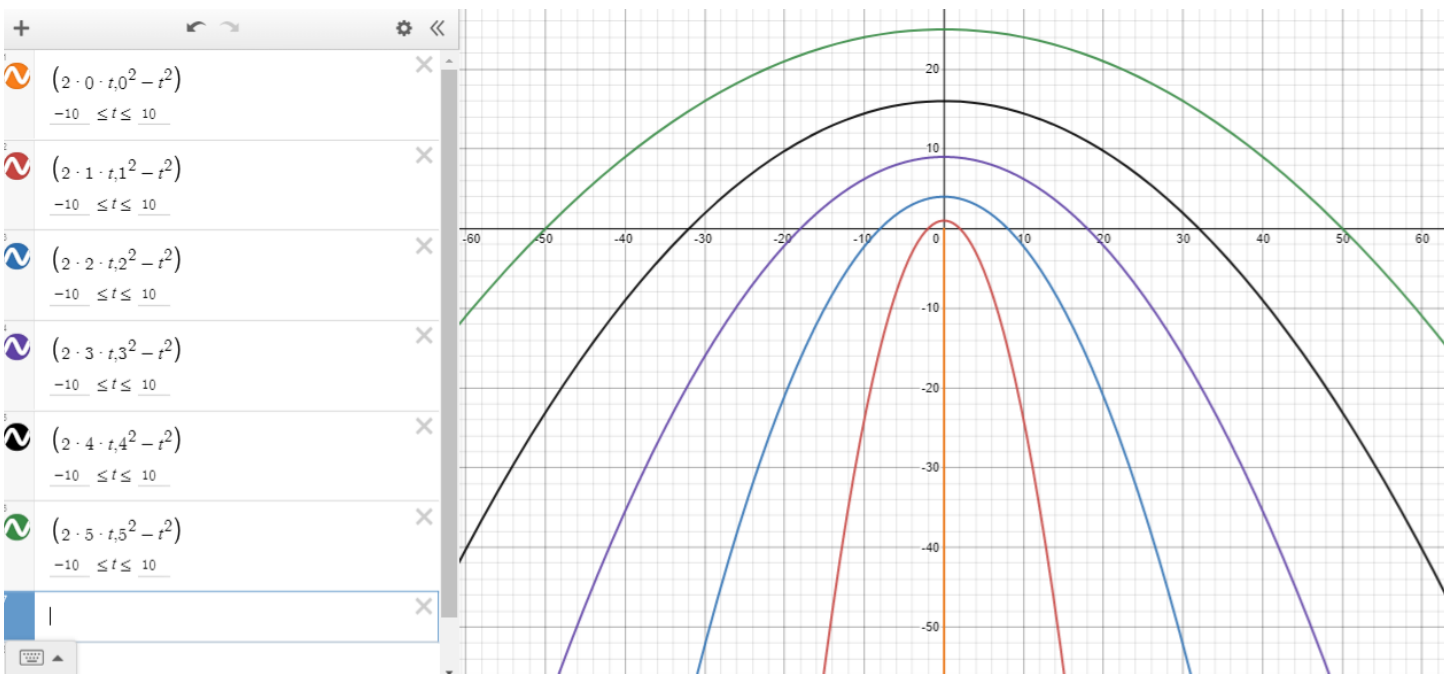


FIGURE 45. The parabolas in the xy-plane corresponding to vertical lines in the uv-plane under the transformation S.

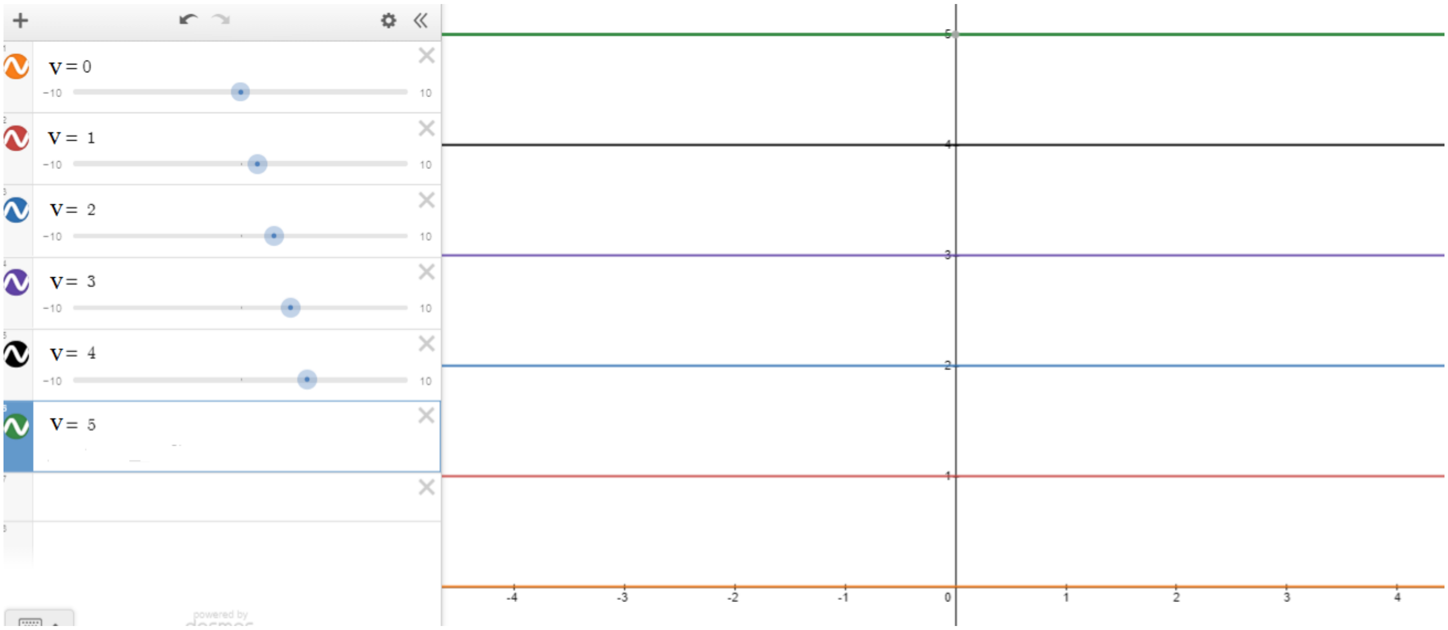


FIGURE 46. Horizontal lines in the uv-plane.

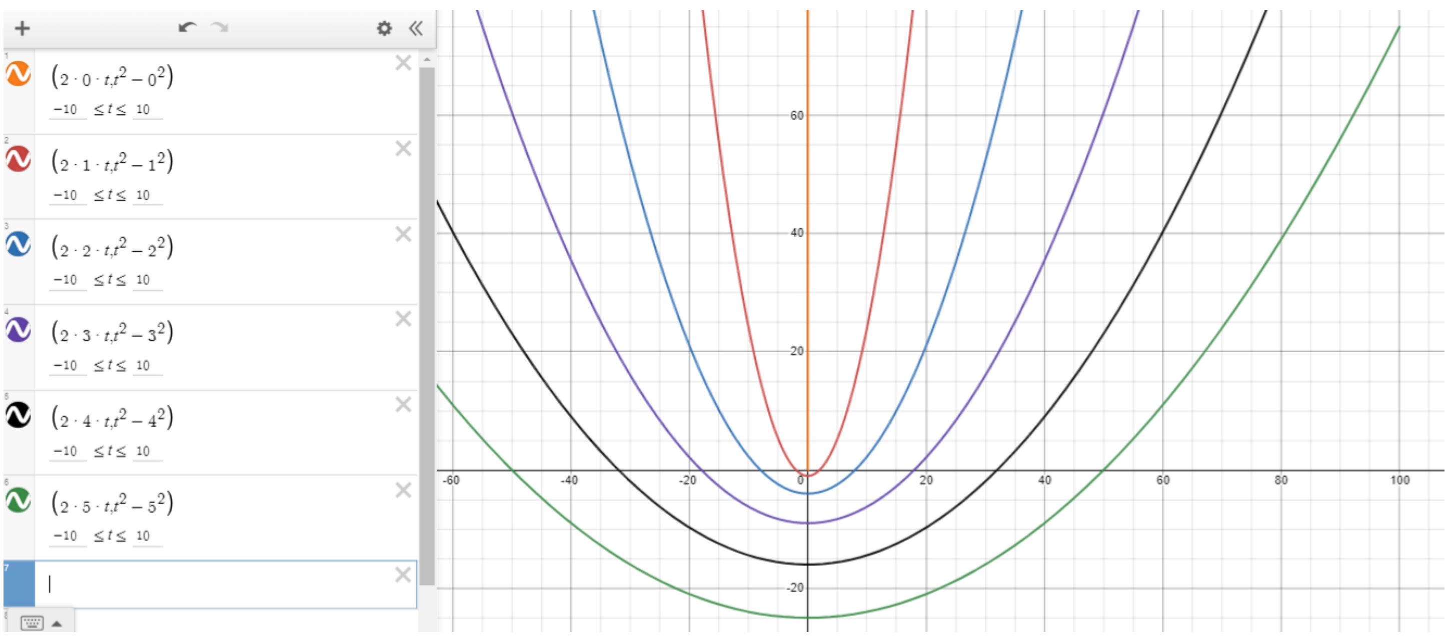


FIGURE 47. The parabolas in the xy-plane corresponding to horizontal lines in the uv-plane under the transformation S.

g. Show that the parabolas that are the images of the lines $u = a$ and $v = b$ under $T(u, v) = (u^2 - v^2, 2uv)$ are orthogonal to eachother.

Solution to part g: We have already seen in parts **a** and **b** that $T(a, v)$ is the parabola $x = a^2 - \frac{1}{4a^2}y^2$ and $T(u, b)$ is the parabola $x = \frac{1}{4b^2}y^2 - b^2$. We will first find the intersection points of these 2 parabolas, then we will calculate the slope of the tangent lines at the intersection points in order to see that the tangent lines (and hence the curves) are orthogonal.

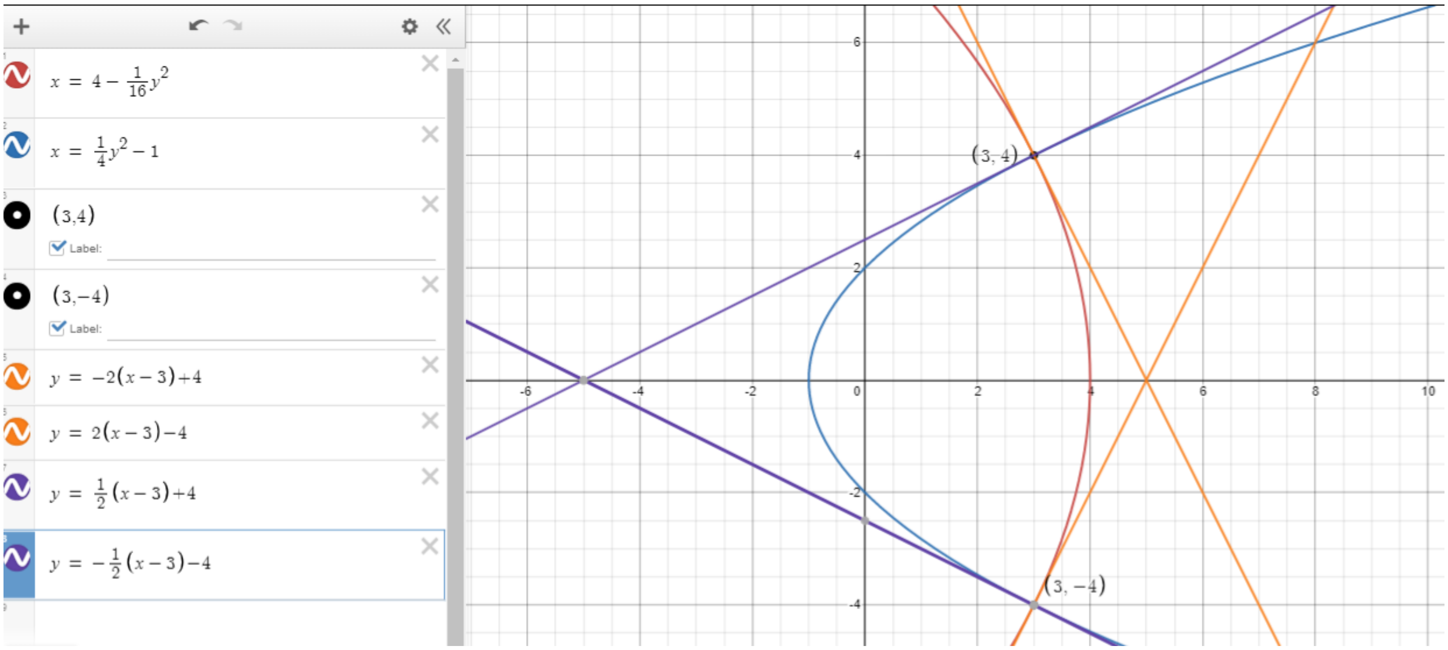


FIGURE 48. A picture of $T(2, v), T(u, 1)$, and the tangent lines to both curves at their intersection points.

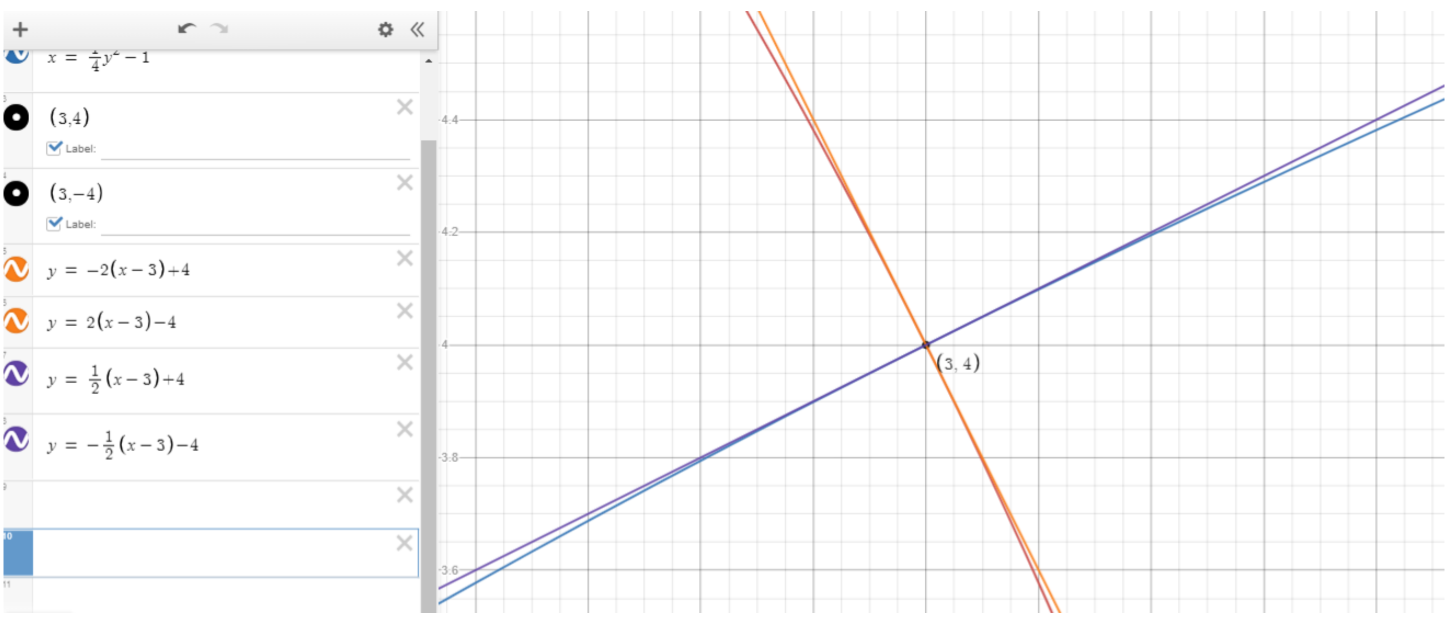


FIGURE 49. A zoomed in shot around the intersection point $(3, 4)$ to show that the tangent lines (and hence the curves) are perpendicular.

To this end, we see that

$$\begin{array}{rcl} x & = & a^2 - \frac{1}{4a^2}y^2 \\ x & = & \frac{1}{4b^2}y^2 - b^2 \end{array} \rightarrow a^2 - \frac{1}{4a^2}y^2 = \frac{1}{4b^2}y^2 - b^2 \rightarrow a^2 + b^2 = (\frac{1}{4a^2} + \frac{1}{4b^2})y^2 \quad (792)$$

$$\rightarrow y^2 = \frac{a^2 + b^2}{\frac{1}{4a^2} + \frac{1}{4b^2}} = 4a^2b^2 \rightarrow y = \pm 2ab \rightarrow x = a^2 - b^2. \quad (793)$$

It follows that $T(a,b) = T(-a,-b) = (a^2 - b^2, 2ab)$ and $T(a,-b) = T(-a,b) = (a^2 - b^2, -2ab)$ are the intersection points of the 2 parabolas. Noting that

$$x = a^2 - \frac{1}{4a^2}y^2 \rightarrow dx = -\frac{1}{2a^2}ydy \rightarrow \frac{dy}{dx} = -2\frac{a^2}{y}, \text{ and} \quad (794)$$

$$x = \frac{1}{4b^2}y^2 - b^2 \rightarrow dx = \frac{1}{2b^2}ydy \rightarrow \frac{dy}{dx} = 2\frac{b^2}{y}, \quad (795)$$

We see that at the point $(a^2 - b^2, 2ab)$, the tangent line to the curve $x = a^2 - \frac{1}{4a^2}y^2$ has a slope of $-\frac{a}{b}$ and the tangent line to the curve $x = \frac{1}{4b^2}y^2 - b^2$ has a slope of $\frac{b}{a}$. Since $-\frac{a}{b} \cdot \frac{b}{a} = -1$, we see that the tangent lines at the point $(a^2 - b^2, 2ab)$ are indeed orthogonal to each other. Similarly, we see that at the point $(a^2 - b^2, -2ab)$, the tangent line to the curve $x = a^2 - \frac{1}{4a^2}y^2$ has a slope of $\frac{a}{b}$ and the tangent line to the curve $x = \frac{1}{4b^2}y^2 - b^2$ has a slope of $-\frac{b}{a}$. Since $\frac{a}{b} \cdot (-\frac{b}{a}) = -1$, we see that the tangent lines at the point $(a^2 - b^2, 2ab)$ are indeed orthogonal to each other.

Remark: We see that the parabolas produced by S in part **f** also share this orthogonality property since orthogonality is preserved under reflections.

Problem 7.5: Consider the vector field $\vec{F} = \langle x, -y\rangle$ and the curve C which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.

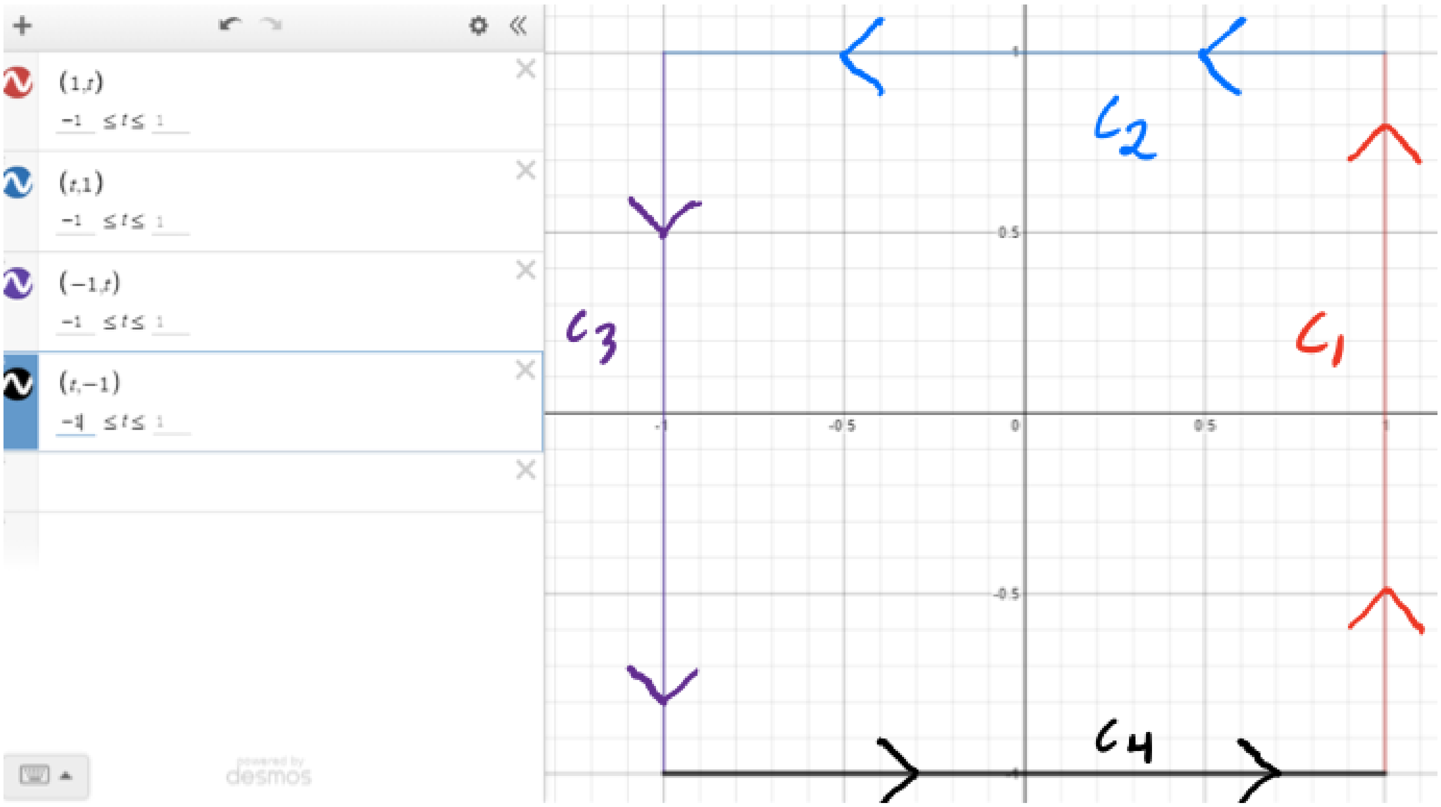


FIGURE 50. The curve C.

(a) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by finding a parametrization $\vec{r}(t)$ for the curve C.
(b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by using the Fundamental Theorem for Line Integrals.

Solution to (a): Letting C_1, C_2, C_3, and C_4 be as in Figure 50, we see that

(796) $$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}.$$

Since

(797) $$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \langle 1, -t\rangle \cdot \langle 0, 1\rangle dt = \int_{-1}^{1} -tdt = -\frac{1}{2}t^2\Big|_{-1}^{1} = 0,$$

(798) $$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{1}^{-1} \langle t, -1\rangle \cdot \langle 1, 0\rangle dt = \int_{1}^{-1} tdt = \frac{1}{2}t^2\Big|_{1}^{-1} = 0,$$

(799) $$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{1}^{-1} \langle -1, -t \rangle \cdot \langle 0, 1 \rangle dt = \int_{1}^{-1} -t dt = -\frac{1}{2}t^2 \Big|_{1}^{-1} = 0,$$

(800) $$\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle t, 1 \rangle \cdot \langle 1, 0 \rangle dt = \int_{-1}^{1} t dt = \frac{1}{2}t^2 \Big|_{-1}^{1} = 0,$$

we see that

(801) $$\int_{C} \vec{F} \cdot d\vec{r} = 0 + 0 + 0 + 0 = \boxed{0}.$$

Solution to (b): Since

(802) $$\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}(-y),$$

we see that $\vec{F} = \langle x, -y \rangle$ is a conservative vector field. We now have 2 ways in which to finish the problem.

Finish 1: Since $\vec{F}$ is a conservative vector field and C is a (simple, piecewise smooth, oriented) closed curve, and $\vec{F}$ is continuous on C and its interior, we see that

(803) $$\int_{C} \vec{F} \cdot d\vec{r} = \boxed{0}.$$

Finish 2: We now want to find a potential function $\varphi(x, y)$ for $\vec{F}$. Since

(804) $$\langle \varphi_x, \varphi_y \rangle = \nabla \varphi = \vec{F} = \langle x, -y \rangle,$$

we see that

(805) $$\varphi_x(x, y) = x \to \varphi(x, y) = \int x dx = \frac{1}{2}x^2 + g(y) \to$$

$$g'(y) = \varphi_y(x,y) = -y \tag{806}$$
$$\rightarrow g(y) = -\frac{1}{2}y^2 + C \rightarrow \varphi(x,y) = \frac{1}{2}(x^2 - y^2) + C.$$

Now let P be any point on the curve C. For example, we may take $P = (1,1)$. Since the curve C can be seen as starting at P and ending at P, the Fundamental Theorem for Line Integrals tells us that

$$\int_C \vec{F} \cdot d\vec{r} = \varphi\left((1,1)\right) - \varphi\left((1,1)\right) = \boxed{0}. \tag{807}$$

Remark: We see that in Finish 2, we did not even need to determine what the function φ was in order to conclude that the final answer is 0.

Problem 7.6: Find the average value of

(808) $$f(x,y)=\sqrt{4+9y^{2/3}}$$

on the curve $y=x^{3/2}$, for $0\le x\le 5$.

Solution: For curves of the form $y=f(x)$, $a\le x\le b$, we have the parameterization $\mathbf{r}(t)=\langle t,f(t)\rangle$, $a\le t\le b$. In this particular problem, we see that $\mathbf{r}(t)=\langle t,t^{\frac{3}{2}}\rangle$, $0\le t\le 5$. It follows that

(809) $$\mathbf{r}'(t)=\langle 1,\frac{3}{2}t^{\frac{1}{2}}\rangle\to|\mathbf{r}'(t)|=\sqrt{1^2+(\frac{3}{2}t^{\frac{1}{2}})^2}=\sqrt{1+\frac{9}{4}t}.$$

Recall that the average value of f over a curve C is given by

(810) $$\text{Av}(f)=\frac{\int_C fds}{\text{Arclength}(C)}=\frac{\int_C fds}{\int_C 1ds}.$$

We begin by calculating $\int_C fds$ and see that

(811) $$\int_C fds=\int_0^5 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt=\int_0^5 f(t,t^{\frac{3}{2}})\sqrt{1+\frac{9}{4}t}dt$$

(812) $$=\int_0^5\sqrt{4+9(t^{\frac{3}{2}})^{\frac{2}{3}}}\sqrt{1+\frac{9}{4}t}dt=\int_0^5\sqrt{4+9t}\sqrt{1+\frac{9}{4}t}dt$$

(813) $$=\int_0^5 2\sqrt{1+\frac{9}{4}t}\sqrt{1+\frac{9}{4}t}dt=2\int_0^5(1+\frac{9}{4}t)dt=2(t+\frac{9}{8}t^2\Big|_0^5)=\frac{265}{4}.$$

We now calculate the arclength of our given curve and see that

(814) $$\text{Arclength}(C)=\int_C 1ds=\int_0^5|\mathbf{r}'(t)|dt=\int_0^5\sqrt{1+\frac{9}{4}t}dt$$

(815) $$\overset{u=1+\frac{9}{4}t}{=}\int_{t=0}^5\sqrt{u}\frac{4}{9}du=\frac{4}{9}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{t=0}^5=\frac{8}{27}(1+\frac{9}{4}t)^{\frac{3}{2}}\Big|_0^5=\frac{335}{27}.$$

It follows that the final answer is

$$\text{Av}(f) = \frac{\int_C f ds}{\text{Arclength}(C)} = \frac{\int_C f ds}{\int_C 1 ds} = \frac{\frac{265}{4}}{\frac{335}{27}} = \boxed{\frac{1431}{268}}. \tag{816}$$

Problem 7.7: Consider

$$\int_{\mathcal{C}} (x^2 + y^2) ds,$$

where $\mathcal{C}$ is the line segment from $(0,0)$ to $(5,5)$.

(1) Find a parametric description for $\mathcal{C}$ in the form $\vec{r}(t) = \langle x(t), y(t) \rangle$. (*Remember to state the domain of the parameter.*)

(2) Evaluate $|\vec{r'}(t)|$.

(3) Convert the line integral to an ordinary integral with respect to the parameter and evaluate it.

Solution to (1): We recall that

$$\vec{r}(t) = \vec{P} + t(\vec{Q} - \vec{P}), 0 \le t \le 1 \tag{817}$$

is one way in which to parameterize the line segment that starts at the point P and ends at the point Q. In this particular problem, we see that

$$\vec{r}(t) = \langle 0, 0 \rangle + t(\langle 5, 5 \rangle - \langle 0, 0 \rangle) = \langle 5t, 5t \rangle, 0 \le t \le 1 \tag{818}$$

is a parameterization for the line segment from $(0,0)$ to $(5,5)$.

Solution to (2): We see that

$$\vec{r'}(t) = \langle 5, 5 \rangle \to |\vec{r'}(t)| = \sqrt{5^2 + 5^2} = \sqrt{50} = 5\sqrt{2}. \tag{819}$$

Solution to (3): We have

$$\int_C (x^2 + y^2) ds = \int_0^1 ((5t)^2 + (5t)^2) 5\sqrt{2} dt = 5\sqrt{2} \int_0^1 (25t^2 + 25t^2) dt \tag{820}$$

$$= 250\sqrt{2} \int_0^1 t^2 dt = \frac{250\sqrt{2}}{3} t^3 \Big|_0^1 = \boxed{\frac{250\sqrt{2}}{3}}. \tag{821}$$

Problem 7.8: Compute

$$\int_{\mathcal{C}} xe^{yz}ds, \tag{822}$$

where $\mathcal{C}$ is $\vec{r}(t) = \langle t, 2t, -4t\rangle$ for $1 \le t \le 2$.

Solution: We observe that

$$\vec{r}\,'(t) = \langle 1, 2, -4\rangle \to |\vec{r}\,'(t)| = \sqrt{1^2 + 2^2 + (-4)^2} = \sqrt{21}, \text{ so} \tag{823}$$

$$\int_{\mathcal{C}} xe^{yz}ds = \int_1^2 te^{2t(-4t)}\sqrt{21}dt = \sqrt{21}\int_1^2 te^{-8t^2}dt \tag{824}$$

$$\overset{u=-8t^2}{=} \sqrt{21}\int_{t=1}^2 e^u(-\frac{1}{16})du = \sqrt{21}(-\frac{1}{16}e^u\Big|_{t=1}^2) \tag{825}$$

$$= -\frac{\sqrt{21}}{16}e^{-8t^2}\Big|_1^2 = \boxed{\frac{\sqrt{21}}{16}(e^{-8} - e^{-32})}. \tag{826}$$

Problem 7.9: Compute

(827) $$\int_{\mathcal{C}} \frac{xy}{z} ds,$$

where $\mathcal{C}$ is the line segment from $(1, 4, 1)$ to $(3, 6, 3)$.

Solution: Firstly, we recall that

(828) $$\mathbf{r}(t) = \vec{P} + t(\vec{Q} - \vec{P}), 0 \leq t \leq 1$$

is one way in which to parameterize the line segment that starts at the point P and ends at the point Q. In this particular problem, we see that

(829) $$\mathbf{r}(t) = \langle 1, 4, 1\rangle + t(\langle 3, 6, 3\rangle - \langle 1, 4, 1\rangle) = \langle 1+2t, 4+2t, 1+2t\rangle, 0 \leq t \leq 1$$

is a parameterization for the line segment from $(1, 4, 1)$ to $(3, 6, 3)$. We note that

(830) $$\mathbf{r}'(t) = \langle 2, 2, 2\rangle \to |\mathbf{r}'(t)| = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}, \text{ so}$$

(831) $$\int_C \frac{xy}{z} ds = \int_0^1 \frac{x(t)y(t)}{z(t)} |\mathbf{r}'(t)| dt = \int_0^1 \frac{(1+2t)(4+2t)}{1+2t} 2\sqrt{3} dt$$

(832) $$= 4\sqrt{3} \int_0^1 (2+t) dt = 4\sqrt{3} \left(2t + \frac{1}{2}t^2 \Big|_{t=0}^{1}\right) = \boxed{10\sqrt{3}}$$

Problem 7.13: Compute the circulation of

$$\vec{F} = \langle y - x, x \rangle$$

on the curve $\mathcal{C}$ which is given by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$ for $0 \leq t \leq 2\pi$.

Solution: We see that

(833) $$\text{Circulation} = \int_C \vec{F} \cdot \hat{T} ds = \int_C \vec{F} \cdot \vec{r}\,'(t) dt$$

(834) $$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \langle -2\sin(t), 2\cos(t) \rangle dt$$

(835) $$= \int_0^{2\pi} \langle \langle \underbrace{2\sin(t)}_{y} - \underbrace{2\cos(t)}_{x}, \underbrace{2\cos(t)}_{x} \rangle \cdot \langle -2\sin(t), 2\cos(t) \rangle dt$$

(836) $$= \int_0^{2\pi} \left(-4\sin^2(t) + 4\cos(t)\sin(t) + 4\cos^2(t) \right) dt$$

(837) $$=^{15} \int_0^{2\pi} (4\cos(2t) + 2\sin(2t))\, dt$$

(838) $$= 2\sin(2t) - \cos(2t) \Big|_0^{2\pi} = \boxed{0}.$$

[15] $\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 = 1 - 2\sin^2(t)$ and $\sin(2t) = 2\sin(t)\cos(t)$.

Problem 7.14: Let a be a positive number. Consider the vector field $\vec{F} = \langle y, x\rangle$ and the curve $\mathcal{C}$ given by $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$ for $0 \le t \le 2\pi$. Compute the flux of $\vec{F}$ across $\mathcal{C}$. (Your answer should be in terms of a.)

Solution: We see that

(839)
$$\vec{r}\,'(t) = \langle -a\sin(t), a\cos(t)\rangle \to |\vec{r}\,'(t)| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2} = a$$

(840)
$$\to \hat{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \langle -\sin(t), \cos(t)\rangle$$

(841)
$$\to \hat{n}(t) = \hat{T}(t) \times \hat{k} = \langle \cos(t), \sin(t)\rangle$$

(842)
$$\text{Flux} = \int_C \vec{F}\cdot\hat{n}ds = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\hat{n}(t)|\vec{r}\,'(t)|dt$$

(843)
$$= \int_0^{2\pi} \langle \underbrace{\sin(t)}_{y}, \underbrace{\cos(t)}_{x}\rangle \cdot \langle\cos(t), \sin(t)\rangle a dt$$

(844)
$$= a\int_0^{2\pi} (\sin(t)\cos(t) + \cos(t)\sin(t))dt$$

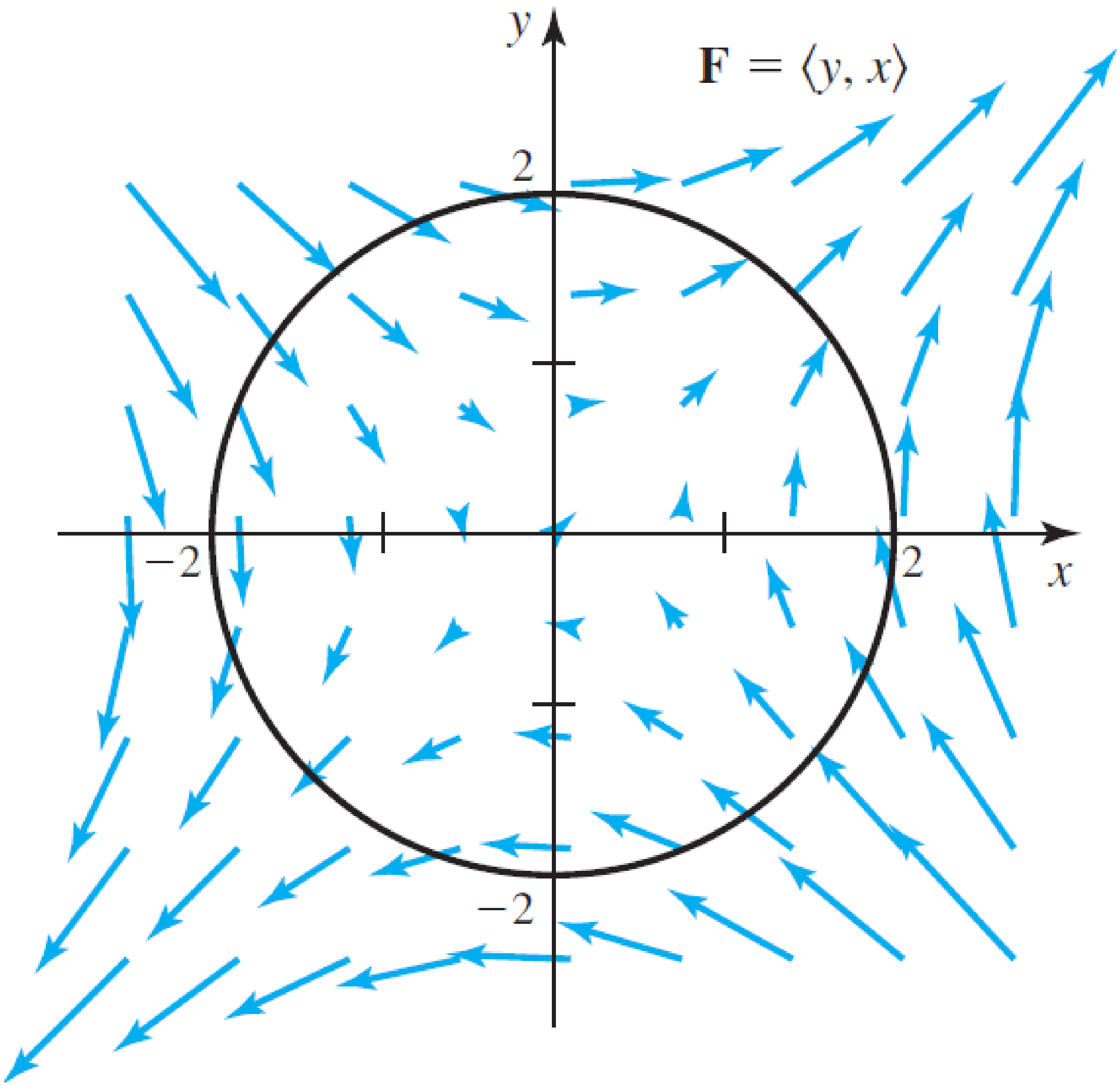
(845)
$$= a\int_0^{2\pi} \sin(2t)dt = -\frac{a}{2}\cos(2t)\Big|_0^{2\pi} = \boxed{0}.$$

Remark: We also could have used the fact that

(846)
$$\hat{n}ds = \hat{n}|\vec{r}\,'(t)|dt = |\vec{r}\,'(t)\times\hat{k}|dt = \langle a\cos(t), a\sin(t)\rangle$$

in order to avoid the calculation of $|\vec{r}\,'(t)|$ and $\hat{T}(t)$ and save a little effort.

Problem 7.15: Consider the flow field $\mathbf{F} = \langle y, x \rangle$ shown in the figure below.



(a) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $0 \le t \le \frac{\pi}{2}$.
(b) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $\frac{\pi}{2} \le t \le \pi$.
(c) Explain why the flux across the quarter circle in the third quadrant equals the flux computed in part (a).
(d) Explain why the flux across the quarter circle in the fourth quadrant equals the flux computed in part (b).
(e) What is the outward flux across the full circle?

Solution to (a): We begin by calculating the unit normal vector $\hat{n}(t)$ at any point on the circle (as opposed to only on the first quadrant). We see that

$$\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle \to |\mathbf{r}'(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} = 2 \tag{847}$$

$$\to \hat{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)} = \frac{\langle -2\sin(t), 2\cos(t)\rangle}{2} = \langle -\sin(t), \cos(t)\rangle \tag{848}$$

$$\to \hat{n}(t) = \hat{T}(t) \times \hat{k} = \langle \cos(t), -(-\sin(t))\rangle = \langle \cos(t), \sin(t)\rangle. \tag{849}$$

We now are able to calculate the desired flux as

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_0^{\frac{\pi}{2}} \mathbf{F}(2\cos(t), 2\sin(t))\cdot\langle\cos(t),\sin(t)\rangle \underbrace{2dt}_{ds} \tag{850}$$

$$= \int_0^{\frac{\pi}{2}} \langle 2\sin(t), 2\cos(t)\rangle\cdot\langle 2\cos(t), 2\sin(t)\rangle dt \tag{851}$$

$$= \int_0^{\frac{\pi}{2}} (4\sin(t)\cos(t) + 4\cos(t)\sin(t))dt \tag{852}$$

$$= \int_0^{\frac{\pi}{2}} 8\sin(t)\cos(t)dt = \int_0^{\frac{\pi}{2}} 4\sin(2t)dt = -2\cos(2t)\Big|_0^{\frac{\pi}{2}} = \boxed{4}. \tag{853}$$

Solution to (b): Since we have already found $\hat{n}(t)$ in part **a**, we proceed directly to the calculation of the flux, which is also similar to the calculation that we did in part **a**.

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(2\cos(t), 2\sin(t))\cdot\langle\cos(t),\sin(t)\rangle 2dt \tag{854}$$

$$= \int_{\frac{\pi}{2}}^{\pi} 8\sin(t)\cos(t)dt = -2\cos(2t)\Big|_{\frac{\pi}{2}}^{\pi} = \boxed{-4}. \tag{855}$$

Solution to (c): The symmetry in the given picture shows us that the flux through the circle in quadrant 1 is the same as the flux through the circle in quadrant 3. To be more detailed, we can observe that the map $(x, y) \mapsto (-x, -y)$ will send the first quadrant to the third quadrant, and the map $\theta \mapsto \theta + \pi$ (which is basically the same map) also maps the first quadrant to the third quadrant. It follows that for each $0 \le t \le \frac{\pi}{2}$ (remembering that t is essentially the angle θ in this situation) we have

(856) $$\mathbf{F}(\mathbf{r}(t+\pi)) = \mathbf{F}(-\mathbf{r}(t)) = -\mathbf{F}(\mathbf{r}(t)), \text{ and}$$

(857) $$\hat{n}(t+\pi) = -\hat{n}(t), \text{ so}$$

(858) $$\text{Flux(Third Quadrant)} = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F} \cdot \hat{n} ds = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}(\mathbf{r}(t)) \cdot \hat{n} ds =$$

(859) $$= \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t+\pi)) \cdot \hat{n}(t+\pi) ds = \int_0^{\frac{\pi}{2}} (-\mathbf{F}(\mathbf{r}(t))) \cdot (-\hat{n}(t)) ds$$

(860) $$= \int_0^{\frac{\pi}{2}} \mathbf{F} \cdot \hat{n}(t) ds = \text{Flux(First Quadrant)}.$$

Solution to (d): Once again the symmetry in the given picture shows us that the flux through the circle in quadrant 2 is the same as the flux through the circle in quadrant 4. To be more detailed, we perform calculations similar to those of part (c) to see that

(861) $$\text{Flux(Fourth Quadrant)} = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F} \cdot \hat{n} ds = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \hat{n} ds$$

(862) $$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(\mathbf{r}(t+\pi)) \cdot \hat{n}(t+\pi) ds = \int_{\frac{\pi}{2}}^{\pi} (-\mathbf{F}(\mathbf{r}(t))) \cdot (-\hat{n}(t)) ds$$

(863) $$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F} \cdot \hat{n}(t) ds = \text{Flux(Second Quadrant)}.$$

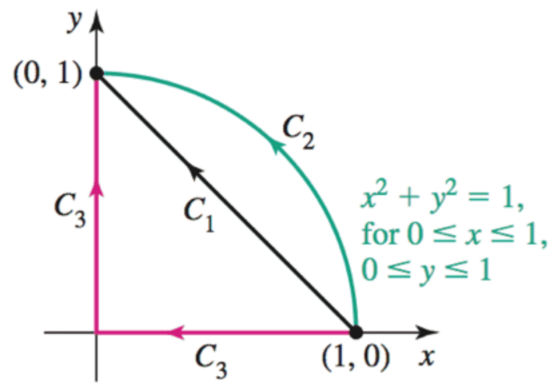
Solution to (e): We could calculate the total flux directly, but to make use of parts **a**−**d**, we observe that

$$\text{Total Flux} = \int_0^{2\pi} \mathbf{F} \cdot \hat{n} ds \tag{864}$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} \mathbf{F} \cdot \hat{n} ds}_{\text{Q1 Flux}} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \mathbf{F} \cdot \hat{n} ds}_{\text{Q2 Flux}} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} \mathbf{F} \cdot \hat{n} ds}_{\text{Q3 Flux}} + \underbrace{\int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F} \cdot \hat{n} ds}_{\text{Q4 Flux}} \tag{865}$$

$$= 4 + (-4) + 4 + (-4) = \boxed{0}. \tag{866}$$

Problem 7.16: Consider the rotation field $\vec{F} = \langle -y, x \rangle$, and the three paths shown in the figure.



(1) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_1$.
(2) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_2$.
(3) Compute the work required in the presence of the force field $\vec{F}$ to move an object on the curve $\mathcal{C}_3$.
(4) Does it appear that the line integral $\int_{\mathcal{C}} \vec{F} \cdot \vec{T} ds$ is independent of the path, where $\mathcal{C}$ is any path from $(1,0)$ to $(0,1)$?

Solution to part 1: We begin by finding a parameterization for our curve $\mathcal{C}_1$. Since $\mathcal{C}_1$ is a line segment, we can use the standard parameterization

(867) $$\vec{r}(t) = \underbrace{\langle 1,0\rangle}_{\text{Start Point}} + t(\underbrace{\langle 0,1\rangle}_{\text{End Point}} - \underbrace{\langle 1,0\rangle}_{\text{Start Point}}) = \langle 1-t, t\rangle, 0 \le t \le 1.$$

We now see that

(868) $$\text{Work} = \int_{\mathcal{C}_1} \vec{F} \cdot \hat{T} ds = \int_{\mathcal{C}_1} \vec{F} \cdot \vec{r}\,' dt$$

(869) $$= \int_0^1 \langle \underbrace{-t}_{-y}, \underbrace{1-t}_{x} \rangle \cdot \langle -1, 1 \rangle dt = \int_0^1 (t + 1 - t) dt = \int_0^1 1 dt = \boxed{1}.$$

Solution to part 2: We begin by finding a parameterization for our curve $\mathcal{C}_2$. Since $\mathcal{C}_2$ is a portion of a circle with the counter clockwise orientation, we can use the standard parameterization

$$\vec{r}(t) = \langle \cos(t), \sin(t) \rangle, 0 \leq t \leq \frac{\pi}{2}. \tag{870}$$

We now see that

$$\text{Work} = \int_{\mathcal{C}_1} \vec{F} \cdot \hat{T} ds = \int_{\mathcal{C}_1} \vec{F} \cdot \vec{r}\,' dt \tag{871}$$

$$= \int_0^{\frac{\pi}{2}} \vec{F}(\vec{r}(t)) \cdot \langle -\sin(t), \cos(t) \rangle dt \tag{872}$$

$$= \int_0^{\frac{\pi}{2}} \langle \underbrace{-\sin(t)}_{-y}, \underbrace{\cos(t)}_{x} \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt \tag{873}$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2(t) + \cos^2(t)) dt = \int_0^{\frac{\pi}{2}} 1 dt = \boxed{\frac{\pi}{2}}. \tag{874}$$

Solution to part 3: Since $\mathcal{C}_3$ is composed of 2 separate smooth curves, we will decompose $\mathcal{C}_3$ into its pieces and handle them separately. Let $\mathcal{C}_4$ denote the line segment from $(1, 0)$ to $(0, 0)$ and let $\mathcal{C}_5$ denote the line segment from $(0, 0)$ to $(0, 1)$. We see that

$$\text{Work} = \int_{\mathcal{C}_3} \vec{F} \cdot \hat{T} ds = \int_{\mathcal{C}_4} \vec{F} \cdot \hat{T} ds + \int_{\mathcal{C}_5} \vec{F} \cdot \hat{T} ds. \tag{875}$$

In light of this observation, we will first calculate the work it takes to move an object across $\mathcal{C}_4$ in the presence of the force field $\vec{F}$, then we will calculate the work it takes to move an object across $\mathcal{C}_5$ in the presence of the force field $\vec{F}$. Since $\mathcal{C}_4$ is a line segment, we see as in part 1 that

$$\vec{r}(t) = \langle -t, 0 \rangle, 0 \leq t \leq 1 \tag{876}$$

is a paramterization for $\mathcal{C}_4$. We now see that

$$\int_{\mathcal{C}_4} \vec{F} \cdot \hat{T} ds = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt \tag{877}$$

$$= \int_0^1 \langle \underbrace{0}_{-y}, \underbrace{-t}_{x} \rangle \cdot \langle -1, 0 \rangle dt = \int_0^1 0 dt = 0. \tag{878}$$

Since $\mathcal{C}_5$ is also a line segment, we see as before that

$$\vec{r}(t) = \langle 0, t \rangle, 0 \leq t \leq 1 \tag{879}$$

is a paramterization for $\mathcal{C}_5$. We now see that

$$\int_{\mathcal{C}_5} \vec{F} \cdot \hat{T} ds = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt \tag{880}$$

$$= \int_0^1 \langle \underbrace{-t}_{-y}, \underbrace{0}_{x} \rangle \cdot \langle 0, 1 \rangle dt = \int_0^1 0 dt = 0. \tag{881}$$

It follows that the total work is $0 + 0 = \boxed{0}$.

Solution to part 4: Since the answers to parts 1, 2, and 3 are all different, we see that the work required to move an object from $(1, 0)$ to $(0, 1)$ in the presence of the force field $\vec{F}$ depends on that path that use. In particular, the vector field $\vec{F}$ is not conservative.

Problem 7.17: Find the work required to move an object along the line segment from $(1,1,1)$ to $(8,4,2)$ through the force field $\vec{F}$ given by

$$\vec{F}=\frac{\langle x,y,z\rangle}{x^2+y^2+z^2}.$$

Solution 1: We note that for $\varphi=\frac{1}{2}\ln(x^2+y^2+z^2)$ we have $\nabla\varphi=\vec{F}$, so we may use the Fundamental Theorem for Line Integrals as follows:

(882) $$\text{Work}=\int_C\vec{F}\cdot d\vec{r}=\int_C\nabla\varphi\cdot d\vec{r}=\varphi\left((8,4,2)\right)-\varphi\left((1,1,1)\right)$$

(883) $$=\frac{1}{2}\ln(8^2+4^2+2^2)-\frac{1}{2}\ln(1^2+1^2+1^2)=\frac{1}{2}\ln(84)-\frac{1}{2}\ln(3)=\boxed{\frac{1}{2}\ln(28)}.$$

Solution 2: Firstly, we recall that one method of parameterizing the line segment that starts at $\vec{p}$ and ends at $\vec{q}$ is to use the parameterization

(884) $$\vec{r}(t)=(1-t)\vec{p}+t\vec{q}=\vec{p}+t(\vec{q}-\vec{p}),\quad 0\le t\le 1.$$

It follows that

(885) $$\vec{r}(t)=\langle 1,1,1\rangle+t\left(\langle 8,4,2\rangle-\langle 1,1,1\rangle\right)=\langle 1+7t,1+3t,1+t\rangle,\quad 0\le t\le 1,$$

is a parameterization of the line segment from $(1,1,1)$ to $(8,4,2)$. We now see that

(886) $$\text{Work}=\int_C\vec{F}\cdot d\vec{r}=\int_0^1\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt$$

. . . .

(887) $$=\int_0^1\underbrace{\frac{\langle 1+7t,1+3t,1+t\rangle}{(1+7t)^2+(1+3t)^2+(1+t)^2}}_{\vec{F}(\vec{r}(t))}\cdot\underbrace{\langle 7,3,1\rangle dt}_{d\vec{r}}$$

. . . .

$$= \int_0^1 \frac{(1+7t)\cdot 7 + (1+3t)\cdot 3 + (1+t)\cdot 1}{1+14t+49t^2+1+6t+9t^2+1+2t+t^2}dt \tag{888}$$

. . . .

$$= \int_0^1 \frac{11+59t}{3+22t+59t^2}dt = \int_0^1 \frac{t+\frac{11}{59}}{t^2+\frac{22}{59}t+\frac{3}{59}}dt = \int_0^1 \frac{t+\frac{11}{59}}{(t+\frac{11}{59})^2+\frac{56}{3481}}dt \tag{889}$$

. . . .

$$= \frac{1}{2}\ln\left((t+\frac{11}{59})^2+\frac{56}{3481}\right)\Big|_0^1 = \boxed{\frac{1}{2}\ln(28)}. \tag{890}$$

Problem 7.18: Given the force field $\mathbf{F} = \langle x, y, z\rangle$, find the work required to move an object around the tilted ellipse that is parameterized by $\mathbf{r}(t) = \langle 4\cos(t), 4\sin(t), 4\cos(t)\rangle$, $0 \le t \le 2\pi$.

Solution: We see that

$$\text{Work} = \int_C \mathbf{F}\cdot\hat{T}ds = \int_0^{2\pi} \mathbf{F}(4\cos(t), 4\sin(t), 4\cos(t))\cdot\vec{r}\,'(t)dt \tag{891}$$

$$= \int_0^{2\pi} \underbrace{\langle 4\cos(t), 4\sin(t), 4\cos(t)\rangle}_{\mathbf{F}(\vec{r}(t))=\vec{r}(t)\text{ coincidentally}}\cdot\langle -4\sin(t), 4\cos(t), -4\sin(t)\rangle dt \tag{892}$$

$$= \int_0^{2\pi} (-16\cos(t)\sin(t) + 16\sin(t)\cos(t) - 16\cos(t)\sin(t))dt \tag{893}$$

$$= \int_0^{2\pi} -16\sin(t)\cos(t)dt = \int_0^{2\pi} -8\sin(2t)dt \tag{894}$$

$$= 4\cos(2t)\Big|_0^{2\pi} = \boxed{0}. \tag{895}$$

Problem 7.19: Evaluate the line integral $\int_C \nabla\phi \cdot d\vec{r}$ for $\phi(x,y) = xy$ and $C : \vec{r}(t) = \langle \cos(t), \sin(t)\rangle$, for $0 \leq t \leq \pi$ in two ways.

(a) Use a parametric description of C and evaluate the integral directly;
(b) Use the Fundamental Theorem for line integrals.

Solution to (a): We see that $\nabla\phi(x,y) = \langle y, x\rangle$, so

(896) $$\int_C \nabla\phi \cdot d\vec{r} = \int_0^\pi \nabla\phi(\vec{r}(t)) \cdot \vec{r}\,'(t)dt$$

(897) $$\int_0^\pi \nabla\phi(\cos(t), \sin(t)) \cdot \langle -\sin(t), \cos(t)\rangle dt$$

(898) $$= \int_0^\pi \langle \sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle dt$$

(899) $$= \int_0^\pi (-\sin^2(t) + \cos^2(t))dt = \int_0^\pi \cos(2t)dt = \frac{1}{2}\sin(2t)\Big|_0^\pi = \boxed{0}.$$

Solution to (b): We see that

(900) $$\int_C \nabla\phi \cdot d\vec{r} = \int_0^\pi \nabla\phi(\vec{r}(t)) \cdot \vec{r}\,'(t)dt$$

(901) $$= \phi(\vec{r}(\pi)) - \phi(\vec{r}(0)) = \phi(-1,0) - \phi(1,0) = 0 - 0 = \boxed{0}.$$

Problem 7.20: Let $\vec{F}$ be the vector field

$$\vec{F} = \langle f(x,y), g(x,y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle.$$

It is a rotational vector field with the graph below

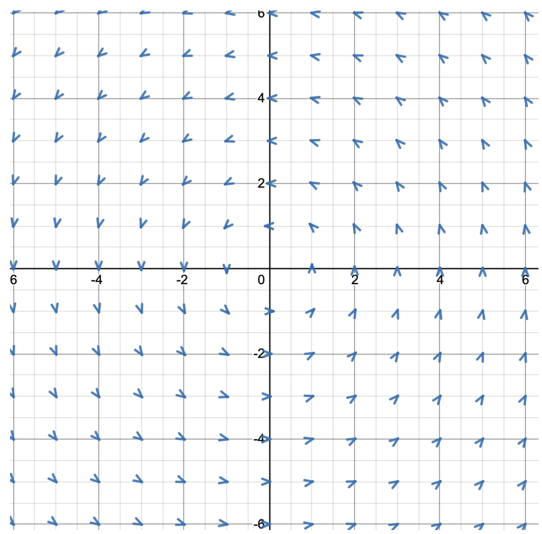


FIGURE 51. vector field $\vec{F}$

(a) Find the domain R of $\vec{F}$.
(b) Is the domain R connected? Is R simply connected?
(c) Show that $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$.
(d) Let C_a be the parameterized circle $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$, $0 \le t < 2\pi$ of radius $a > 0$. Show that the integral

$$\int_{C_a} \vec{F}\cdot d\vec{r} = 2\pi.$$

(e) Is $\vec{F}$ a conservative vector field on R? If so, please explain. Otherwise, please explain why it doesn't contradict the result in (3).
(f) Let R_1 be the region $R_1 = \{1 \le x \le 2, 1 \le y \le 2\}$. Is $\vec{F}$ a conservative vector field on R_1? Please explain.

Solution to part (a): The domain of $\vec{F}$ consists of all points in $\mathbb{R}^2$ at which $\vec{F}$ is defined. We see that the only time that $\vec{F}$ is undefined is when $x^2+y^2 = 0$, as we cannot divide by 0, but $x^2+y^2 = 0$ is only satisfed by $(x,y) = (0,0)$, so the domain of $\vec{F}$ is $\boxed{R = \mathbb{R}^2 \setminus \{(0,0)\}}$.

Solution to part (b): The domain of R is connected since it is actually path connected[16]. Given any 2 points in R, there exists a path consisting of either 1 or 2 straight line segments that connects the 2 points.

Solution to part (c): We see that

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = -\frac{x}{(x^2+y^2)^2}\cdot 2x + \frac{1}{x^2+y^2} \tag{902}$$

$$= -\frac{2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \text{ and} \tag{903}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = -\frac{-y}{(x^2+y^2)^2}\cdot 2y + \frac{-1}{x^2+y^2} \tag{904}$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial f}{\partial x}. \tag{905}$$

Solution to part (d): We see that

$$\int_{C_a} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt = \int_0^{2\pi} \vec{F}(a\cos(t), a\sin(t))\cdot\langle -a\sin(t), a\cos(t)\rangle dt \tag{906}$$

$$= \int_0^{2\pi} \left\langle \frac{-a\sin(t)}{(a\cos(t))^2+(a\sin(t))^2}, \frac{a\cos(t)}{(a\cos(t))^2+(a\sin(t))^2}\right\rangle\cdot\langle -a\sin(t), a\cos(t)\rangle dt \tag{907}$$

$$= \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle\cdot\langle -\sin(t), \cos(t)\rangle dt = \int_0^{2\pi} (\sin^2(t)+\cos^2(t))dt \tag{908}$$

$$= \int_0^{2\pi} 1dt = \boxed{2\pi}. \tag{909}$$

[16] Being path connected is a stronger condition than just being connected, but you probably won't study the difference between the 2 notions unless you go on to take a course in real analysis or topology.

Solution to part (e): Since C_a is a closed loop inside of R for any radius $a > 0$, and $\int_{C_a} \vec{F} \cdot d\vec{r} = 2\pi \neq 0$, we see (Theorem 15.6 on page 1118) that $\vec{F}$ is not a conservative vector field on R. Our calcuations in part (c) cannot be used alongside Theorem 15.3 (on page 1113) to conclude that the vector field $\vec{F}$ is conservative, because Theorem 15.3 requires that the vector field $\vec{F}$ be defined on a simply connected region D.

Solution to part (f): Since the region R_1 is simply connected (it has no holes) and $\vec{F}$ is continuous on R_1, we may use the result of part (c) to conclude that the vector field $\vec{F}$ is conservative on the region R_1.

Problem 7.10: Let $f(x, y) = x$ and consider the segment of the parabola $y = x^2$ joining $O(0, 0)$ and $P(1, 1)$.

(1) Let $\mathcal{C}_1$ be the segment from O to P. Find a parametrization of $\mathcal{C}_1$, then evaluate $\int_{\mathcal{C}_1} f ds$.
(2) Let $\mathcal{C}_2$ be the segment from P to O. Find a parametrization of $\mathcal{C}_2$, then evaluate $\int_{\mathcal{C}_2} f ds$.
(3) Compare the results of (1) and (2).

Solution to (1): We see that $\mathbf{r}(t) = \langle t, t^2\rangle$, $0 \leq t \leq 1$ is a parameterization of the segement of the parabola $y = x^2$ from $O(0, 0)$ to $P(1, 1)$. We see that

(910) $$\mathbf{r}'(t) = \langle 1, 2t\rangle \to |\mathbf{r}'(t)| = \sqrt{1^2 + 4t^2} = \sqrt{1 + 4t^2}, \text{ so}$$

(911) $$\int_C f ds = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt - \int_0^1 f(t, t^2)\sqrt{1 + 4t^2}dt = \int_0^1 t\sqrt{1 + 4t^2}dt$$

(912) $$\overset{u=1+4t^2}{=} \int_{t=0}^1 \sqrt{u}\frac{1}{8}du = \frac{1}{12}u^{\frac{3}{2}}\Big|_{t=0}^1 = \frac{1}{12}(1 + 4t^2)^{\frac{3}{2}}\Big|_0^1 = \boxed{\frac{5^{\frac{3}{2}} - 1}{12}}.$$

Solution to (2): We see that if we replace t by $1 - t$, then the parameterization starts at $P(1, 1)$ and ends at $O(0, 0)$, so $\mathbf{r}(t) = \langle 1 - t, (1 - t)^2\rangle$ is a parameterization of the segment of the parabola $y = x^2$ from $P(1, 1)$ to $O(0, 0)$. We see that

(913) $$\mathbf{r}'(t) = \langle -1, -2(1 - t)\rangle$$

(914) $$\to |\mathbf{r}'(t)| = \sqrt{(-1)^2 + (-2(1 - t))^2} = \sqrt{1 + 4(1 - t)^2}, \text{ so}$$

(915) $$\int_C f ds = \int_0^1 f(\mathbf{r}(t))|\mathbf{r}'(t)|dt = \int_0^1 f(1-t, (1-t)^2)\sqrt{1 + 4(1 - t)^2}dt$$

$$= \int_0^1 (1-t)\sqrt{1+4(1-t)^2}dt \overset{u=1+4(1-t)^2}{=} \int_{t=0}^1 \sqrt{u}(-\frac{1}{8})du = -\frac{1}{12}u^{\frac{3}{2}}\Big|_{t=0}^1 \tag{916}$$

$$= -\frac{1}{12}(1+4(1-t)^2)^{\frac{3}{2}}\Big|_{t=0}^1 = \boxed{\frac{5^{\frac{3}{2}}-1}{12}}. \tag{917}$$

Solution to (3): We see that the answers to parts (1) and (2) are the same. This makes sense because we are integrating the same function values over the same region. This should be compared to the fact that $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Note that the reason that we do not obtain a negative sign in part (2) is because $ds = |\mathbf{r}'(t)|dt$, and the absolute values absorb the negative sign. To see this fact in action back in the one dimensional case, we note that $\mathbf{r_1}(t) = \langle (b-a)t + a\rangle, 0 \le t \le 1$ is a parameterization of the line segment from $x = a$ to $x = b$, and $\mathbf{r_2}(t) = \langle (a-b)t + b\rangle, 0 \le t \le 1$ is a parameterization of the line segment from $x = b$ t $x = a$. We see that $\mathbf{r_1}'(t) = \langle b-a\rangle = -\langle a-b\rangle = \mathbf{r_2}'(t)$, so $ds = |\mathbf{r_1}'(t)| = |\mathbf{r_2}'(t)| = b-a$ is the same for both parameterizations.

Problem 7.11: Find the average value of the function $f(x,y) = x + 2y$ on the line segment from $(1,1)$ to $(2,5)$.

Solution: Firstly, we recall that the average value of a function f over a curve C is given by

$$\text{Av}(f) = \frac{\int_C f ds}{\text{Arclength}(c)} = \frac{\int_C f ds}{\int_C 1 ds}. \tag{918}$$

In order to calculate the relevant line integrals, we begin by parameterizing the line segment from $(1,1)$ to $(2,5)$. We see that

$$\vec{r}(t) = \langle 1,1\rangle + t(\langle 2,5\rangle - \langle 1,1\rangle) = \langle 1+t, 1+4t\rangle, 0 \leq t \leq 1, \tag{919}$$

is a parameterization of the line segment from $(1,1)$ to $(2,5)$. It follows that

$$\vec{r}\,'(t) = \langle 1,4\rangle \to |\vec{r}\,'(t)| - \sqrt{1^2+4^2} = \sqrt{17}. \tag{920}$$

We are now able to calculate both of the relevant line integrals.

$$\int_C f ds = \int_0^1 f(\vec{r}(t))|\vec{r}\,'(t)|dt = \int_0^1 f(1+t, 1+4t)\sqrt{17}dt \tag{921}$$

$$= \int_0^1 \Big((1+t) + 2(1+4t)\Big)\sqrt{17}dt = \sqrt{17}\int_0^1 (3+9t)dt \tag{922}$$

$$= \sqrt{17}(3t + \frac{9}{2}t^2\Big|_0^1) = \frac{15\sqrt{17}}{2}. \tag{923}$$

$$\int_C 1ds = \int_0^1 1 \cdot |\vec{r}\,'(t)|dt = \int_0^1 \sqrt{17}dt = \sqrt{17} \tag{924}$$

$$\to \text{Av}(f) = \frac{\int_C f ds}{\int_C 1ds} = \frac{\frac{15\sqrt{17}}{2}}{\sqrt{17}} = \boxed{\frac{15}{2}}. \tag{925}$$

Problem 7.12: Find the average value of the function $f(x, y, z) = x$ over the curve $\mathcal{C}$ that is paramterized by

$$\vec{r}(t) = \langle 20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2}\rangle, 0 \leq t \leq 4\pi. \tag{926}$$

Solution: We begin by evaluating $\int_{\mathcal{C}} f ds$ and finding the arclength of $\mathcal{C}$. Since $ds = |\vec{r}\,'(t)|dt$, we first observe that

$$\vec{r}\,'(t) = \langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle \tag{927}$$

$$\to |\vec{r}\,'(t)| = \sqrt{(5\cos(\frac{t}{4}))^2 + (5\sin(\frac{t}{4}))^2 + (\frac{1}{2})^2} \tag{928}$$

$$= \sqrt{25\cos^2(\frac{t}{4}) + 25\sin^2(\frac{t}{4}) + \frac{1}{4}} = \sqrt{25 + \frac{1}{4}} = \frac{1}{2}\sqrt{101}. \tag{929}$$

We now see that

$$\int_{\mathcal{C}} f ds = \int_0^{4\pi} f(\vec{r}(t)) \cdot |\vec{r}\,'(t)|dt \tag{930}$$

$$= \int_0^{4\pi} f(20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2})\frac{1}{2}\sqrt{101}dt = \int_0^{4\pi} \underbrace{20\sin(\frac{t}{4})}_{f(x,y,z)=x} \cdot \frac{1}{2}\sqrt{101}dt \tag{931}$$

$$= 10\sqrt{101}\int_0^{4\pi} \sin(\frac{t}{4})dt = 10\sqrt{101}\Big(-4\cos(\frac{t}{4})\Big|_0^{4\pi}\Big) = 80\sqrt{101}, \text{ and} \tag{932}$$

$$\text{Arclength of } \mathcal{C} = \int_{\mathcal{C}} 1ds = \int_0^{4\pi} |\vec{r}\,'(t)|dt = \int_0^{4\pi} \frac{1}{2}\sqrt{101}dt = 2\sqrt{101}\pi. \tag{933}$$

. . . .

Putting everything together, we see that

$$\text{Average value of } f \text{ over } \mathcal{C} = \frac{\int_{\mathcal{C}} f ds}{\text{Arclength of } \mathcal{C}} = \frac{80\sqrt{101}}{2\sqrt{101}\pi} = \boxed{\frac{40}{\pi}}. \tag{934}$$

Problem 9.4: Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -3 & 1 & -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{935}$$

a) Determine conditions on b_1, b_2, and b_3 that are necessary and sufficient for the system of equations $A\vec{x} = \vec{b}$ to be consistent.

b) For each of the following choices of $\vec{b}$, either show that the system $A\vec{x} = \vec{b}$ is inconsistent or exhibit the solution.

$$\textbf{i) } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \textbf{ii) } \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \quad \textbf{iii) } \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \quad \textbf{iv) } \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution to a: We begin by representing the equation $A\vec{x} = \vec{b}$ as an augmented matrix that we will proceed to row reduce into reduced echelon form.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & b_1 \\ 2 & -1 & 1 & b_2 \\ -3 & 1 & -3 & b_3 \end{array}\right] \overset{\substack{R_2 - 2R_1 \\ R_3 + 3R_1}}{\to} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & b_1 & & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & -2 & -6 & 3b_1 & & +b_3 \end{array}\right] \tag{936}$$

$$\overset{R_3+2R_2}{\to} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & b_1 & & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & 0 & 0 & -b_1 & +2b_2 & +b_3 \end{array}\right] \tag{937}$$

At this point you can already deduce when the system is consistent.

$$\overset{R_1+R_2}{\to} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -b_1 & +b_2 & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & 0 & 0 & -b_1 & +2b_2 & +b_3 \end{array}\right] \tag{938}$$

From the third row of the augmented matrix in equation (938), we see that

$$-b_1 + 2b_2 + b_3 = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0, \tag{939}$$

and that the system of equations $A\vec{x} = \vec{b}$ is consistent if and only if equation (939) is true. Furthermore, in the event that equation (939) is true, we see that equations represented in equation (938) are

(940)
$$\begin{array}{ccccccccc} x_1 & & & + & 2x_3 & = & -b_1 & + & b_2 \\ & & x_2 & + & x_3 & = & -2b_1 & + & b_2 \end{array}$$

(941)
$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free.}$$

Solution to b: In part **a** we obtained a formula for $\vec{x}$ in terms of $\vec{b}$, so we will now apply that formula to each of the vectors.

$$\textbf{i: } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 2 \neq 0 \rightarrow \boxed{\text{The system is inconsistent}}.$$

$$\textbf{ii: } \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 0$$

(942)
$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free}$$

(943)
$$\rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3 \\ -x_3 - 8 \\ x_3 \end{bmatrix}, x_3 \text{ is free}}.$$

$$\textbf{iii: } \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 0$$

(944)
$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free}$$

(945)
$$\rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 - 4 \\ -x_3 - 11 \\ x_3 \end{bmatrix}, x_3 \text{ is free}}.$$

iv: $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \to -b_1 + 2b_2 + b_3 = 4 \neq 0 \to \boxed{\text{ The system is inconsistent}}.$

Problem 9.5: Find the inverse of

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix} \tag{946}$$

Solution: We reduce the 3 by 6 matrix $[A|I_3]$ until the left half is in reduced echelon form, which will be I_3 since A is invertible.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \tag{947}$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \tag{948}$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right) \tag{949}$$

$$\xrightarrow{R_2 + \frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right) \xrightarrow{R_1 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 4 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right). \tag{950}$$

To check our work, we note that

$$AA^{-1} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & -1 & 4 \\ -5 & -2 & 5 \\ -2 & -1 & 2 \end{pmatrix} \tag{951}$$

$$= \begin{pmatrix} 1\cdot(-3)+(-2)\cdot(-5)+3\cdot(-2) & 1\cdot(-1)+(-2)\cdot(-2)+3\cdot(-1) & 1\cdot 4+(-2)\cdot 5+3\cdot 2 \\ 0\cdot(-3)+2\cdot(-5)+(-5)\cdot(-2) & 0\cdot(-1)+2\cdot(-2)+(-5)\cdot(-1) & 0\cdot 4+2\cdot 5+(-5)\cdot 2 \\ 1\cdot(-3)+(-1)\cdot(-5)+1\cdot(-2) & 1\cdot(-1)+(-1)\cdot(-2)+1\cdot(-1) & 1\cdot 4+(-1)\cdot 5+1\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{952}$$

Remark: We only have to check that $A^{-1}A = I_3$ **OR** $AA^{-1} = I_3$. We do not have to check both.

Problem 9.6: Consider the matrices A, D ane E given by

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } E = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}. \tag{953}$$

Find matrices B and C for which $AB = D$ and $CA = E$.

Solution: We see that

$$A^{-1}D = A^{-1}(AB) = (A^{-1}A)B = I_2B = B, \text{so} \tag{954}$$

$$B = A^{-1}D = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \tag{955}$$

$$= \begin{bmatrix} 3 \cdot (-1) + 1 \cdot 1 & 3 \cdot 2 + 1 \cdot 0 & 3 \cdot 3 + 1 \cdot 2 \\ 0 \cdot (-1) + 2 \cdot 1 & 0 \cdot 2 + 2 \cdot 0 & 0 \cdot 3 + 2 \cdot 2 \end{bmatrix} \tag{956}$$

$$= \boxed{\begin{bmatrix} -2 & 6 & 11 \\ 2 & 0 & 4 \end{bmatrix}}. \tag{957}$$

Similarly, we see that

$$EA^{-1} = (CA)A^{-1} = C(AA^{-1}) = CI_2 = C, \text{ so} \tag{958}$$

$$C = EA^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + (-1) \cdot 0 & 2 \cdot 1 + (-1) \cdot 2 \\ 1 \cdot 3 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 2 \\ 0 \cdot 3 + 3 \cdot 0 & 0 \cdot 1 + 3 \cdot 2 \end{bmatrix} \tag{959}$$

$$= \boxed{\begin{bmatrix} 6 & 0 \\ 3 & 3 \\ 0 & 6 \end{bmatrix}}. \tag{960}$$

Problem 9.7: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$, and let I_n denote the $(n \times n)$ identity matrix. Let $A = I_n + \vec{u}\vec{v}^T$, and suppose that $\vec{v}^T\vec{u} \neq -1$. Show that

$$A^{-1} = I_n - a\vec{u}\vec{v}^T, \text{ where } a = \frac{1}{1 + \vec{v}^T\vec{u}}. \tag{961}$$

This result is known as the Sherman-Woodberry formula.

Example: If $n = 3$,

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ then} \tag{962}$$

$$\vec{v}^T\vec{u} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (-1) \cdot 1 + 1 \cdot 2 + 0 \cdot 3 = 1 \neq -1 \text{ and} \tag{963}$$

$$A = I_3 + \vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \tag{964}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \cdot (-1) & 1 \cdot 1 & 1 \cdot 0 \\ 2 \cdot (-1) & 2 \cdot 1 & 2 \cdot 0 \\ 3 \cdot (-1) & 3 \cdot 1 & 3 \cdot 0 \end{pmatrix} \tag{965}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix}. \tag{966}$$

We also saw that

$$\vec{v}^T\vec{u} = 1 \text{ and } \vec{u}\vec{v}^T = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} \text{ so} \tag{967}$$

$$a = \frac{1}{1 + \vec{v}^T\vec{u}} = \frac{1}{1+1} = \frac{1}{2} \text{ and} \tag{968}$$

$$A^{-1} = I_3 - a\vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}. \tag{969}$$

Indeed, we see that

$$AA^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix} \tag{970}$$

$$= \begin{pmatrix} 0\cdot\frac{3}{2}+1\cdot 1+0\cdot\frac{3}{2} & 0\cdot(-\frac{1}{2})+1\cdot 0+0\cdot(-\frac{3}{2}) & 0\cdot 0+1\cdot 0+0\cdot 1 \\ (-2)\cdot\frac{3}{2}+3\cdot 1+0\cdot\frac{3}{2} & (-2)\cdot(-\frac{1}{2})+3\cdot 0+0\cdot(-\frac{3}{2}) & (-2)\cdot 0+3\cdot 0+0\cdot 1 \\ (-3)\cdot\frac{3}{2}+3\cdot 1+1\cdot\frac{3}{2} & (-3)\cdot(-\frac{1}{2})+3\cdot 0+1\cdot(-\frac{3}{2}) & (-3)\cdot 0+3\cdot 0+1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{971}$$

Solution: The inverse of a matrix (if it exists) is unique, so for

$$B = I_n - a\vec{u}\vec{v}^T, \tag{972}$$

we only have to verify that

$$AB = I_n \text{ or } BA = I_n, \tag{973}$$

as we will then know that A is invertible, and that $A^{-1} = B$. Since $\vec{v}^T\vec{u}$ is a scalar, let us simplify our notation by letting

$$b = \vec{v}^T\vec{u} \text{ so that } a = \frac{1}{1+b}. \tag{974}$$

We see that

$$AB = (I_n + \vec{u}\vec{v}^T)(I_n - a\vec{u}\vec{v}^T) = I_nI_n + \vec{u}\vec{v}^TI_n + I_n(-a\vec{u}\vec{v}^T) + \vec{u}\vec{v}^T(-a\vec{u}\vec{v}^T) \tag{975}$$

$$= I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a(\vec{u}\vec{v}^T)(\vec{u}\vec{v}^T) = I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(\vec{v}^T\vec{u})\vec{v}^T \tag{976}$$

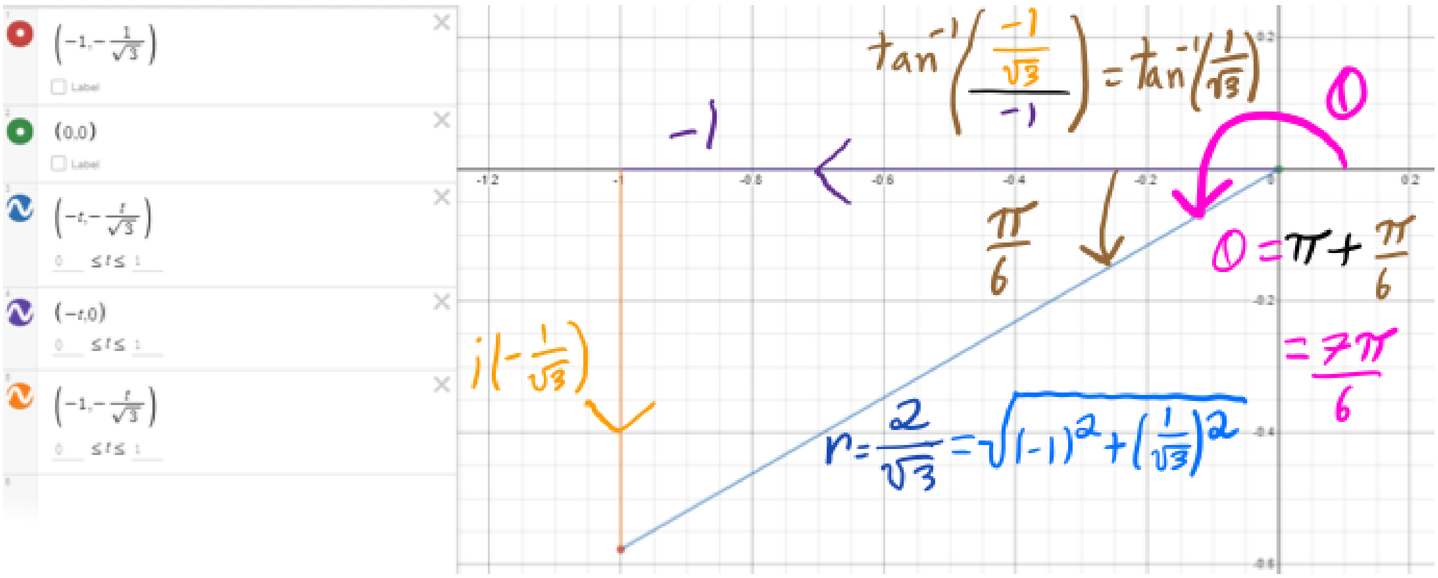
$$\overset{\text{By (974)}}{=} I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(b)\vec{v}^T = I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - ab\vec{u}\vec{v}^T \tag{977}$$

$$= I_n + (1 - a - ab)\vec{u}\vec{v}^T \overset{\text{By (974)}}{=} I_n + (1 - \frac{1}{1+b} - \frac{b}{1+b})\vec{u}\vec{v}^T \tag{978}$$

$$= I_n + 0 \cdot \vec{u}\vec{v}^T = I_n. \tag{979}$$

Problem 10.1: Plot $z = -1 - \frac{1}{\sqrt{3}}i$ in the complex plane. Then find the modulus and argument of z, and express z in the form $z = re^{i\theta}$.

Solution: Based on the diagram below, we see that $-1 - \frac{1}{\sqrt{3}}i = \boxed{\frac{2}{\sqrt{3}}e^{i\frac{7\pi}{6}}}$.



Problem 10.2: For $z = -1 + 4i$ and $w = 5 + 2i$ evaluate $\left|\frac{z}{2w}\right|$.

Solution 1: We see that

$$\frac{z}{2w} = \frac{-1+4i}{2(5+2i)} = \frac{-1+4i}{10+4i} = \frac{-1+4i}{10+4i} \cdot \underbrace{\frac{10-4i}{10-4i}}_{1} = \frac{(-1+4i)(10-4i)}{(10+4i)(10-4i)} \tag{980}$$

$$= \frac{-10+40i+4i-16i^2}{100+40i-40i-16i^2} \overset{i^2=-1}{=} \frac{-10+40i+4i+16}{100+40i-40i+16} \tag{981}$$

$$= \frac{6+44i}{116} = \frac{3+22i}{58} \tag{982}$$

$$\rightarrow \left|\frac{z}{2w}\right| = \left|\frac{3+22i}{58}\right| = \frac{1}{58}|3+22i| = \frac{1}{58}\sqrt{3^2+22^2} = \boxed{\frac{\sqrt{493}}{58}}. \tag{983}$$

Solution 2: We see that

$$\left|\frac{z}{2w}\right| = \frac{|z|}{|2w|} = \frac{|z|}{2|w|} = \frac{|-1+4i|}{2|5+2i|} \tag{984}$$

$$= \frac{\sqrt{(-1)^2+4^2}}{2\sqrt{5^2+2^2}} = \boxed{\frac{\sqrt{17}}{2\sqrt{29}} = \frac{\sqrt{493}}{58}}. \tag{985}$$

Problem 10.3: Evaluate $i(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}})$.

Solution: Recalling Euler's formula

$$e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)), \text{ we see that} \tag{986}$$

$$i(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}}) = i\left(\left(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})\right) - \left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)\right) \tag{987}$$

$$= i\left(\left(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})\right) - \left(\cos(\frac{\pi}{6}) - i\sin(\frac{\pi}{6})\right)\right) = i\left(2i\sin(\frac{\pi}{6})\right) \tag{988}$$

$$= i(2i \cdot \frac{1}{2}) = i^2 = \boxed{-1}. \tag{989}$$

Problem 10.4: Let $z = -1+i$ and $w = 1+i\sqrt{3}$ be the two complex numbers.
(1) Compute directly $z \cdot w$ and $\frac{z}{w}$ and express the answer in Cartesian form, i.e., the form $x + iy$, where x and y are real numbers.
(2) Express z and w in polar form. Compute $z \cdot w$ and $\frac{z}{w}$ in polar forms. Compare your answer with part (1).
(3) Draw the four complex numbers w, z, $z \cdot w$ and $\frac{z}{w}$ in the following coordinate. Explain what multiplication by w and division by w do to the complex number z in terms of argument and modulus.

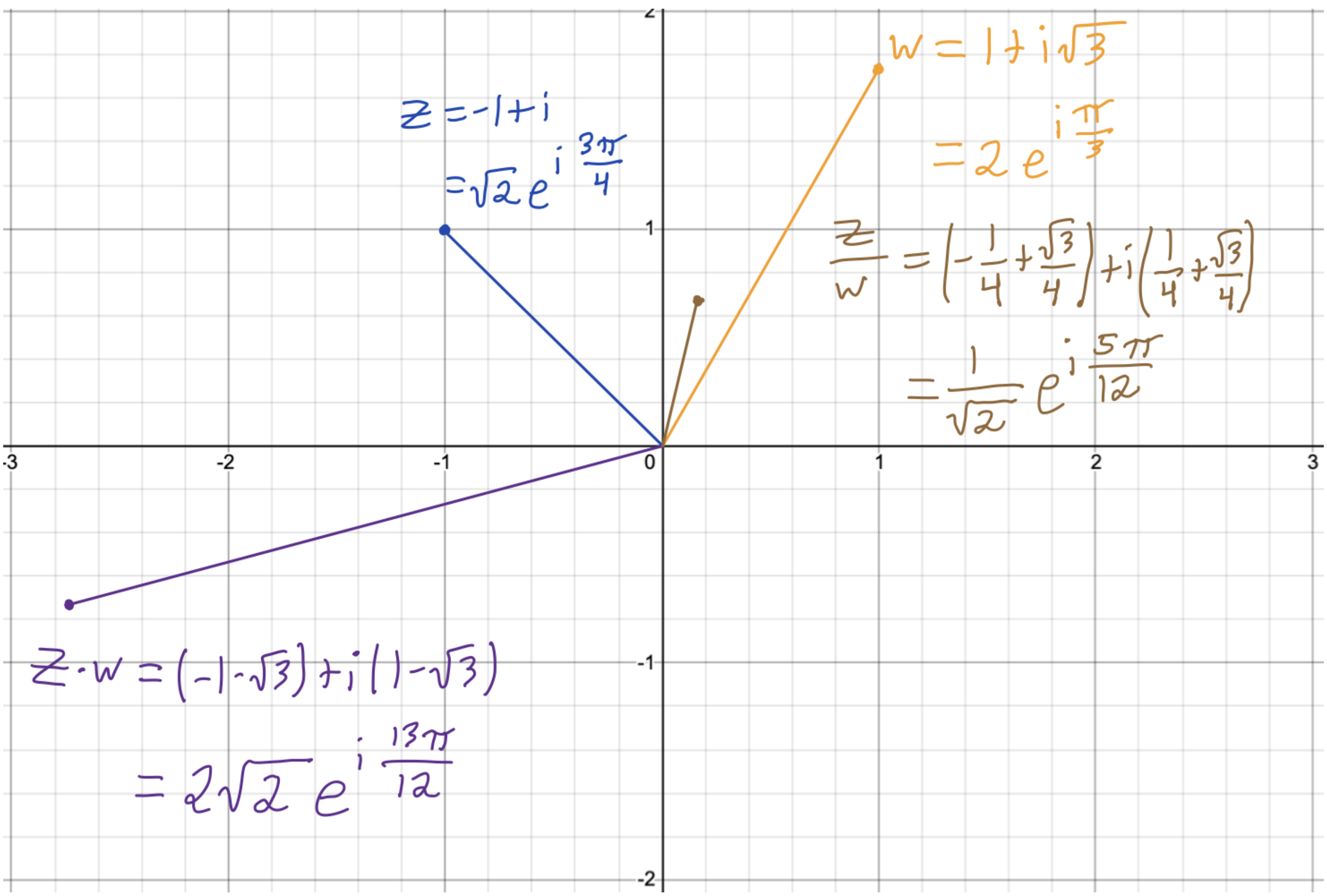


FIGURE 52. A grid to help you plot $w, z, z \cdot w$, and $\frac{z}{w}$.

Solution to Part (1): Firstly, we see that

(990) $$z \cdot w = (-1+i)(1+i\sqrt{3}) = -1 \cdot 1 + -1 \cdot i\sqrt{3} + i \cdot 1 + \underbrace{i \cdot i}_{i^2=-1}\sqrt{3}$$

(991) $$= -1 - i\sqrt{3} + i - \sqrt{3} = \boxed{(-1-\sqrt{3}) + i(1-\sqrt{3})}, \text{ and}$$

(992) $$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2} = \frac{(-1+i)(1-i\sqrt{3})}{1^2+\sqrt{3}^2}$$

(993) $$= \frac{1}{4}(-1\cdot 1 + (-1)\cdot(-i\sqrt{3}) + i\cdot 1 + \underbrace{i\cdot(-i}_{-i^2=1}\sqrt{3})) = \frac{1}{4}(-1+i\sqrt{3}+i+\sqrt{3})$$

(994) $$= \boxed{(-\frac{1}{4}+\frac{\sqrt{3}}{4}) + i(\frac{1}{4}+\frac{\sqrt{3}}{4})}.$$

Solution to Part (2): We see that $|z| = \sqrt{1^2+1^2} = \sqrt{2}$, and that $|w| = \sqrt{1^2+\sqrt{3}^2} = 2$. It follows that
Since z is in the third quadrant and w is in the first quadrant, we see that

(995) $$\theta_z = \tan^{-1}(\frac{\text{Im(z)}}{\text{Re}(z)}) + \pi = \tan^{-1}(\frac{1}{-1}) + \pi = \tan^{-1}(-1) + \pi = \frac{3\pi}{4}, \text{ and}$$

(996) $$\theta_w = \tan^{-1}(\frac{\text{Im}(w)}{\text{Re}(w)}) = \tan^{-1}(\frac{\sqrt{3}}{1}) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}.$$

Recalling that $z = |z|e^{i\theta_z}$ and $w = |w|e^{i\theta_w}$ are the polar forms of z and w respectively, we see that $z = \sqrt{2}e^{\frac{3\pi i}{4}}$ and $w = 2e^{\frac{\pi i}{3}}$. We now see that

(997) $$z\cdot w = \sqrt{2}e^{\frac{3\pi i}{4}}\cdot 2e^{\frac{\pi i}{3}} = 2\sqrt{2}e^{\frac{3\pi i}{4}+\frac{\pi i}{3}} = \boxed{2\sqrt{2}e^{\frac{13\pi i}{12}}}, \text{ and}$$

(998) $$\frac{z}{w} = \frac{\sqrt{2}e^{\frac{3\pi i}{4}}}{2e^{\frac{\pi i}{3}}} = \frac{1}{\sqrt{2}}e^{\frac{3\pi i}{4}-\frac{\pi i}{3}} = \boxed{\frac{1}{\sqrt{2}}e^{\frac{5\pi i}{12}}}.$$

Using a computer algebra system such as wolfram alpha, we can confirm that

(999) $$z\cdot w = 2\sqrt{2}e^{\frac{13\pi i}{12}} = 2\sqrt{2}(\cos(\frac{13\pi}{12}) + i\sin(\frac{13\pi}{12})) = (-1-\sqrt{3}) + i(1-\sqrt{3})$$

and

(1000) $$\frac{z}{w} = \frac{1}{\sqrt{2}}e^{\frac{5\pi i}{12}} = \frac{1}{\sqrt{2}}(\cos(\frac{5\pi}{12}) + i\sin(\frac{5\pi}{12})) = (-\frac{1}{4}+\frac{\sqrt{3}}{4}) + i(\frac{1}{4}+\frac{\sqrt{3}}{4}).$$

Solution to Part (3): The calcuations from Part (2) show us that when we multiply 2 complex numbers we multiply their magnitudes and add their arguments, and when we divide 2 complex numbers, we divide their magnitudes and subtract their arguments. In particular, if we multiply a complex number (such as z) by w, then we will double the magnitude and add $\frac{5\pi}{12}$ to the argument, and if we divide a complex number (such as z) by w, then we will half the magnitude and subtract $\frac{5\pi}{12}$ from the argument.

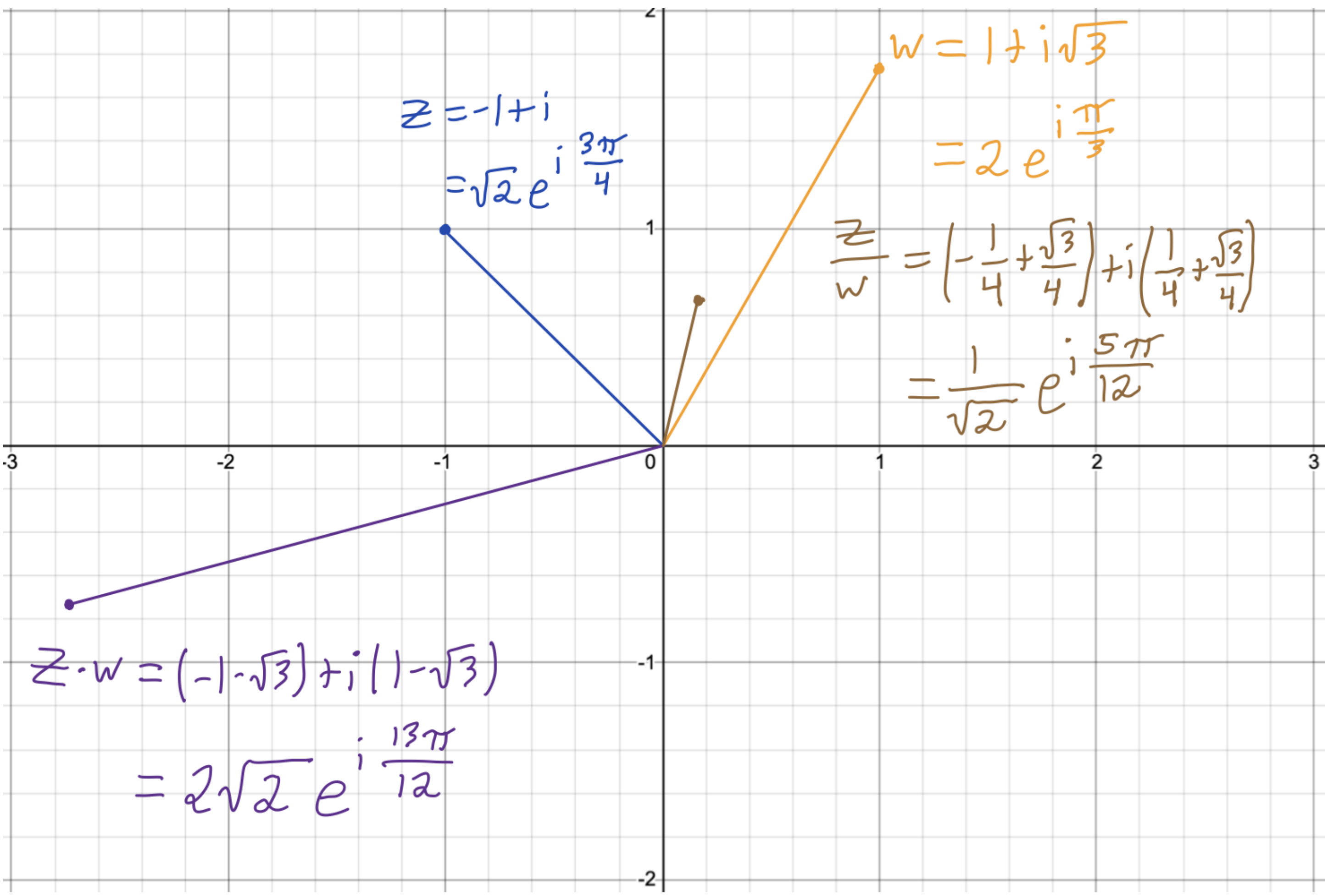


FIGURE 53. A grid to help you plot $w, z, z \cdot w$, and $\frac{z}{w}$.

Problem 10.5: (Appendix C. 29, 30)
(1) Equate the real and imaginary parts of both sides of the identity

$$e^{i(a-b)} = e^{ia}e^{-ib}$$

to prove that

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b);$$
$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b).$$

(2) Equate the real and imaginary parts of both sides of the identity

$$e^{i2\theta} = e^{i\theta} \cdot e^{i\theta}$$

to prove that

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \text{ and } \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Solution to Part (1): Using Euler's formula, we see that

(1001) $$e^{i(a-b)} = \cos(a-b) + i\sin(a-b), e^{ia} = \cos(a) + i\sin(a), \text{ and}$$

(1002) $$e^{-ib} = e^{i(-b)} = \cos(-b) + i\sin(-b) = \cos(b) - i\sin(b), \text{ so}$$

(1003) $$\cos(a-b) + i\sin(a-b) = e^{i(a-b)} = e^{ia}e^{-ib}$$

(1004) $$= (\cos(a) + i\sin(a))(\cos(b) - i\sin(b))$$

(1005) $$= \cos(a)\cos(b) - i\cos(a)\sin(b) + i\sin(a)\cos(b) - \underbrace{i^2}_{i^2=-1}\sin(a)\sin(b)$$

(1006) $$= \cos(a)\cos(b) + \sin(a)\sin(b) + i(\sin(a)\cos(b) - \cos(a)\sin(b))$$

$$(1007) \qquad \rightarrow \begin{matrix} \cos(a-b) & = & \cos(a)\cos(b)+\sin(a)\sin(b) \\ \sin(a-b) & = & \sin(a)\cos(b)-\cos(a)\sin(b) \end{matrix}.$$

Solution to Part (2): We could deduce the result in part (2) by letting $a = -b = \theta$, but we will instead prove the result as a corollary to Euler's formula once again just for the extra practice. We once again begin the problem by using Euler's formula to see that

$$(1008) \qquad e^{i2\theta} = \cos(2\theta) + i\sin(2\theta) \text{ and } e^{i\theta} = \cos(\theta) + i\sin(\theta), \text{ so}$$

$$(1009) \qquad e^{i2\theta} = e^{2i\theta} = (e^{i\theta})^2 = (\cos(\theta) + i\sin(\theta))^2$$

$$(1010) \qquad = \cos^2(\theta) + 2i\cos(\theta)\sin(\theta) + \underbrace{i^2}_{i^2=-1}\sin^2(\theta)$$

$$(1011) \qquad = \cos^2(\theta) - \sin^2(\theta) + i2\cos(\theta)\sin(\theta)$$

$$(1012) \qquad \rightarrow \begin{matrix} \cos(2\theta) & = & \cos^2(\theta)-\sin^2(\theta) \\ \sin(2\theta) & = & 2\sin(\theta)\cos(\theta) \end{matrix}.$$

Problem 10.6: Find all possible fourth roots of -16. Equivalently, find all possible values of $(-16)^{\frac{1}{4}}$.

Solution: We see that

$$-16 = 16\cdot(-1) = 16e^{i\pi} = 16e^{i(\pi+2n\pi)} \text{ (where } n \text{ is an integer)} \tag{1013}$$

$$\to (-16)^{\frac{1}{4}} = \left(16e^{i(\pi+2n\pi)}\right)^{\frac{1}{4}} = 16^{\frac{1}{4}}\left(e^{i(\pi+2n\pi)}\right)^{\frac{1}{4}} \tag{1014}$$

$$= 2e^{i(\frac{\pi}{4}+\frac{n}{2}\pi)} \text{ (where } n \text{ is an integer)} \tag{1015}$$

$$\to (-16)^{\frac{1}{4}} \in \{2e^{i\frac{\pi}{4}}, 2e^{i\frac{3\pi}{4}}, 2e^{i\frac{5\pi}{4}}, 2e^{i\frac{7\pi}{4}}\}. \tag{1016}$$

Making use of Euler's formula, we see that

$$2e^{i\frac{\pi}{4}} = 2\left(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right) = 2(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \sqrt{2}+\sqrt{2}i, \tag{1017}$$

$$2e^{i\frac{3\pi}{4}} = 2\left(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right) = 2(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = -\sqrt{2}+\sqrt{2}i, \tag{1018}$$

$$2e^{i\frac{5\pi}{4}} = 2\left(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})\right) = 2(-\frac{1}{\sqrt{2}} + i(-\frac{1}{\sqrt{2}})) = -\sqrt{2}-\sqrt{2}i, \tag{1019}$$

$$2e^{i\frac{7\pi}{4}} = 2\left(\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4})\right) = 2(\frac{1}{\sqrt{2}} + i(-\frac{1}{\sqrt{2}})) = \sqrt{2}-\sqrt{2}i, \tag{1020}$$

$$\to (-16)^{\frac{1}{4}} \in \boxed{\{\sqrt{2}+\sqrt{2}i, -\sqrt{2}+\sqrt{2}i, -\sqrt{2}-\sqrt{2}i, \sqrt{2}-\sqrt{2}i\}}. \tag{1021}$$

Problem 10.7: Determine A, ω, and φ for which

(1022) $$-3\sin(4t) + 3\cos(4t) = A\sin(\omega t + \varphi).$$

Solution: Firstly, we use the angle-addition formula for sin to see that

(1023) $$A\sin(\omega t + \varphi) = A\sin(\omega t)\cos(\varphi) + A\sin(\varphi)\cos(\omega t), \text{ so}$$

(1024) $$-3\sin(4t) + 3\cos(4t) = A\cos(\varphi)\sin(\omega t) + A\sin(\varphi)\cos(\omega t).$$

We now see that $\omega = 4$, and that

(1025) $$\begin{array}{rcl} A\cos(\varphi) & = & -3 \\ A\sin(\varphi) & = & 3 \end{array}$$

(1026) $$\to A^2 = A^2\cos^2(\varphi) + A^2\sin^2(\varphi) = (-3)^2 + 3^2 = 18 \to A = \pm 3\sqrt{2}$$

(1027) $$\to \begin{array}{rcl} \cos(\varphi) & = & \mp\frac{1}{\sqrt{2}} \\ \sin(\varphi) & = & \pm\frac{1}{\sqrt{2}} \end{array} \to \varphi = \frac{3\pi}{4}, -\frac{\pi}{4}$$

(1028)
$$\to -3\sin(4t) + 3\cos(4t) = \underbrace{\boxed{3\sqrt{2}\sin(4t + \frac{3\pi}{4})}}_{\text{This is amplitude-phase form since A is positive.}} = \boxed{-3\sqrt{2}\sin(4t - \frac{\pi}{4})}.$$

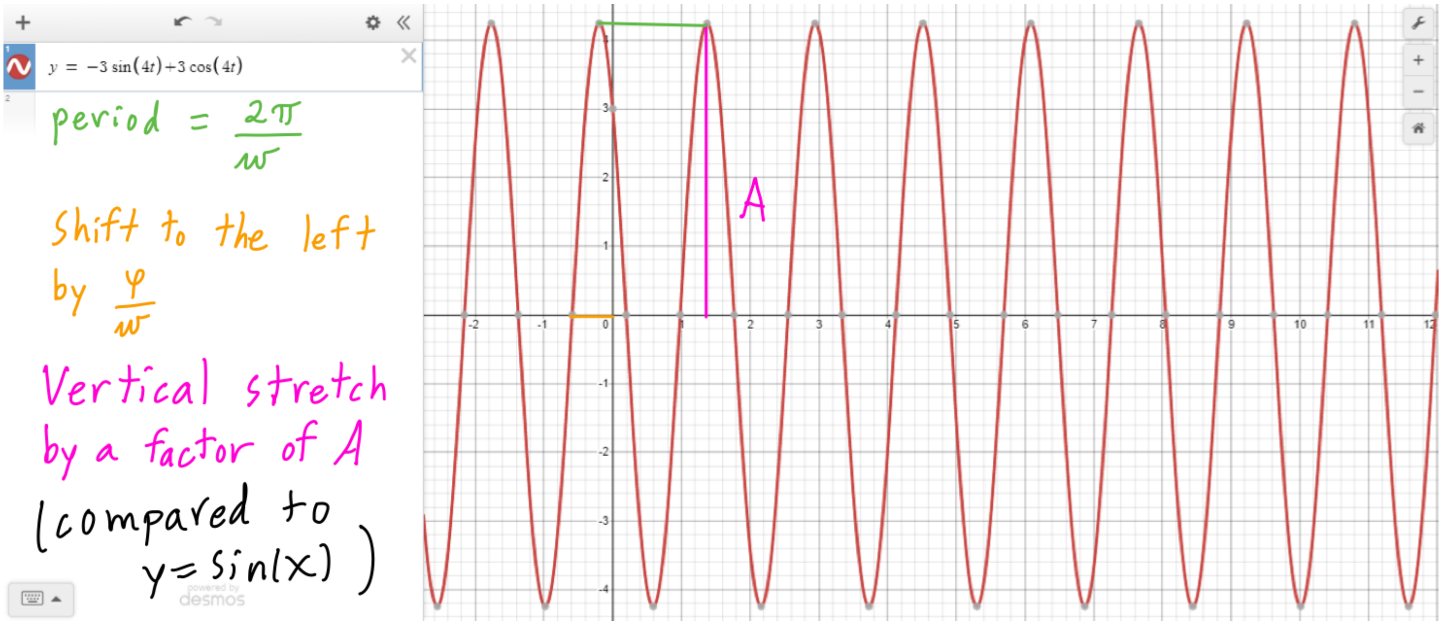


FIGURE 54. The graph of $y = -3\sin(4t) + 3\cos(4t)$ showing the Amplitude of A, Phase Shift of φ, and the period of $\frac{2\pi}{\omega}$.

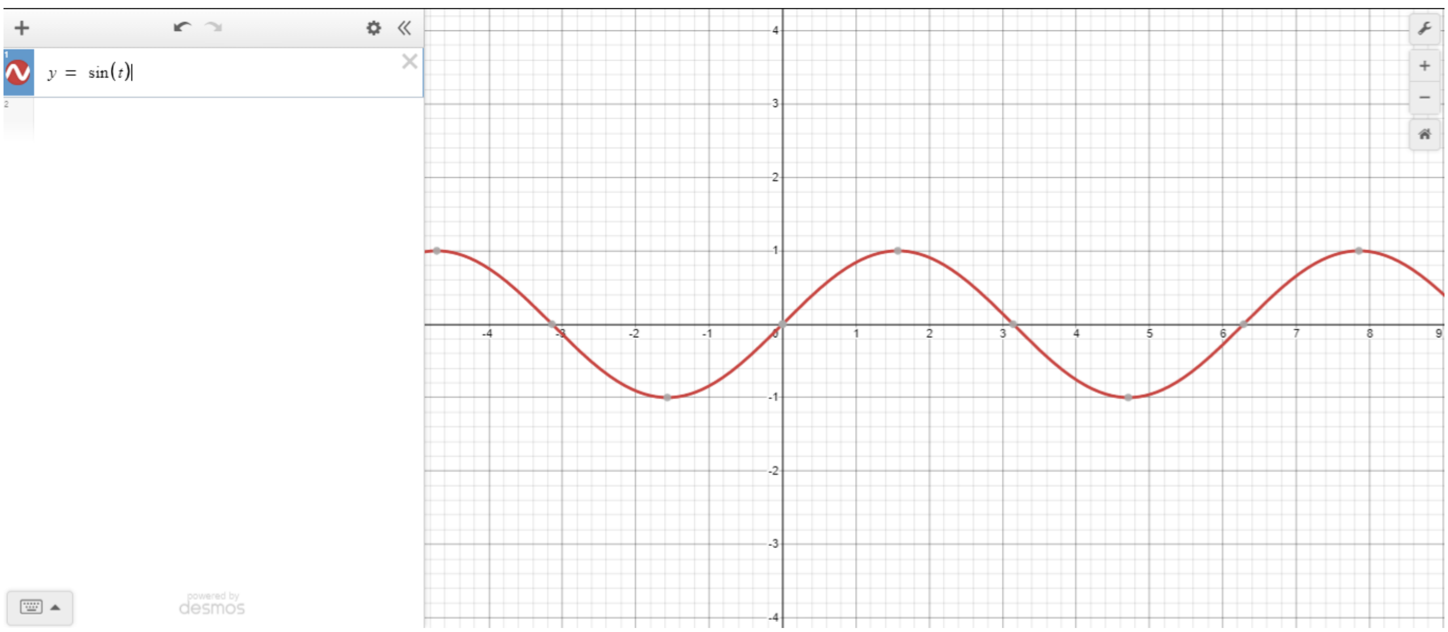


FIGURE 55. A graph of $y = \sin(t)$ for comparison.

Problem 10.8: Determine R, δ, and ω_0 for which

(1029) $$-2\cos(\pi t) - 3\sin(\pi t) = R\cos(\omega_0 t - \delta).$$

Solution: Using the cosine subtraction formula of

(1030) $$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y),$$

we see that

(1031) $$R\cos(\omega_0 t - \delta) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta),$$

so we want to find R, ω_0, and δ for which

(1032) $$-2\cos(\pi t) - 3\sin(\pi t) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta).$$

Comparing the functions that have t in them, i.e., $\cos(\pi t), \cos(\omega_0 t), \sin(\pi t)$, and $\sin(\omega_0 t)$, we see that $\omega_0 = \pi$. We now want to find R and δ for which

(1033) $$-2\cos(\pi t) - 3\sin(\pi t) = R\cos(\pi t)\cos(\delta) + R\sin(\pi t)\sin(\delta),$$

which is the same as finding R and δ for which

(1034) $$R\cos(\delta) = -2 \text{ and } R\sin(\delta) = -3.$$

We now see that

(1035) $$R^2 = R^2(\cos^2(\delta) + \sin^2(\delta)) = (-2)^2 + (-3)^2 = 13$$

(1036) $$\rightarrow R = \pm\sqrt{13}.$$

We may pick $R = \sqrt{13}$ or $R = -\sqrt{13}$, so we will pick $R = \sqrt{13}$ for convenience. We now see that

(1037) $$\cos(\delta) = -\frac{2}{\sqrt{13}} \text{ and } \sin(\delta) = -\frac{3}{\sqrt{13}},$$

so δ is in the third quadrant, i.e., $\pi < \delta < \frac{3\pi}{2}$. We now see that

(1038) $$\delta = 2\pi - \cos^{-1}(-\frac{2}{\sqrt{13}}) \approx 4.12.$$

In conclusion,

(1039) $$-2\cos(\pi t) - 3\sin(\pi t) = \boxed{\sqrt{13}\cos(\pi t - 2\pi + \cos^{-1}(-\frac{2}{\sqrt{13}}))}$$

(1040) $$\approx \sqrt{13}\cos(\pi t - 4.12)).$$

Problem 11.1: Solve the following initial value problem.

$$y'' - 3y' - 18y = 0; \quad y(0) = 0, y'(0) = 4. \tag{1041}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Solution: We see that the characteristic polynomial of equation (1041) is

$$0 = r^2 - 3r - 18 = (r-6)(r+3), \tag{1042}$$

which has roots $r = -3, 6$. It follows that the general solutions to equation (1041) is

$$y(t) = c_1e^{-3t} + c_2e^{6t}. \tag{1043}$$

Using the initial conditions, we see that

$$\begin{array}{ccccccc} 0 & = & y(0) & = & c_1e^{-3\cdot 0} + c_2e^{6\cdot 0} & = & c_1 + c_2 \\ 4 & = & y'(0) & = & -3c_1e^{3\cdot 0} + 6c_2e^{6\cdot 0} & = & -3c_1 + 6c_2 \end{array} \tag{1044}$$

. . . .

$$\rightarrow \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ -3c_1 & + & 6c_2 & = & 4 \end{array} \overset{R_2+3R_1}{\rightarrow} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & 9c_2 & = & 4 \end{array} \tag{1045}$$

. . . .

$$\overset{\frac{1}{9}R_2}{\rightarrow} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & c_2 & = & \frac{4}{9} \end{array} \overset{R_1-R_2}{\rightarrow} \begin{array}{rcl} c_1 & = & -\frac{4}{9} \\ c_2 & = & \frac{4}{9} \end{array} \tag{1046}$$

. . . .

$$\rightarrow \boxed{y(t) = -\frac{4}{9}e^{-3t} + \frac{4}{9}e^{6t}}. \tag{1047}$$

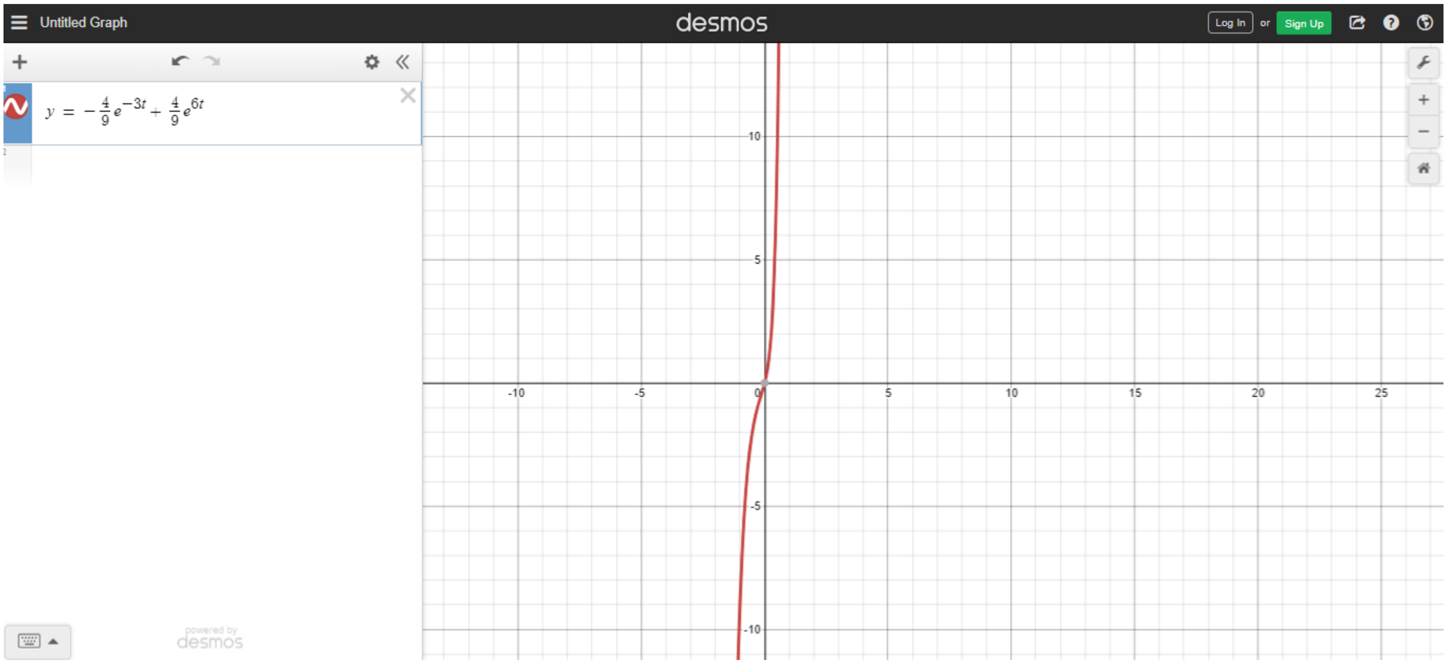


FIGURE 56. A graph of the solution to the initial value problem in (1041).

Problem 11.4: Let a be a real number.

(a) Find the general solution to equation (1048) in terms of a.

$$y'' - (a+2)y' + 2ay = 0. \tag{1048}$$

(b) Solve the initial value problem given in (1049).

$$y'' - 25y' + 46y; \quad y(0) = 0,\ y'(0) = 21. \tag{1049}$$

Solution to (a): By examining the characteristic equation of equation (1048), we see that

$$0 = r^2 - (a+2)r + 2a = (r-a)(r-2) \tag{1050}$$

$$\to y(t) = \boxed{\begin{cases} c_1e^{at} + c_2e^{2t} & \text{if } a \neq 2 \\ c_2e^{2t} + c_2te^{2t} & \text{if } a = 2 \end{cases}}. \tag{1051}$$

Solution to (b): We saw in part (a) that the general solution to equation (1049) is $y(t) = c_1e^{23t} + c_2e^{2t}$. It follows that $y'(t) = 23c_2e^{23t} + 2c_2e^{2t}$. Making use of the given initial conditions, we see that

$$\begin{array}{rcccccc} 0 & = & y(0) & = & c_1e^{23\cdot 0} + c_2e^{2\cdot 0} & = & c_1 + c_2 \\ 21 & = & y'(0) & = & 23c_1e^{23\cdot 0} + 2c_2e^{2\cdot 0} & = & 23c_1 + 2c_2 \end{array} \tag{1052}$$

$$\to c_1 = \frac{1}{21}\cdot 21c_1 = \frac{1}{21}\Big((23c_1 + 2c_2) - 2(c_1 + c_2)\Big) = \frac{1}{21}(21 - 2\cdot 0) = 1 \tag{1053}$$

$$\to c_2 = -c_1 = -1 \to y(t) = \boxed{e^{23t} - e^{2t}}. \tag{1054}$$

Problem 11.2: Solve the following initial value problem.

$$y'' - y' + \frac{1}{4}y = 0; \quad y(0) = 1, y'(0) = 2. \tag{1055}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Solution: We see that the characteristic polynomial of equation (1055) is

$$0 = r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2, \tag{1056}$$

which has $r = \frac{1}{2}$ as a double root. It follows that the general solutions to equation (1055) is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}. \tag{1057}$$

Noting that

$$y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}} = (\frac{1}{2}c_1 + c_2)e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}}, \tag{1058}$$

we can use the initial conditions, to see that

$$\begin{array}{rcrcccc} 1 & = & y(0) & = & c_1 e^{\frac{0}{2}} + c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & c_1 \\ 2 & = & y'(0) & = & (\frac{1}{2}c_1 + c_2)e^{\frac{0}{2}} + \frac{1}{2}c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & \frac{1}{2}c_1 + c_2 \end{array} \tag{1059}$$

$$\rightarrow \begin{array}{rcrcl} c_1 & & & = & 1 \\ \frac{1}{2}c_1 & + & c_2 & = & 2 \end{array} \rightarrow \begin{array}{rcccl} c_1 & = & & & 1 \\ c_2 & = & 2 - \frac{1}{2} \cdot 1 & = & \frac{3}{2} \end{array} \tag{1060}$$

$$\rightarrow \boxed{y(t) = e^{\frac{t}{2}} + \frac{3}{2}t e^{\frac{t}{2}}}. \tag{1061}$$

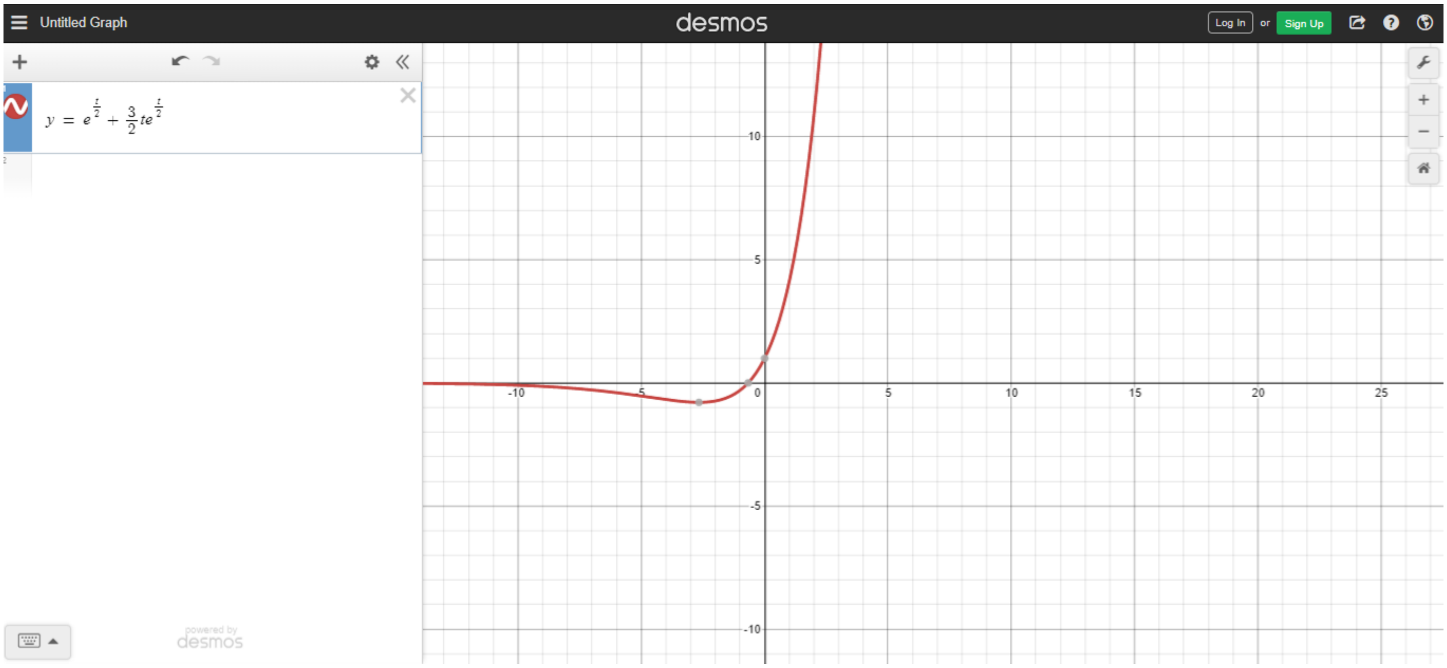


FIGURE 57. A graph of the solution to the initial value problem in (1055).

Problem 11.3: Solve the following initial value problem.

$$y'' + 6y' + 10y = 0; \quad y(0) = 0, y'(0) = 6. \tag{1062}$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Solution: We see that the characteristic polynomial of equation (1062) is

$$0 = r^2+6r+10 \to r = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i, \tag{1063}$$

It follows that the general solutions to equation (1062) is

$$y(t) = c_1' e^{(-3+i)t} + c_2' e^{(-3-i)t} = c_1 \sin(t)e^{-3t} + c_2 \cos(t)e^{-3t}. \tag{1064}$$

Noting that

$$y'(t) = c_1 \cos(t)e^{-3t} - 3c_1 \sin(t)e^{-3t} - c_2 \sin(t)e^{-3t} - 3c_2 \cos(t)e^{-3t} \tag{1065}$$

$$= (-3c_1 - c_2) \sin(t)e^{-3t} + (c_1 - 3c_2) \cos(t)e^{-3t}, \tag{1066}$$

we can use the initial conditions to see that

$$\begin{array}{rcrcl} 0 & = & y(0) & = & c_1 \sin(0)e^{-3\cdot 0} + c_2 \cos(0)e^{-3\cdot 0} \\ 6 & = & y'(0) & = & (-3c_1 - c_2) \sin(0)e^{-3\cdot 0} + (c_1 - 3c_2) \cos(0)e^{-3\cdot 0} \end{array} \tag{1067}$$

. . . .

$$\to \begin{array}{rcl} 0 & = & c_2 \\ 6 & = & c_1 - 3c_2 \end{array} \to \begin{array}{rcrcl} c_2 & = & & & 0 \\ c_1 & = & 6 + 3c_2 & = & 6 \end{array} \tag{1068}$$

. . . .

$$\to \boxed{y(t) = 6 \sin(t)e^{-3t}}. \tag{1069}$$

FIGURE 58. A graph of the solution to the initial value problem in (1062).

Problem 11.5: Solve the following initial value problem.

(1070) $$t^2y'' + 6ty' + 6y = 0;\ y(1) = 0, y'(1) = -4.$$

Draw the graph of the solution. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Remark: The solution below is a detailed solutions 'from first principles' that also demonstrates how to perform a change of variables in a differential equation. For a homework or an exam, you are permitted to use the fact that the characteristic equation of $t^2y''+aty'+by=0$ is given by $r^2+(a-1)r+b=0$, and that the general solution is of the form $y(t) = c_1t^{r_1} + c_2t^{r_2}$ if the roots are distinct[17], or $y(t) = c_1t^r + c_2t^r\ln(t)$ if $r_1 = r_2 = r$.

Solution: We perform a substitution (or a change of variables) in order to convert equation (1070) into a constant coefficient differential equation, which will then be straight-forward to solve. Letting $x = \ln(t)$, we see that $t = e^x$, and we may define $h(x) = y(e^x) = y(t)$. We see that

(1071) $$h'(x) = \frac{d}{dx}h(x) = \frac{d}{dx}y(e^x) = y'(e^x)\cdot\frac{d}{dx}e^x = y'(e^x)\cdot e^x = ty'(t), \text{ and}$$

(1072) $$h''(x) = \frac{d}{dx}h'(x) = \frac{d}{dx}(e^xy'(e^x)) = \frac{d}{dx}(e^x)\cdot y'(e^x) + e^x\cdot\frac{d}{dx}y'(e^x)$$

(1073) $$= e^xy'(e^x) + e^x\cdot e^xy''(e^x) = e^xy'(e^x) + e^{2x}y''(e^x) = ty'(t) + t^2y''(t).$$

We now see that

(1074) $$0 = t^2y'' + 6ty' + 6y = (t^2y'' + ty') + 5ty' + 6y$$

(1075) $$= (t^2y''(t) + ty'(t)) + 5ty'(t) + 6y(t)$$

[17] This general form is still correct if r_1 and r_2 are distinct complex numbers, but it is usually not the preferred form of the general solution. If $r \pm si$ are the distinct complex roots of the characteristic equation, then the preferred form the general solution is $y(t) = c_1t^r\cos(s\ln(t)) + c_2t^r\sin(s\ln(t))$.

. . . .

$$= h''(x) + 5h'(x) + 6h(x) = h'' + 5h' + 6h. \tag{1076}$$

We see that the characteristic equation of our converted equation is

$$0 = r^2 + 5r + 6 = (r+2)(r+3), \tag{1077}$$

and has solutions $r = -3, -2$. It follows that the general solution to our converted equation is

$$h(x) = c_1e^{-2x} + c_2e^{-3x}. \tag{1078}$$

Recalling that $x = \ln(t)$, we see that the general solution to equation (1070) is

$$y(t) = h(x) = c_1e^{-2x} + c_2e^{-3x} = c_1e^{-2\ln(t)} + c_2e^{-3\ln(t)} = c_1t^{-2} + c_2t^{-3}. \tag{1079}$$

Making use of the initial conditions, we see that

$$\begin{array}{rcrcrcr} 0 & = & y(1) & = & c_1 \cdot 1^{-2} + c_2 \cdot 1^{-3} & = & c_1 + c_2 \\ -4 & = & y'(1) & = & -2c_1 \cdot 1^{-3} - 3c_2 \cdot 1^{-4} & = & -2c_1 - 3c_2 \end{array} \tag{1080}$$

. . . .

$$\rightarrow \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ -2c_1 & - & 3c_2 & = & -4 \end{array} \overset{R_2+2R_1}{\longrightarrow} \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ & & -c_2 & = & -4 \end{array} \tag{1081}$$

. . . .

$$\overset{\frac{1}{5}R_2}{\longrightarrow} \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ & & c_2 & = & 4 \end{array} \overset{R_1-R_2}{\longrightarrow} \begin{array}{rcr} c_1 & = & -4 \\ c_2 & = & 4 \end{array} \tag{1082}$$

. . . .

$$\rightarrow \boxed{y(t) = -4t^{-2} + 4t^{-3}}. \tag{1083}$$

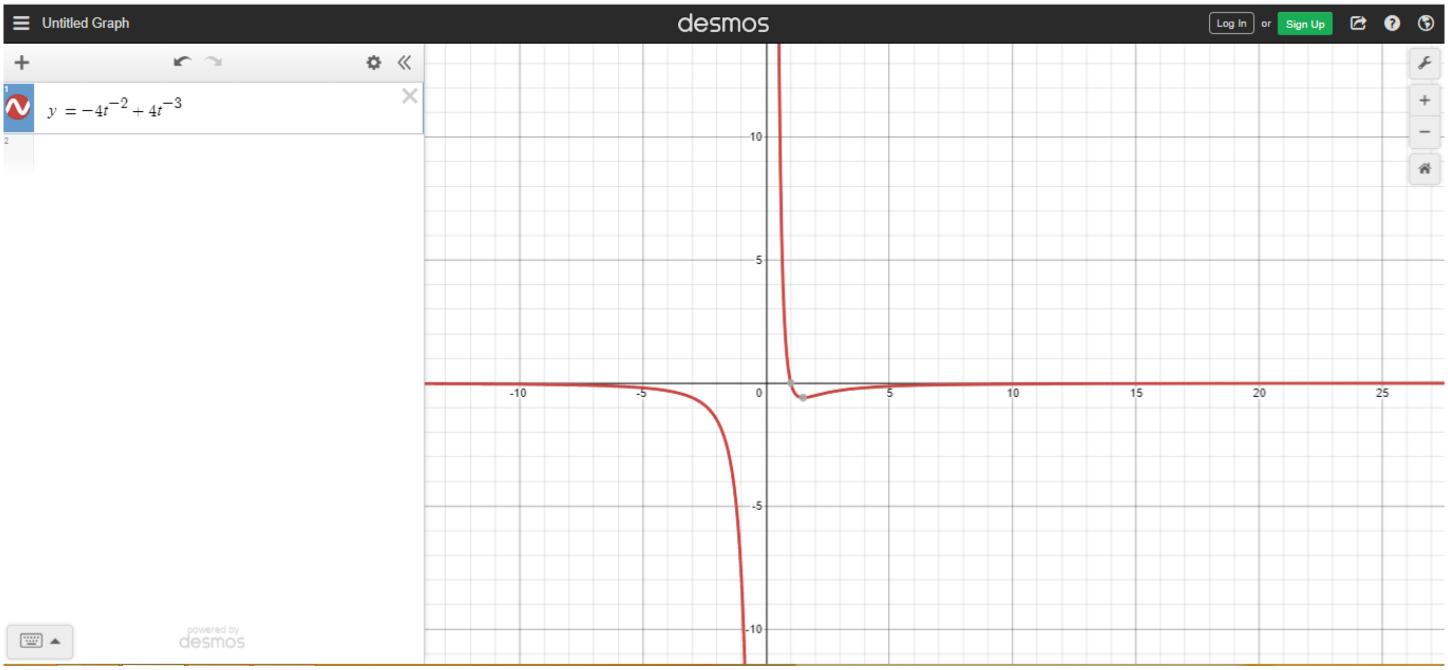


FIGURE 59. A graph of the solution to the initial value problem in (1070).

Problem 11.6: Find the general solution of the equation

$$y''y' = 1. \tag{1084}$$

Solution: We note that $y(t)$ is not present in equation (1084), so we perform the substitution $v(t) = y'(t)$. We see that $v'(t) = y''(t)$, so equation (1084) becomes

$$1 = vv' = v\frac{dv}{dt} \to dt = vdv \tag{1085}$$

$$\to t = \int dt = \int vdv = \frac{1}{2}v^2 + c_1 = \frac{1}{2}(y')^2 + c_1 \tag{1086}$$

$$\to \pm\sqrt{2t - 2c_1} = y' = \frac{dy}{dt} \to dy = \pm\sqrt{2t - 2c_1}dt \tag{1087}$$

$$y = \int dy = \int \pm\sqrt{2t - 2c_1}dt = \pm\frac{1}{3}(2t - 2c_1)^{\frac{3}{2}} + c_2 \tag{1088}$$

$$\to \boxed{y(t) = \pm\frac{1}{3}(2t - 2c_1)^{\frac{3}{2}} + c_2}. \tag{1089}$$

Remark: If we had initial conditions, then we could use them to try and determine values for c_1 and c_2. We should also note that this solution is only defined when $t > c_1$. We also note that the form of the general solution looks completely different from the form of the general solution to a linear differential equation. The constants c_1 and c_2 are NOT coefficients in a linear combination, and we have 2 completely disjoint sets of solutions (the positive solutions and the negative solutions each have 2 degrees of freedom).

Problem 11.7: Solve the differential equation

$$y'' = e^{-y'}. \tag{1090}$$

Solution: We note that $y(t)$ is not present in equation (1090), so we perform the substitution $v(t) = y'(t)$. We see that $v'(t) = y''(t)$, so equation (1090) becomes

$$v' = e^{-v} \to 1 = e^v v' = e^v \frac{dv}{dt} \to dt = e^v dv \tag{1091}$$

$$\to \int dt = \int e^v dv \to t + c_1 = e^v = e^{y'} \tag{1092}$$

$$\to \ln(t + c_1) = y' = \frac{dy}{dt} \to dy = \ln(t + c_1)dt \tag{1093}$$

$$\to y = \int dy = \int \ln(t + c_1)dt = (t + c_1)\ln(t + c_1) - t + c_2. \tag{1094}$$

$$\to \boxed{y(t) = (t + c_1)\ln(t + c_1) - t + c_2}. \tag{1095}$$

Remark: If we had initial conditions, then we could use them to try and determine values for c_1 and c_2. We should also note that this solution is only defined when $t > -c_1$. We also note that the form of the general solution looks completely different from the form of the general solution to a linear differential equation. The constants c_1 and c_2 are NOT coefficients in a linear combination.

For the following problems use the method of undetermined coefficients in order to find the general form of the solution to the given differential equation.

$$y'' + y = \cos(2t) + t^3. \tag{1096}$$

$$y'' + 4y = \cos(2t). \tag{1097}$$

$$2y'' - 8y' + 8y = 4e^{2t}. \tag{1098}$$

$$y'' - y = 25te^{-t}\sin(3t). \tag{1099}$$

$$y^{(4)} - 3y'' + 2y = 6te^{2t}. \tag{1100}$$

Solution: Solution to equation (1096): We see that the homogeneous equation corresponding to equation (1096) is

$$y'' + y = 0, \tag{1101}$$

and has characteristic equation

$$0 = r^2 + 1 = (r + i)(r - i) \tag{1102}$$

It follows that the general solution to equation (1101) is

$$y(t) = c_1e^{-it} + c_2e^{it} = c_3\sin(t) + c_4\cos(t). \tag{1103}$$

We now see that the right hand side of equation (1096) is not related to the solutions of equation (1101), so we may use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = A\cos(2t) + B\sin(2t) + Ct^3 + Dt^2 + Et + F}. \tag{1104}$$

Solution to equation (1097): We see that the homogeneous equation corresponding to equation (1097) is

(1105) $$y'' + 4y = 0,$$

and has characteristic equation

(1106) $$0 = r^2 + 4 = (r + 2i)(r - 2i).$$

It follows that the general solution to equation (1105) is

(1107) $$y(t) = c_1e^{-2it} + c_2e^{2it} = c_3\sin(2t) + c_4\cos(2t)$$

We now see that the right hand side of equation (1097) **is** related to the solutions of equation (1105), so we have to adjust the standard form of the general solution in the method of undetermined coefficients. Originally, we would have used

(1108) $$y(t) = A\sin(2t) + B\cos(2t),$$

but we saw that $\sin(2t)$ and $\cos(2t)$ are solutions to equation (1105), so we then adjust our answer by multiplying by t to get

(1109) $$\boxed{y(t) = At\sin(2t) + Bt\cos(2t)}.$$

Solution to equation (1098)**:** We see that the homogeneous equation corresponding to equation (1098) is

(1110) $$2y'' - 8y'' + 8y = 0 \to y'' - 4y' + 4y = 0,$$

and has characteristic equation

(1111) $$0 = r^2 - 4r + 4 = (r - 2)^2.$$

It follows that the general solution to equation (1110) is

(1112) $$y(t) = (c_1t + c_2)e^{2t}.$$

We now see that the right hand side of equation (1098) **is** related to the solutions of equation (1110), so we have to adjust the standard form of the general solution in the method of undetermined coefficients. Originally, we would have used

$$y(t) = Ae^{2t}, \tag{1113}$$

but we saw that e^{2t} is a solution to equation (1110), so we would then adjust our answer by multiplying by t to get

$$y(t) = Ate^{2t}, \tag{1114}$$

but we see that te^{2t} is also a solution to equation (1110) (which should not surprise us since 2 was a double root of the characteristic equation), so we adjust our answer by multiplying by t once again to get

$$\boxed{y(t) = At^2e^{2t}}. \tag{1115}$$

Solution to equation (1099): We see that the homogeneous equation corresponding to equation (1099) is

$$y'' - y = 0, \tag{1116}$$

and has characteristic equation

$$0 = r^2 - 1 = (r-1)(r+1). \tag{1117}$$

It follows that the general solution to equation (1116) is

$$y(t) = c_1e^t + c_2e^{-t}. \tag{1118}$$

Recalling that

$$e^{-t}\sin(3t) = -\frac{i}{2}(e^{(-1+3i)t} - e^{(-1-3i)t}), \tag{1119}$$

we see that the right hand side of equation (1099) is not related to the solutions of equation (1116), so we may proceed to use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = (At + B)e^{-t}\sin(3t) + (Ct + D)e^{-t}\cos(3t)}. \tag{1120}$$

Solution to equation (1100): We see that the homogeneous equation corresponding to equation (1100) is

$$y^{(4)} - 3y'' + 2y = 0, \tag{1121}$$

and has characteristic equation

$$0 = r^4 - 3r^2 + 2 = (r^2 - 2)(r^2 - 1) = (r - \sqrt{2})(r + \sqrt{2})(r - 1)(r + 1). \tag{1122}$$

It follows that the general solution to equation (1121) is

$$y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + c_3 e^t + c_4 e^{-t}. \tag{1123}$$

We now see that the right hand side of equation (1100) is not related to the solutions of equation (1121), so we may proceed to use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = (At + B)e^{2t}}. \tag{1124}$$

Problem 11.9: Find a particular solution of the following equation.

$$y'' - y' - 6y = \sin(t) + 3\cos(t). \tag{1125}$$

Solution: We see that the corresponding homogeneous equation of equation (1125) is

$$y_c'' - y_c' - 6y_c = 0. \tag{1126}$$

By examining the characteristic equation of equation (1126), we see that

$$0 = r^2 - r - 6 = (r-3)(r+2) \to y_c(t) = c_1e^{-2t} + c_2e^{3t}. \tag{1127}$$

Since $\sin(t) + 3\cos(t)$ is unrelated to $y_c(t)$, we may proceed to use the method of undetermined coefficients without any adjustments. We use the trial solution of $y_p(t) = A\sin(t) + B\cos(t)$ since $\{\sin(t), \cos(t)\}$ is a linearly independent set of functions that 'generates' any function which is a derivative of $\sin(t) + 3\cos(t)$. We also note that $y_p'(t) = A\cos(t) - B\sin(t)$, and $y_p''(t) = -A\sin(t) - B\cos(t)$. Plugging y_p, y_p', and y_p'' into equation (1125) we see that

$$\sin(t) + 3\cos(t) = y_p'' - y_p' - 6y_p \tag{1128}$$

$$= \underbrace{-A\sin(t) - B\cos(t)}_{y_p''(t)} - (\underbrace{A\cos(t) - B\sin(t)}_{y_p'(t)}) - 6(\underbrace{A\sin(t) + B\cos(t)}_{y_p(t)}) \tag{1129}$$

$$= (-7A + B)\sin(t) + (-A - 7B)\cos(t) \to \begin{aligned} -7A + B &= 1 \\ -A - 7B &= 3 \end{aligned} \tag{1130}$$

$$\to A = -\frac{1}{50}\Big((-A - 7B) + 7(-7A + B)\Big) = -\frac{1}{50}(3 + 7 \cdot 1) = -\frac{1}{5} \tag{1131}$$

$$\to B = 1 + 7A = -\frac{2}{5} \to y_p(t) = \boxed{-\frac{1}{5}\sin(t) - \frac{2}{5}\cos(t)}. \tag{1132}$$

Problem 11.10: Find a particular solution of the following equation.

(1133) $$y'' + y = \cos(2t) + t^3.$$

Solution: We see that the corresponding homogeneous equation for (1133) is

(1134) $$y_c'' + y_c = 0.$$

By examining the characteristic equation of equation (1134), we see that

(1135) $$0 = r^2 + 1 \to r = \pm i \to y_c(t) = c_1' e^{-it} + c_2' e^{it} = c_1 \sin(t) + c_2 \cos(t).$$

Since $\cos(2t) + t^3$ is unrelated to $y_c(t)$, we may proceed to use the method of undetermined coefficients without any adjustments. We use a trial solution of

(1136) $y_p(t) = A\sin(2t) + B\cos(2t) + Ct^3 + Dt^2 + Et + F$, and observe that

. . . .

(1137) $y_p'(t) = 2A\cos(2t) - 2B\sin(2t) + 3Ct^2 + 2Dt + E$, and

. . . .

(1138) $$y_p''(t) = -4A\sin(2t) - 4B\cos(2t) + 6Ct + 2D.$$

Plugging y_p, y_p', and y_p'' into equation (11.10), we see that

. . . .

(1139) $$\cos(2t) + t^3 = y_p'' + y_p = \underbrace{-4A\sin(2t) - 4B\cos(2t) + 6Ct + 2D}_{y_p''(t)} + \underbrace{A\sin(2t) + B\cos(2t) + Ct^3 + Dt^2 + Et + F}_{y_p(t)}$$

. . . .

(1140) $$= -3A\sin(2t) - 3B\cos(2t) + Ct^3 + Dt^2 + (E + 6C)t + (2D + F)$$

. . . .

$$\rightarrow \begin{aligned} -3A &= 0 \text{ (by comparing the } \sin(2t) \text{ terms)} \\ -3B &= 1 \text{ (by comparing the } \cos(2t) \text{ terms)} \\ C &= 1 \text{ (by comparing the } t^3 \text{ terms)} \\ D &= 0 \text{ (by comparing the } t^2 \text{ terms)} \\ E + 6C &= 0 \text{ (by comparing the } t \text{ terms)} \\ 2D + F &= 0 \text{ (by comparing the constant terms)} \end{aligned} \tag{1141}$$

. . . .

$$\rightarrow (A, B, C, D, E, F) = (0, -\frac{1}{3}, 1, 0, -6, 0) \tag{1142}$$

. . . .

$$\rightarrow y_p(t) = \boxed{-\frac{1}{3}\cos(2t) + t^3 - 6t}. \tag{1143}$$

Problem 11.11: (related to the Modified Ungraded problem 16.3.35 (4)) Find a particular solution of the following equation.

(1144) $$y'' + 4y = \cos(2t).$$

Solution: We see that the corresponding homogeneous equation of equation (1144) is

(1145) $$y_c'' + 4y_c = 0.$$

By examining the characteristic equation of equation (1145), we see that

(1146) $$0 = r^2 + 4 \to r = \pm 2i \to y_c(t) = c_1' e^{-2it} + c_2' e^{2it} = c_1 \sin(2t) + c_2 \cos(2t).$$

Normally, the method of undetermined coefficients would have us use the trial solution $y_p(t) = A\sin(2t) + B\cos(2t)$. However, we see in this case that $y_p(t) = y_c(t)$ (after renaming some variables), so we have to adjust our trial solution to obtain $y_p(t) = ty_p(t) = At\sin(2t) + Bt\cos(2t)$. Noting that

(1147) $$y_p'(t) = A\sin(2t) + 2At\cos(2t) + B\cos(2t) - 2Bt\sin(2t)$$

(1148) $$= (-2Bt + A)\sin(2t) + (2At + B)\cos(2t), \text{ and}$$

(1149) $$y_p''(t) = -2B\sin(2t) + 2(-2Bt + A)\cos(2t) + 2A\cos(2t) - 2(2At + B)\sin(2t)$$

(1150) $$= (-4At - 4B)\sin(2t) + (-4Bt + 4A)\cos(2t).$$

Plugging $y_p(t), y_p'(t),$ and $y_p''(t)$ into (1144), we see that

(1151) $$\cos(2t) = y_p'' + 4y_p$$

(1152)
$$= \underbrace{(-4At - 4B)\sin(2t) + (-4Bt + 4A)\cos(2t)}_{y_p''(t)} + 4\underbrace{At\sin(2t) + Bt\cos(2t)}_{y_p(t)}$$

. . . .

(1153)
$$-4B\sin(2t) + 4A\cos(2t)$$

. . . .

(1154)
$$\rightarrow \begin{array}{rcl} -4B &=& 0 \\ 4A &=& 1 \end{array} \rightarrow (A, B) = (\frac{1}{4}, 0) \rightarrow y_p(t) = \boxed{\frac{1}{4}t\sin(2t)}.$$

Problem 11.12: Find a particular solution to equation (1155).

$$y'' + 4y = t\sin(2t). \tag{1155}$$

Solution: We begin by finding the general solutions of the corresponding homogeneous equation, which is the equation

$$y_c'' + 4y_c = 0. \tag{1156}$$

By examining the characteristic equation for equation (1156), we see that

$$r^2 + 4 = 0 \to r = \pm 2i \to y_c(t) = c_1\cos(2t) + c_2\sin(2t). \tag{1157}$$

Normally, the method of undetermined coefficients would tell us to use the trial solution $y_p(t) = (At + B)\cos(2t) + (Ct + D)\sin(2t)$, since $\{\cos(2t), \sin(2t), t\cos(2t), t\sin(2t)\}$ is a linearly independent set of functions that can 'generate' any function that is a derivative of $t\sin(2t)$. However, since $\cos(2t)$ and $\sin(2t)$ are a solutions to equation (1156), we see that plugging $y_p(t)$ into the left hand side of equation (1155) will result in an expression of the form $E\cos(2t) + F\sin(2t)$, which is not what we want. We consequently have to adjust our trial solution and use $y_p(t) = ty_p(t) = (At^2 + Bt)\cos(2t) + (Ct^2 + Dt)\sin(2t)$. We now see that

$$y_p'(t) = (2At + B)\cos(2t) - 2(At^2 + Bt)\sin(2t) + (2Ct + D)\sin(2t) + 2(Ct^2 + Dt)\cos(2t) \tag{1158}$$

$$= (2Ct^2 + 2At + 2Dt + B)\cos(2t) + (-2At^2 + 2Ct - 2Bt + D)\sin(2t), \text{ and} \tag{1159}$$

$$y_p''(t) = (4Ct + 2A + 2D)\cos(2t) - 2(2Ct^2 + 2At + 2Dt + B)\sin(2t) + (-4At + 2C - 2B)\sin(2t) + 2(-2At^2 + 2Ct - Bt + D)\cos(2t) \tag{1160}$$

$$= (-4At^2 + 8Ct - 4Bt + 2A + 4D)\cos(2t) + (-4Ct^2 - 8At - 4Dt + 2C - 4B)\sin(2t). \tag{1161}$$

We may now plug $y_p(t), y_p'(t)$, and $y_p''(t)$ into equation (1155) in order to solve for A, B, C and D.

$$t\sin(2t) = y_p''(t) + 4y_p(t) \tag{1162}$$

$$= \underbrace{(-4At^2 + 8Ct - 4Bt + 2A + 4D)\cos(2t) + (-4Ct^2 - 8At - 4Dt + 2C - 4B)\sin(2t)}_{y_p''(t)} + 4(\underbrace{(At^2 + Bt)\cos(2t) + (Ct^2 + Dt)\sin(2t)}_{y_p(t)}) \tag{1163}$$

$$= (8Ct + 2A + 4D)\cos(2t) + (-8At + 2C - 4B)\sin(2t) \tag{1164}$$

$$\rightarrow \begin{aligned} 8C &= 0 \text{ (by considering the } t\cos(2t) \text{ term)} \\ 2A + 4D &= 0 \text{ (by considering the } \cos(2t) \text{ term)} \\ -8A &= 1 \text{ (by considering the } t\sin(2t) \text{ term)} \\ 2C - 4B &= 0 \text{ (by considering the } \sin(2t) \text{ term)} \end{aligned} \rightarrow \tag{1165}$$

$$\begin{matrix} C = 0 \\ A = -\frac{1}{8} \end{matrix} \rightarrow \begin{matrix} D = -\frac{1}{2}A = \frac{1}{16} \\ B = \frac{1}{2}C = 0 \end{matrix} \tag{1166}$$

$$\rightarrow y_p(t) = \boxed{-\frac{1}{8}t^2\cos(2t) + \frac{1}{16}t\sin(2t)}. \tag{1167}$$

Problem 11.13: Find the general solution of the following equation and solve the given initial value problem.

$$y'' + y = 4\sin(2t); \quad y(0) = 1, y'(0) = 0. \tag{1168}$$

Draw the graph of the solution and determine the period of the function. (You may seek help from graphing website/software. Think about why the graph behave in that way and how is that related to the solution function.)

Solution: We see that the corresponding homogeneous equation for (1168) is

$$y_c'' + y_c = 0. \tag{1169}$$

By examining the characteristic equation of equation (1169), we see that

$$0 = r^2 + 1 \to r = \pm i \to y_c(t) = c_1' e^{-it} + c_2' e^{it} = c_1 \sin(t) + c_2 \cos(t). \tag{1170}$$

Since $4\sin(2t)$ is unrelated to $y_c(t)$, we may proceed to use the method of undetermined coefficients without any adjustments.Using a trial solution of $y_p(t) = A\sin(2t) + B\cos(2t)$, we observe that $y_p'(t) = 2A\cos(2t) - 2B\sin(2t)$ and $y_p''(t) = -4A\sin(2t) - 4B\cos(2t)$. Plugging y_p, y_p', and y_p'' into equation (1168), we see that

. . . .

$$4\sin(2t) = y_p'' + y_p = \underbrace{-4A\sin(2t) - 4B\cos(2t)}_{y_p''(t)} + \underbrace{A\sin(2t) + B\cos(2t)}_{y_p(t)}. \tag{1171}$$

. . . .

$$= -3A\sin(2t) - 3B\cos(2t) \to \begin{matrix} -3A & = & 4 \\ -3B & = & 0 \end{matrix} \tag{1172}$$

. . . .

$$\to (A, B) = (-\frac{4}{3}, 0) \to y_p(t) = -\frac{4}{3}\sin(2t). \tag{1173}$$

. . . .

Now that we have found a particular solution $y_p(t)$ to equation (1168), we see that $y(t) = y_p(t) + y_c(t)$ is the general solution to equation (1168). After explicitly writing down the general solution $y(t)$, we will make use of the given initial values to finish the given initial value problem.

. . . .

$$\rightarrow y(t) = y_p(t) + y_c(t) = -\frac{4}{3}\sin(2t) + c_1\sin(t) + c_2\cos(t) \tag{1174}$$

. . . .

$$\rightarrow y'(t) = -\frac{8}{3}\cos(2t) + c_1\cos(t) - c_2\sin(t) \tag{1175}$$

. . . .

$$\begin{array}{rcccl} 1 & = & y(0) & = & c_2 \\ 0 & = & y'(0) & = & -\frac{8}{3} + c_1 \end{array} \rightarrow (c_1, c_2) = (\frac{8}{3}, 1) \tag{1176}$$

. . . .

$$\rightarrow y(t) = \boxed{-\frac{4}{3}\sin(2t) + \frac{8}{3}\sin(t) + \cos(t)}. \tag{1177}$$

. . . .

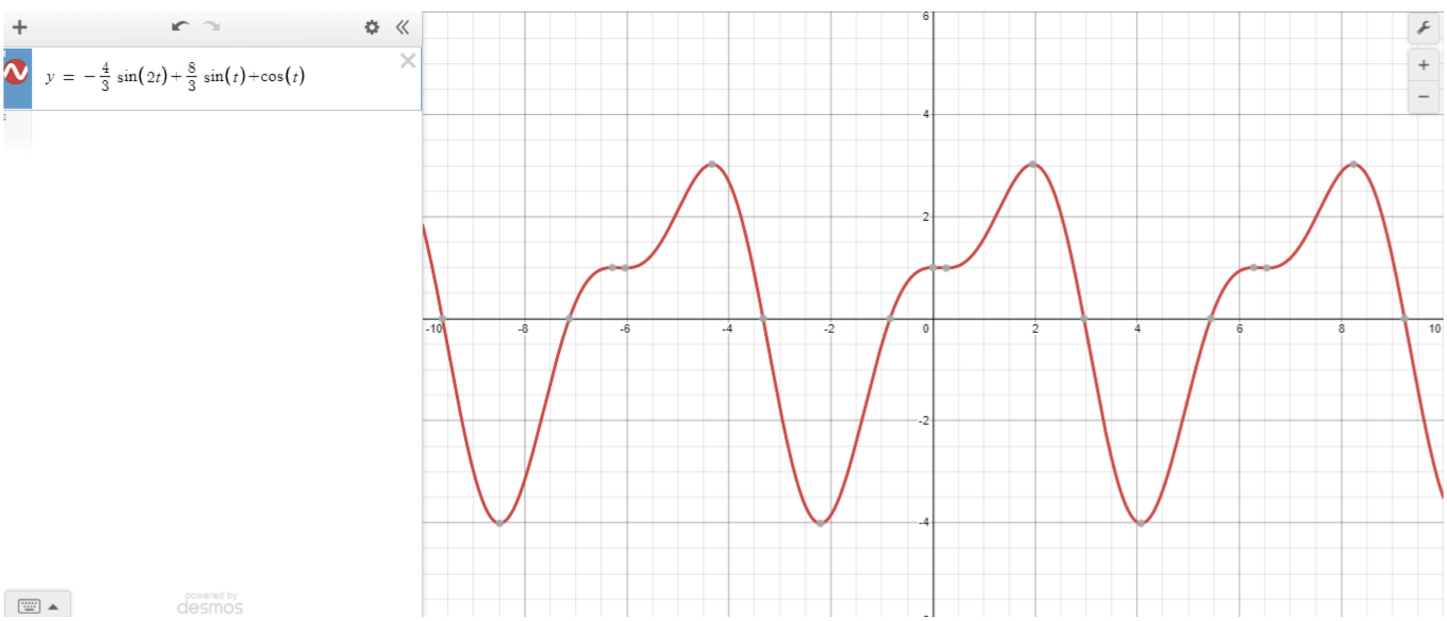


FIGURE 60. A graph of the solution to the initial value problem given in (1168)

It is clear that $y(t+2\pi) = y(t)$, so $y(t)$ is periodic with a period of at most 2π. Based on the graph of $y(t)$, we see that the period of $y(t)$ is not smaller than 2π, so the period of $y(t)$ must be exactly 2π.

Problem 11.14: Use the method of undetermined coefficients to find the general solution to the differential equation

(1178) $$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t).$$

Solution: We will first find a particular solution $y_1(t)$ for

(1179) $$y'' + 3y' = 2t^4,$$

a particular solution $y_2(t)$ for

(1180) $$y'' + 3y' = t^2e^{-3t},$$

and a particular solution $y_3(t)$ for

(1181) $$y'' + 3y' = \sin(3t).$$

Once $y_1(t), y_2(t)$, and $y_3(t)$ are all found, the linearity of equation (1178) lets us see that $y_1(t) + y_2(t) + y_3(t)$ is a particular solution of (1178). To find $y_1(t)$ we begin with

(1182) $$y_1(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$$

but we then notice that $y(t) = 1$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (1178), so we have to modify this initial guess to become

(1183) $$y_1(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t.$$

Since

(1184) $$y_1'(t) = 5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1 \text{ and}$$

(1185) $$y_1''(t) = 20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2,$$

we see that

$$2t^4 = y_1'' + 3y_1' \tag{1186}$$

$$= (20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2) + 3(5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1) \tag{1187}$$

$$= 15a_5t^4 + (12a_4 + 20a_5)t^3 + (9a_3 + 12a_4)t^2 + (6a_2 + 6a_3)t + (3a_1 + 2a_2) \tag{1188}$$

$$\rightarrow \begin{array}{ccccc} & & 15a_5 & = & 2 \\ 12a_4 & + & 20a_5 & = & 0 \\ 9a_3 & + & 12a_4 & = & 0 \\ 6a_2 & + & 6a_3 & = & 0 \\ 3a_1 & + & 2a_2 & = & 0 \end{array} \tag{1189}$$

$$\rightarrow (a_1, a_2, a_3, a_4, a_5) = (\frac{16}{81}, -\frac{8}{27}, \frac{8}{27}, -\frac{2}{9}, \frac{2}{15}). \tag{1190}$$

To find $y_2(t)$ we begin with

$$y_2(t) = (a_0 + a_1t + a_2t^2)e^{-3t} \tag{1191}$$

but we then notice that $y(t) = e^{-3t}$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (1178), so we have to modify this initial guess to become

$$y_2(t) = (a_1t + a_2t^2 + a_3t^3)e^{-3t}. \tag{1192}$$

Since

$$y_2'(t) = (a_1t + a_2t^2 + a_3t^3)'e^{-3t} + (a_1t + a_2t^2 + a_3t^3)(-3e^{-3t}) \tag{1193}$$

$$= (a_1 + 2a_2t + 3a_3t^2)e^{-3t} + (-3a_1t - 3a_2t^2 - 3a_3t^3)e^{-3t} \tag{1194}$$

$$= \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)e^{-3t} \text{ and} \tag{1195}$$

$$y_2''(t) = \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)' e^{-3t} + \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)(-3e^{-3t}) \tag{1196}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} + \left(-3a_1 + (9a_1 - 6a_2)t + (9a_2 - 9a_3)t^2 + 9a_3t^3\right) e^{-3t} \tag{1197}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t}, \tag{1198}$$

we see that

$$t^2e^{-3t} = y_2'' + 3y_2' \tag{1199}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t} + 3\left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right) e^{-3t} \tag{1200}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} \tag{1201}$$

$$\rightarrow \begin{array}{rcrcl} & & -9a_3 & = & 1 \\ -6a_2 & + & 6a_3 & = & 0 \\ -3a_1 & + & 2a_2 & = & 0 \end{array} \rightarrow (a_1, a_2, a_3) = (-\frac{2}{27}, -\frac{1}{9}, -\frac{1}{9}). \tag{1202}$$

Lastly, to find $y_3(t)$ we use

$$y_3(t) = A\sin(3t) + B\cos(3t). \tag{1203}$$

Since

$$y_3'(t) = 3A\cos(3t) - 3B\sin(3t) \text{ and} \tag{1204}$$

$$y_3''(t) = -9A\sin(3t) - 9B\cos(3t), \tag{1205}$$

we see that

$$\begin{aligned}\sin(3t) = y_3'' + 3y_3' = (-9A\sin(3t) - 9B\cos(3t)) \\ + 3(3A\cos(3t) - 3B\sin(3t))\end{aligned} \tag{1206}$$

$$= (-9A - 9B)\sin(3t) + (9A - 9B)\cos(3t) \tag{1207}$$

$$\to \begin{array}{rcrcl} -9A & - & 9B & = & 1 \\ 9A & - & 9B & = & 0 \end{array} \to (A, B) = (-\frac{1}{18}, -\frac{1}{18}). \tag{1208}$$

Recalling that the general solution to the equation

$$y'' + 3y' = 0 \tag{1209}$$

is given by $y(t) = c_1 + c_2e^{-3t}$, we see that the general solution to equation (1178) is

$$\begin{aligned}y(t) = c_1 + c_2e^{-3t} - \frac{2}{27}te^{-3t} - \frac{1}{9}t^2e^{-3t} - \frac{1}{9}t^3e^{-3t} \\ + \frac{16}{81}t - \frac{8}{27}t^2 + \frac{8}{27}t^3 - \frac{2}{9}t^4 + \frac{2}{15}t^5 - \frac{1}{18}\sin(3t) - \frac{1}{18}\cos(3t).\end{aligned} \tag{1210}$$

Remark: In the beginning, we could have also directly guessed that the general form of a particular solution is

$$\begin{aligned}y(t) = (c_1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) \\ + (c_2 + b_1t + b_2t^2 + b_3t^3)e^{-3t} + A\sin(3t) + B\cos(3t),\end{aligned} \tag{1211}$$

but when attempting to calculate the coefficients by hand (instead of using a computer algebra system) it is useful to break up the work into smaller chunks as we did here.

Problem 11.15: Solve the initial value problem

(1212) $$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}, \quad y(\pi) = 0,\ t > 0.$$

Solution: We see that this first order differential equation is given to us in the standard form of

(1213) $$y' + p(t)y = g(t),$$

so our integrating factor is just

(1214) $$\nu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln(t)} = t^2,$$

where we have chosen the constant of integration to be 0 for convenience. Multiplying both sides of equation (1212) by our integrating factor $\nu(t)$ gives us

(1215) $$\cos(t) = t^2y' + 2ty = (t^2y)'$$

(1216) $$\to t^2y = \int \cos(t)dt = \sin(t) + C$$

(1217) $$\to y(t) = y = \frac{\sin(t) + C}{t^2}.$$

We will now use our initial condition of $y(\pi) = 0$ in order to solve for the constant C. We see that

(1218) $$0 = y(\pi) = \frac{\sin(\pi) + C}{\pi^2} = \frac{C}{\pi^2} \to C = 0.$$

In conclusion, we see that the solution to the initial value problem is

(1219) $$\boxed{\frac{\sin(t)}{t^2}, \quad t > 0}.$$

Problem 11.16: Show that if a and λ are positive constants and b is any real number, then every solution of the equation

(1220) $$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.

Solution: Just as in problem 2.1.16, we see that the differential equation is already given to us in standard form, so our integrating factor is

(1221) $$\nu(t) = e^{\int a dt} = e^{at},$$

where we have once again chosen our constant of integration to be 0 for convenience. Multiplying both sides of equation (1221) by our integrating factor $\nu(t)$ gives us

(1222) $$be^{(a-\lambda)t} = be^{-\lambda t}e^{at} = e^{at}y' + ae^{at}y = (e^{at}y)'$$

(1223) $$\to e^{at}y = \int be^{(a-\lambda)t}dt = \begin{cases} \frac{b}{a-\lambda}e^{(a-\lambda)t} + C & \text{if } a \neq \lambda \\ bt + C & \text{if } a = \lambda \end{cases}$$

(1224) $$y(t) = y = \begin{cases} \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at} & \text{if } a \neq \lambda \\ bte^{-at} + Ce^{-at} & \text{if } a = \lambda \end{cases}.$$

Since $a > 0$, we see that

(1225) $$\lim_{t\to\infty} Ce^{-at} = \lim_{t\to\infty} bte^{-at} = 0,$$

so when $a = \lambda$ we have

(1226) $$\lim_{t\to\infty} y(t) = 0.$$

Similarly, since $\lambda > 0$, we see that if $a \neq \lambda$ then

(1227) $$\lim_{t\to\infty} \frac{b}{a-\lambda}e^{-\lambda t} = 0,$$

which shows us that in this case we also have

$$\lim_{t\to\infty} y(t) = 0. \tag{1228}$$

Problem 11.17: Solve the initial value problem

$$y' = \frac{3x^2 - e^x}{2y - 5}, \quad y(0) = 1. \tag{1229}$$

Solution: This differential equation is not linear, but it is separable, so we will separate the variables and integrate in order to solve it. In this case, all we have to do to separate the variables is multiple both sides of equation (1229) by $(2y - 5)$ to obtain

$$(2y - 5)y' = 3x^2 - e^x \tag{1230}$$

$$\rightarrow (2y - 5)dy = (3x^2 - e^x)dx \tag{1231}$$

$$\int (2y - 5)dy = \int (3x^2 - e^x)dx \tag{1232}$$

$$y^2 - 5y = x^3 - e^x + C. \tag{1233}$$

To solve for C, we use the initial condition $y(0) = 1$ to obtain

$$1^2 - 5 \times 1 = 0^3 - e^0 + C \tag{1234}$$

$$\rightarrow C = 1 - 5 + e^0 = -3 \tag{1235}$$

$$\rightarrow y^2 - 5y = x^3 - e^x - 3. \tag{1236}$$

We currently have an implicit relationship betwen x and y. Luckily, in this case we can just apply the quadratic formula to obtain an explicit relationship between x and y. We see that

$$y^2 - 5y + (e^x + 3 - x^3) = 0 \tag{1237}$$

$$\rightarrow y = \frac{5 \pm \sqrt{25 - 4(e^x + 3 - x^3)}}{2} = \frac{5 \pm \sqrt{13 - 4e^x + 4x^3}}{2}. \tag{1238}$$

Recalling that $y(0) = 1$, we see that

$$\boxed{y(x) = \frac{5 - \sqrt{13 - 4e^x + 4x^3}}{2}}. \tag{1239}$$

We see that the solution is defined when

$$13 - 4e^x + 4x^3 \geq 0. \tag{1240}$$

We see that inequality (1240) holds when $x \in (-1, 1)$ (the details of this are left as an exercise to the reader), so we know that our solution exists on this interval. The solution actually exists on an interval larger than $(-1, 1)$, but it is difficult to calculate the entire interval on which the solution exists, so we will settle for this approximation.

Problem 11.18:
Part a: Verify that $y_1(t) = 1 - t$ and $y_2(t) = -\frac{t^2}{4}$ are both solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1. \tag{1241}$$

Where are these solutions valid?

Part b: Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2 of the 10th edition of 'Elementary Differential Equations' by W.E. Boyce and R.C. DiPrima.

Part c: Show that $y(t) = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, then the initial condition is also satisfied and the solution $y = y_1(t)$ is obtained. Show that no other choice of c gives a second solution. Note that no choice of c gives the solution $y = y_2(t)$.

Solution to (a): We see that $y_1(2) = y_2(2) = -1$. We also see that

$$y_1' = -1 \text{ and} \tag{1242}$$

$$\frac{-t + \sqrt{t^2 + 4(1-t)}}{2} = \frac{-t + \sqrt{t^2 - 4t + 4}}{2} = \frac{-t + \sqrt{(t-2)^2}}{2} \tag{1243}$$

$$\stackrel{*}{=} \frac{-t + (t-2)}{2} = -1, \tag{1244}$$

so $y_1(t)$ is indeed a solution to the initial value problem in equation (1241) that is valid for $t \in [2, \infty)$ (as seen from equation (*)). Lastly, we see that

$$y_2' = -\frac{t}{2} \text{ and} \tag{1245}$$

$$\frac{-t + \sqrt{t^2 + 4(-\frac{t^2}{4})}}{2} = \frac{-t}{2}, \tag{1246}$$

so $y_2(t)$ is also a solution to the initial value problem in equation (1241) that is valid for all $t \in (-\infty, \infty)$.

Solution to (b): We see that in this problem we have

$$f = f(t,y) = \frac{-t + \sqrt{t^2 + 4y}}{2}, \tag{1247}$$

so

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{t^2 + 4y}}. \tag{1248}$$

Since $\frac{\partial f}{\partial y}(2, -1)$ is not defined, $\frac{\partial f}{\partial y}$ is not continuous in any open rectangle containing $(2, -1)$, so the conditions of Theorem 2.4.2 are not satisfied, which means that we cannot apply the uniqueness part of Theorem 2.4.2.

Solution to (c): Letting c be any real number and letting $y(t) = ct + c^2$ we see that

$$y' = c \text{ and} \tag{1249}$$

$$\frac{-t + \sqrt{t^2 + 4(ct + c^2)}}{2} = \frac{-t + \sqrt{t^2 + 4ct + 4c^2}}{2} = \frac{-t + \sqrt{(t + 2c)^2}}{2} \tag{1250}$$

$$\overset{*}{=} \frac{-t + t + 2c}{2} = c, \tag{1251}$$

so $y(t)$ is a solution to the differential equation in (1241). In order to satisfy the initial condition of $y(2) = -1$, we see that we must have

$$-1 = 2c + c^2 \to 0 = 1 + 2c + c^2 = (1 + c)^2 \to c = -1. \tag{1252}$$

When $c = -1$, we see that we do indeed recover the solution $y_1(t)$. Furthermore, we see that $y_2(t)$ is a solution to the initial value problem in equation (1241) that does not come from $y(t)$ for any choice of c.

Problem 11.19: Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}. \tag{1253}$$

Solution: Letting

$$F(x,y) = \frac{x^2+xy+y^2}{x^2}, \tag{1254}$$

we see that for any real number c we have

$$F(cx,cy) = \frac{(cx)^2+(cx)(cy)+(cy)^2}{(cx)^2} = \frac{c^2x^2+c^2xy+c^2y^2}{c^2x^2} \tag{1255}$$

$$= \frac{x^2+xy+y^2}{x^2} = F(x,y), \tag{1256}$$

so equation (1254) is a homogeneous equation. Letting $v = \frac{y}{x}$, we see that

$$v' = \frac{dv}{dx} = \frac{y'}{x} - \frac{y}{x^2} = \frac{y'}{x} - \frac{v}{x} \tag{1257}$$

$$\rightarrow xv' + v = y'. \tag{1258}$$

We may now rewrite equation (1254) as a differential equation in v. Observe that

$$xv' + v = y' = \frac{x^2+xy+y^2}{x^2} = \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2} \tag{1259}$$

$$= 1 + \frac{y}{x} + (\frac{y}{x})^2 = 1 + v + v^2 \tag{1260}$$

$$\rightarrow xv' = 1 + v^2. \tag{1261}$$

We see that equation (1261) is a separable differential equation, so we may go ahead and solve it by separating the variables. We see that

$$\frac{dv}{1+v^2} = \frac{dx}{x} \tag{1262}$$

(1263) $$\rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

(1264) $$\rightarrow \tan^{-1}(v) = \ln(x) + C.$$

(1265) $$\rightarrow \tan^{-1}(\frac{y}{x}) = \ln(x) + C$$

(1266) $$\rightarrow \frac{y}{x} = \tan(\ln(x) + C)$$

(1267) $$\rightarrow \boxed{y(x) = y = x\tan(\ln(x) + C)},$$

Since there were no initial values, we did not need to solve for C, but we do need to find an interval on which the solution is valid. We see that we need $x \neq 0$ in order for equation (1254) to be well defined, $x > 0$ in order for the $\ln(x)$ in equation (1267) to be well defined, and we need $\ln(x) + C$ to be contained between 2 consecutive odd multiples of $\frac{\pi}{2}$ in order for the tan in equation (1267) to be well defined. This last conditions results in the following calculations.

(1268)
$$\ln(x) + C \in (\frac{2n-1}{2}\pi, \frac{2n+1}{2}\pi) \Leftrightarrow \ln(x) \in (\frac{2n-1}{2}\pi - C, \frac{2n+1}{2}\pi - C)$$

(1269) $$\Leftrightarrow \boxed{x \in (e^{\frac{2n-1}{2}\pi - C}, e^{\frac{2n+1}{2}\pi - C}) \text{ (for some integer n)}}.$$

Problem 11.20: Find and classify (stable, unstable, semistable) the equilibrium points of the differential equation

$$\frac{dy}{dt} = y(1 - y^2), \quad -\infty < y_0 < \infty. \tag{1270}$$

Solution: We first examine the direction field of equation (1270) and examine some integral curves.

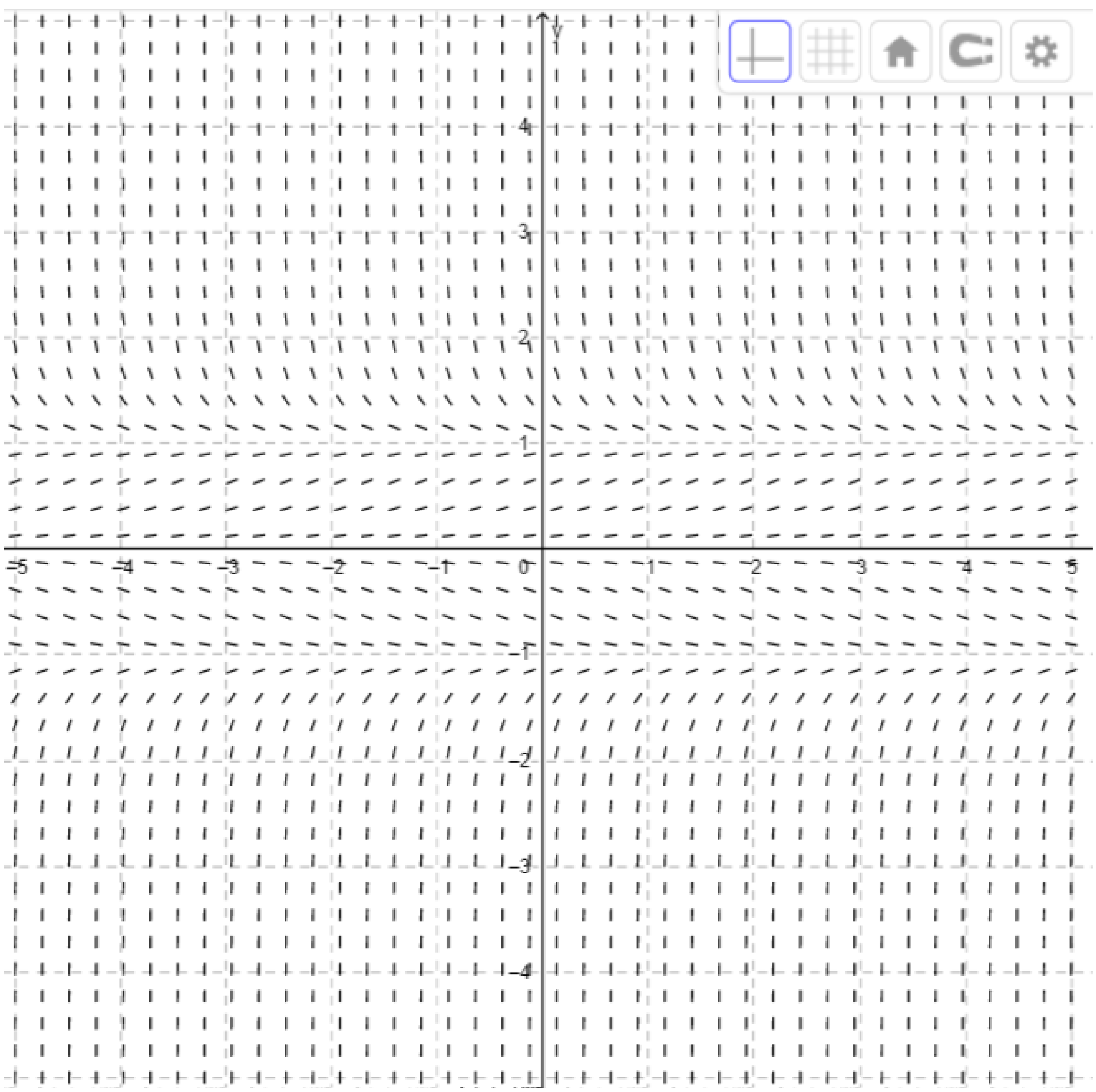


FIGURE 61. The direction field for equation (1270).

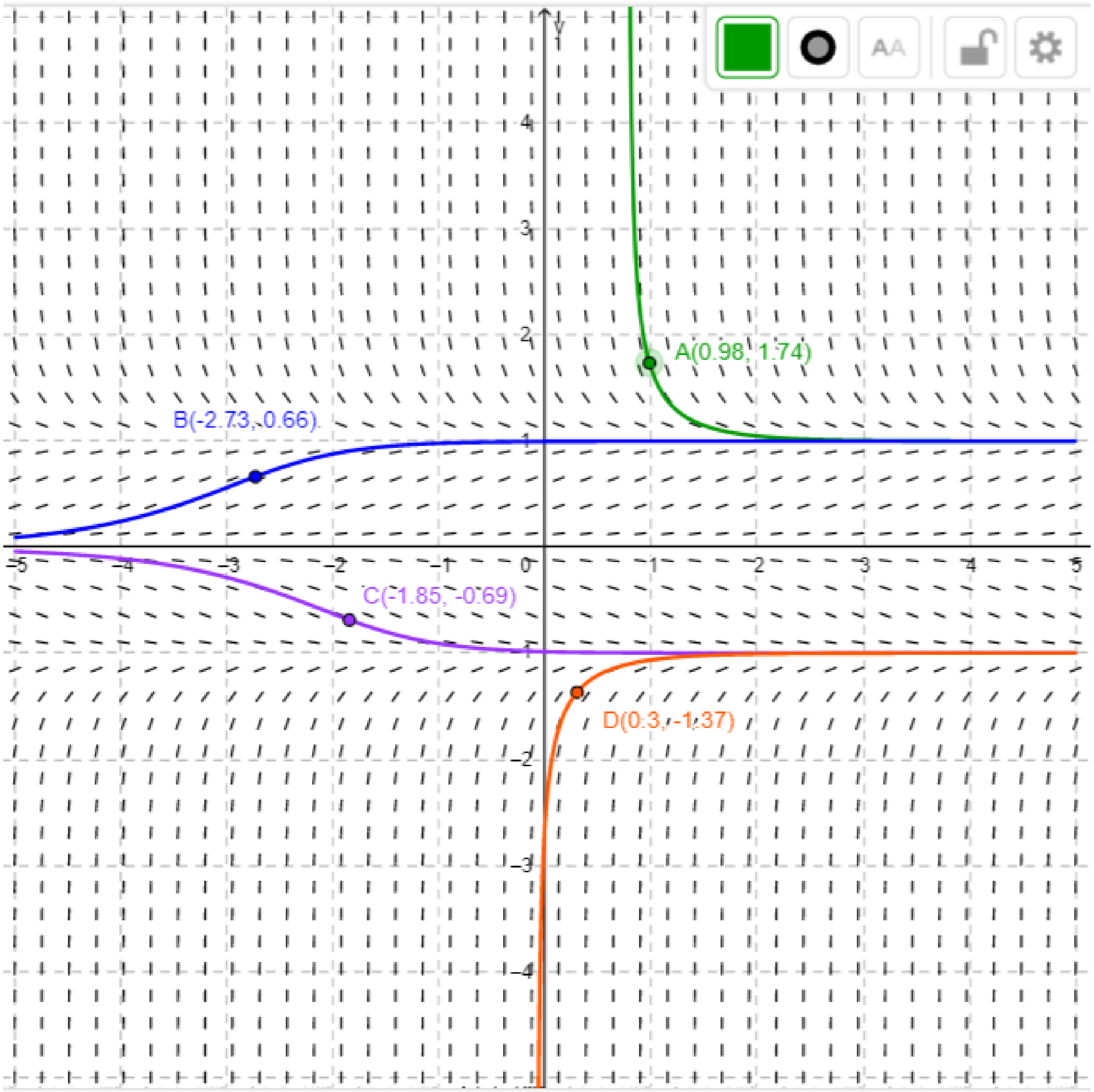


FIGURE 62. Some integral curves for equation (1270).

From the integral curves, we see that $y = -1$ and $y = 1$ are stable equilibrium points and $y = 0$ is an unstable equilibrium point. We will now rigorously verify that this is the case. Let $f(y) = y(1 - y^2)$.

We see that if $y > 1$ then $f(y) < 0$ and if $0 < y < 1$ then $f(y) > 0$. Since $y' = f(y)$, we see that when y is larger than 1, it will decrease, and when y is between 0 and 1 it will increase, so $y = 1$ is a stable equilibrium point.

Similarly, when $y < -1$ we have $f(y) > 0$ and when $-1 < y < 0$ we have $f(y) < 0$. Since $y' = f(y)$, we see that when y is smaller than -1 it will increase, and when y is between -1 and 0 it will decrease, so $y = -1$ is a stable equilibrium point.

Lastly, to see that $y = 0$ is an unstable equilibrium point, we simply recall that if y is between 0 and 1 then it will increase towards 1, and if y is between -1 and 0 then it will decrease towards -1.

Problem 11.21: Find and classify (stable, unstable, semistable) the equilibrium points of the differential equation

$$\frac{dy}{dt} = y^2(4 - y^2), \quad -\infty < y_0 < \infty. \tag{1271}$$

Solution: We first examine the direction field of equation (1271) and examine some integral curves.

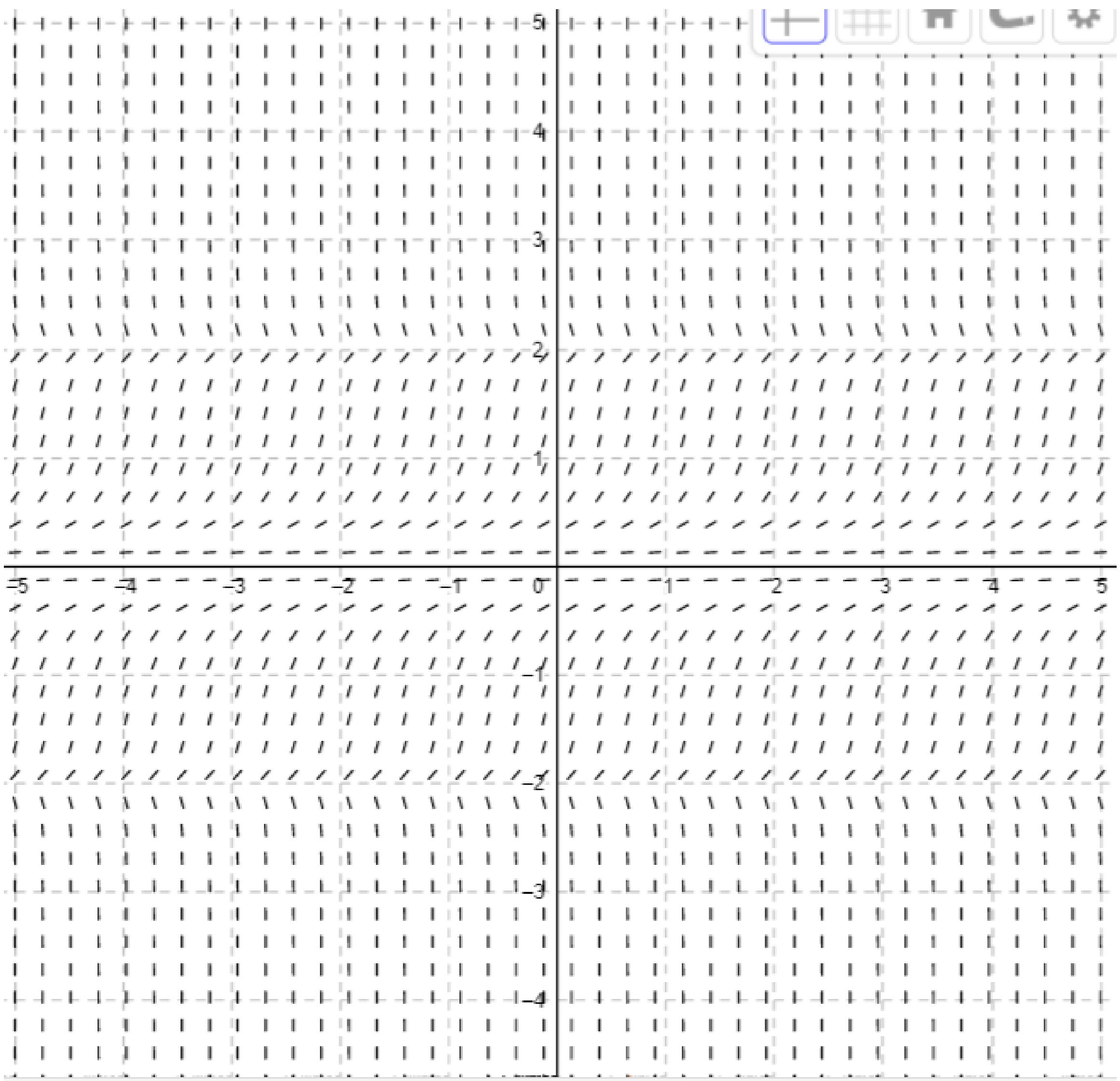


FIGURE 63. The direction field for equation (1271)

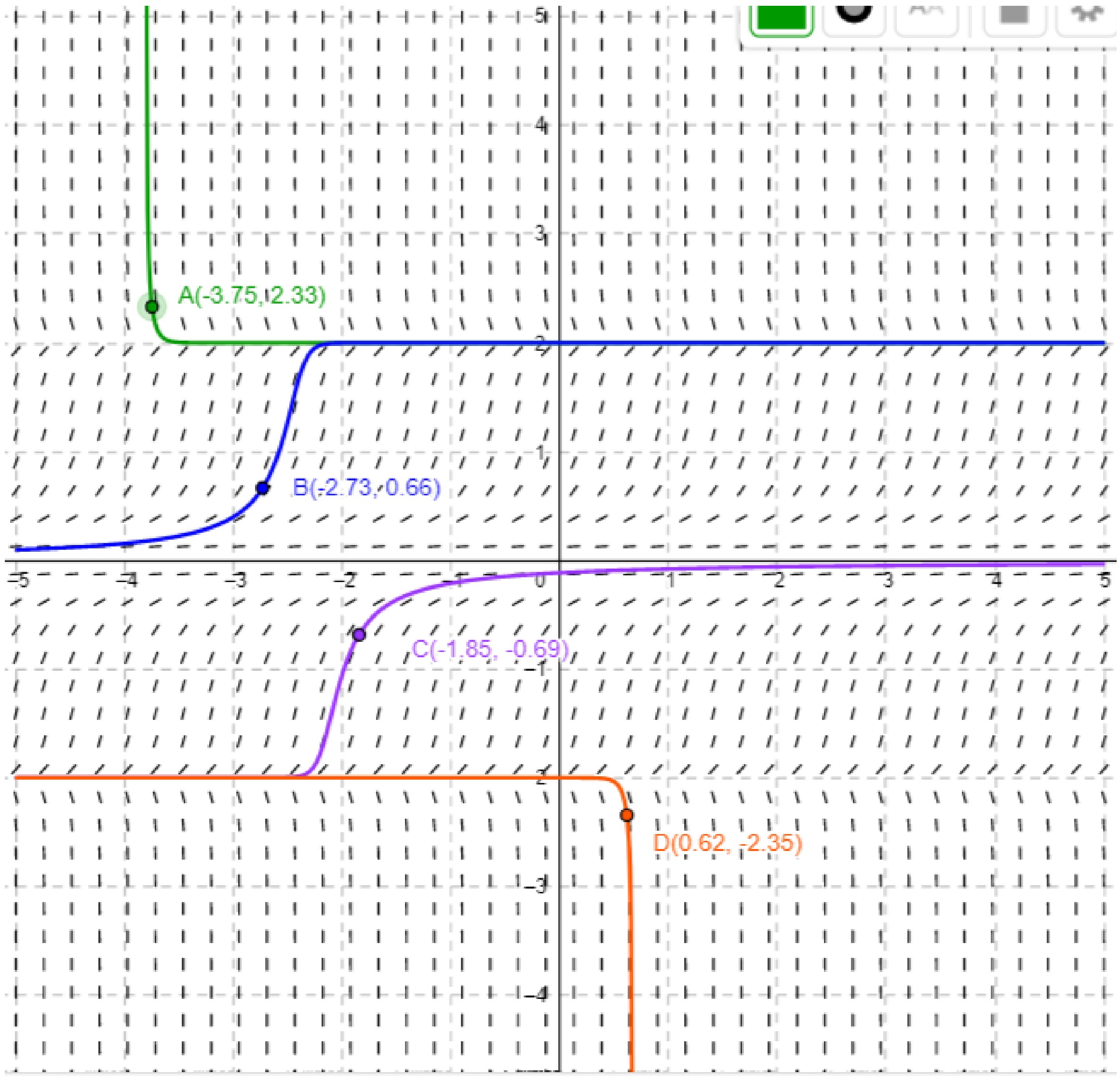


FIGURE 64. Some integral curves for equation (1271)

From the integral curves, we see that $y = 2$ is a stable equilibrium point, $y = 0$ is a semistable equilibrium point and $y = -2$ is an unstable equilibrium point. We will now rigorously verify that this is the case. Let $f(y) = y^2(4 - y^2)$.

We see that if $y > 2$ then $f(y) < 0$ and if $0 < y < 2$ then $f(y) > 0$. Since $y' = f(y)$, we see that when y is larger than 2, it will decrease, and when y is between 0 and 2 it will increase, so $y = 2$ is a stable equilibrium point.

Similarly, when $y < -2$ we have $f(y) < 0$ and when $-2 < y < 0$ we have $f(y) > 0$. Since $y' = f(y)$, we see that when y is smaller than -2 it will decrease, and when y is between -2 and 0 it will increase, so $y = -2$ is an unstable equilibrium point.

Lastly, to see that $y = 0$ is a semistable equilibrium point, we simply recall that if y is between 0 and 2 then it will increase towards 2, and if y is between -2 and 0 then it will decrease towards 0.

Problem 11.22: Find and classify (stable, unstable, semistable) the equilibrium points of the differential equation

$$\frac{dy}{dt} = y^2(1-y)^2, \quad -\infty < y_0 < \infty. \tag{1272}$$

Solution: We first examine the direction field of equation (1272) and examine some integral curves.

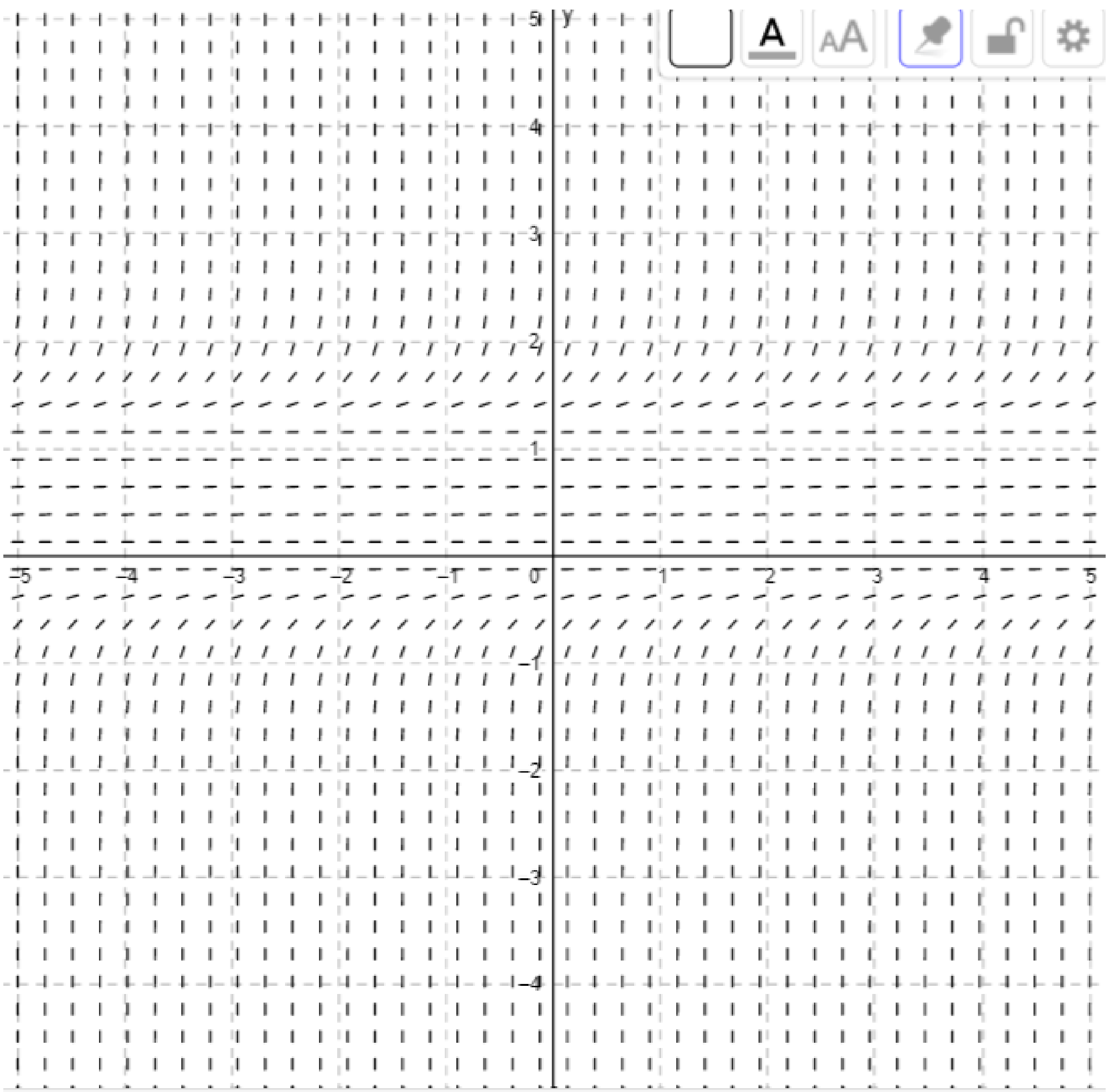


FIGURE 65. The direction field for equation (1272).

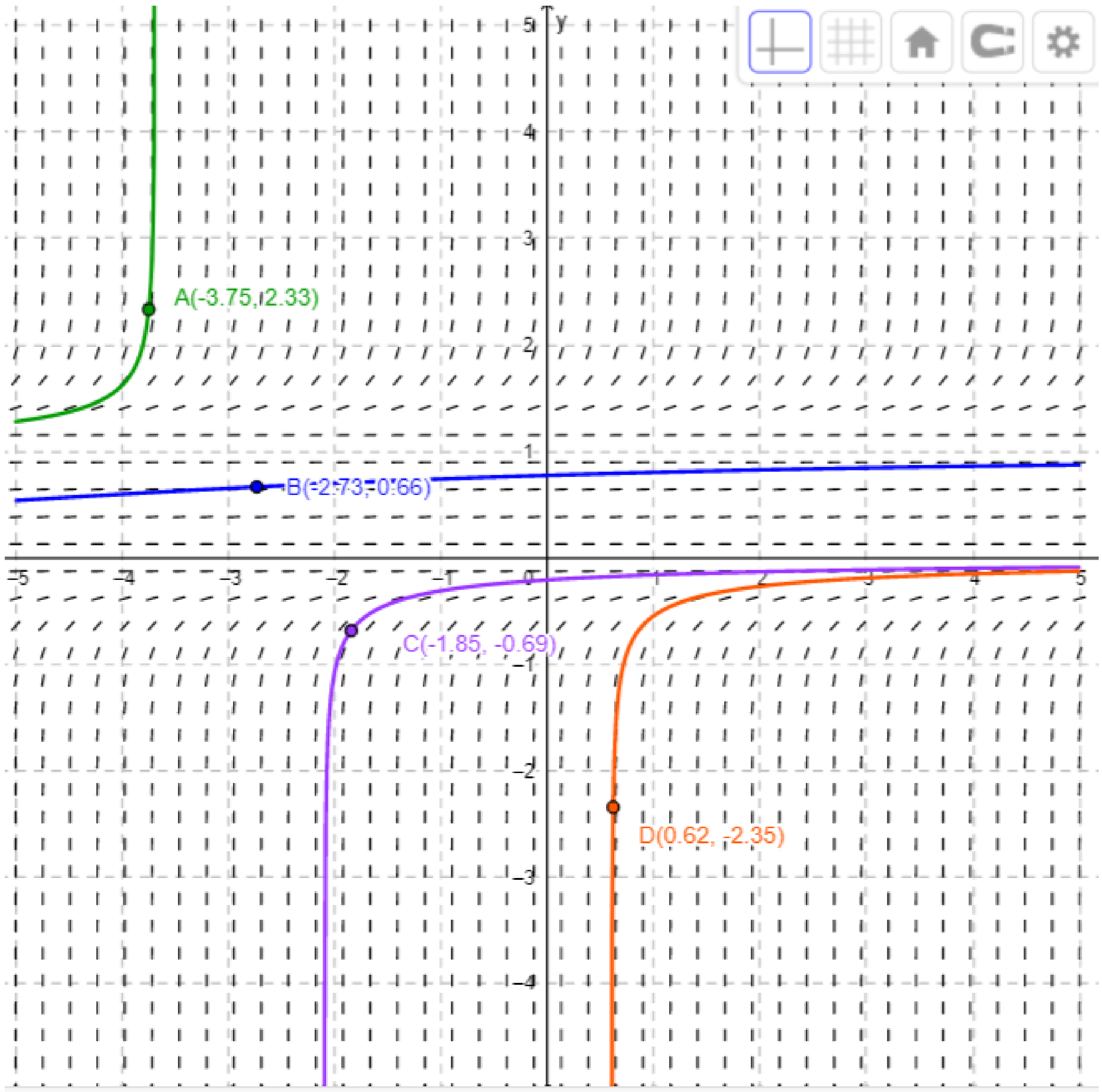


FIGURE 66. Some integral curves for equation (1272).

From the integral curves, we see that $y = 0$ and $y = 1$ are both semistable equilibrium points. We will now rigorously verify that this is the case. Let $f(y) = y^2(1-y)^2$.

We see that if $y > 1$ then $f(y) > 0$ and if $0 < y < 1$ then $f(y) > 0$. Since $y' = f(y)$, we see that when y is larger than 1, it will increase, and when y is between 0 and 1 it will increase, so $y = 1$ is a semistable equilibrium point.

Similarly, when $y < 0$ we have $f(y) > 0$. Since $y' = f(y)$, we see that when y is smaller than 0 it will increase, and we already saw that if y is between 0 and 1 it will increase, so $y = 0$ is also a semistable equilibrium point.

Lastly, to see that $y = 0$ is a semistable equilibrium point, we simply recall that if y is between 0 and 2 then it will increase towards 2, and if y is between -2 and 0 then it will decrease towards 0.

Problem 11.23: Solve the following initial value problem and find **an** interval on which the solution is valid.

$$(2x - y) + (2y - x)y' = 0, \quad y(1) = 3. \tag{1273}$$

Solution: First, we will check whether or not equation (1273) is an exact equation. Letting $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$, we see that $M_y(x,y) = -1 = N_x(x,y)$, so (1273) is an exact equation. This means that there exists a function $\psi(x,y)$ for which $\psi_x(x,y) = M(x,y)$ and $\psi_y(x,y) = N(x,y)$. We now see that

$$\psi(x,y) = \int M(x,y)dx + h(y) = \int (2x-y)dx + h(y) = x^2 - xy + h(y) \tag{1274}$$

$$\rightarrow 2y - x = N(x,y) = \psi_y(x,y) = -x + h'(y) \tag{1275}$$

$$\rightarrow h'(y) = 2y \rightarrow h(y) = y^2 + c_1 \tag{1276}$$

$$\rightarrow \psi(x,y) = x^2 - xy + y^2 + c_1. \tag{1277}$$

If $y = y(x)$ is a solution to equation (1273), then

$$0 = (2x - y(x)) + (2y(x) - x)\, y'(x) = M\,(x, y(x)) + N\,(x, y(x))\, y'(x) \tag{1278}$$

$$= \psi_x\,(x, y(x)) + \psi_y\,(x, y(x))\, y'(x) = \frac{d}{dx}\psi\,(x, y(x)) \tag{1279}$$

$$\rightarrow \psi\,(x, y(x)) = c_2 \rightarrow x^2 - xy + y^2 = c_3 := c_2 - c_1. \tag{1280}$$

To determine the value of c_3, we simply use the initial condition of $y(1) = 3$ to see that

$$c_3 = 1^2 - 1 \cdot 3 + 3^2 = 7. \tag{1281}$$

It follows the the implicit relationship between x and y is given by

$$x^2 - xy + y^2 = 7. \tag{1282}$$

Luckily, in this case we can explicitly solve for y in terms of x by using the quadratic formula. We note that

$$y^2 + (-x)y + (x^2 - 7) = 0 \tag{1283}$$

$$\rightarrow y(x) = y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2} = \frac{x \pm \sqrt{28 - 3x^2}}{2}. \tag{1284}$$

Once again recalling that $y(1) = 3$, we see that

$$\boxed{y = \frac{x + \sqrt{28 - x^2}}{2}}. \tag{1285}$$

We see that the solution is well defined for $x \in [-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}]$, and that all of the terms of equation (1273) are well defined on the interval $(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}})$ (consider y'), so our solution is valid on $\boxed{(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}})}$.

Remark: We see that in equation (1276) we could have simply taken $c_1 = 0$ and $c_2 = c_3$ so that we only ever have to manage 1 constant term. In the future, we will do this.

Problem 11.24: Find the general solution of the differential equation

$$1+\left(\frac{x}{y}-\sin(y)\right)y'=0. \tag{1286}$$

Solution: We begin by checking whether or not equation (1286) is an exact equation. Letting $M(x,y)=1$ and $N(x,y)=\frac{x}{y}-\sin(y)$, we see that $M_y(x,y)=0\neq\frac{1}{y}=N_x(x,y)$. However, we see that $M_y(x,y)-N_x(x,y)=-\frac{1}{y}$. Since $M_y(x,y)-N_x(x,y)$ is a function of a single variable, we can multiply equation (1286) by an integrating factor $\mu(x,y)$ to turn it into an exact equation. We recall that an integrating factor $\mu(x,y)$ satisfies equation (26) of chapter 2.6 of the textbook, namely,

$$M\mu_y-N\mu_x+(M_y-N_x)\mu=0. \tag{1287}$$

Since M_y-N_x is a function only of y, we can make equation (1287) a separable equation if we set $\mu_x=0$, which will happen if we assume that $\mu=\mu(y)$ is a function only of y instead of a function of x and y. After making this assumption, we see that

$$0=M\mu_y+(M_y-N_x)\mu=\mu_y-\frac{1}{y}\mu \tag{1288}$$

$$\to\frac{1}{y}=\frac{\mu_y}{\mu}=\frac{\frac{d\mu}{dy}}{\mu} \tag{1289}$$

$$\to\frac{dy}{y}=\frac{d\mu}{\mu}\to\int\frac{dy}{y}=\int\frac{d\mu}{\mu} \tag{1290}$$

$$\to\ln(y)=\ln(u)+c\to y=A\mu. \tag{1291}$$

Since we can take μ to be any solution of equation (1287), we may set $A=1$ to obtain $\mu=y$. Multiplying both sides of equation (1286) by y yields

$$y+(x-y\sin(y))\,y'=0, \tag{1292}$$

which is easily checked to be an exact equation. We now proceed as we did in problem 2.6.13. We see that

$$\psi(x,y) = \int M(x,y)dx + h(y) = \int ydx + h(y) = xy + h(y) \tag{1293}$$

$$\rightarrow x - y\sin(y) = N(x,y) = \psi_y(x,y) = x + h'(y) \tag{1294}$$

$$\rightarrow h'(y) = -y\sin(y) \rightarrow h(y) = y\cos(y) - \sin(y) \tag{1295}$$

$$\rightarrow \psi(x,y) = xy + y\cos(y) - \sin(y). \tag{1296}$$

Using the same reasoning as in equations (1278)-(1280) from problem 2.6.13, we see that there is a constant c for which

$$\boxed{c = \psi(x,y) = xy + y\cos(y) - \sin(y)}. \tag{1297}$$

We settle for the implicit solution since we are not able to explicitly solve for y in terms of x.

Problem 11.25: Use Euler's method to approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$, and 0.4 with $h = 0.1$.

(1298) $$y' = 0.5 - t + 2y, \quad y(0) = 1.$$

Solution: We apply Euler's method as instructed.

(1299) $$y(0.1) \approx y(0) + 0.1 \cdot y'(0) = 1 + 0.1 \cdot (0.5 - 0 + 2 \cdot 1) = 1.25.$$

(1300) $$y(0.2) \approx y(\underbrace{0.1}_{t}) + \underbrace{0.1}_{h} \cdot y'(\underbrace{0.1}_{t}) \approx 1.25 + \underbrace{0.1}_{h} \cdot (0.5 - \underbrace{0.1}_{t} + 2 \cdot 1.25) = 1.54.$$

(1301) $$y(0.3) \approx y(0.2) + 0.1 \cdot y'(0.2) \approx 1.54 + 0.1 \cdot (0.5 - 0.2 - 2 \cdot 1.54) = 1.878.$$

(1302) $$y(0.4) \approx y(0.3) + 0.1 \cdot y'(0.3) \approx 1.878 + 0.1 \cdot (0.5 - 0.3 + 2 \cdot 1.878) = 2.2736.$$

Problem 11.26: A homebuyer takes out a mortgage of \$100,000 with an interest rate of 9%. What monthly payment is required to pay off the loan in 30 years? In 20 years? What is the total amount paid during the term of the loan in each of these cases?

Solution: We will assume that at the end of the month the interest is applied first, and the monthly payment is paid afterwards. Let p denote the monthly payment of the homebuyer and let u_n denote the debt that remains at the end of the n^{th} month after the interest has been applied and the monthly payment has been paid. By convention, we set $u_0 = \$100,000$. We see that the sequence u_n satisfies the recurrence relation

$$u_{n+1} = (1 + \frac{0.09}{12})u_n - p = 1.0075u_n - p, \quad \text{for } n \geq 0. \tag{1303}$$

Please note that if interest was applied after the monthly payment is paid, then we would instead have the recurrence

$$u_{n+1} = (1 + \frac{0.09}{12})(u_n - p) = 1.0075u_n - 1.0075p, \quad \text{for } n \geq 0. \tag{1304}$$

For convenience, let $r = 1.0075$. This step is not necessary, but I think that it makes the work we are about to do much cleaner and more understandable. In order to find a general formula for u_n, let us calculate u_1, u_2, and u_3 to see if we can detect a pattern. We see that

$$u_1 = ru_0 - p, \tag{1305}$$

$$u_2 = ru_1 - p = r(ru_0 - p) - p = r^2u_0 - rp - p, \text{ and} \tag{1306}$$

$$u_3 = ru_2 - p = r(r^2u_0 - rp - p) - p = r^3u_0 - r^2p - rp - p. \tag{1307}$$

This leads us to the conjecture that

(1308) $$u_n = r^n u_0 - \sum_{j=0}^{n-1} r^j p, \quad \text{for } n \geq 0,$$

and we can verify this conjecture using the method of induction. We see that the induction hypothesis (equation (1308)) holds for $n = 0$ by convention. If the convention is bothersome, then it is also sufficient to note that the induction hypothesis holds for $n = 1$. For the inductive step, we will assume that the hypothesis is true for $n = N$, and show that this implies that the hypothesis is true for $n = N + 1$. We see that

(1309) $$u_{N+1} = ru_N - p = r(r^N u_0 - \sum_{j=0}^{N-1} r^j p) - p$$

(1310) $$= r^{N+1}u_0 - (r\sum_{j=0}^{N-1} r^j p) - p = r^{N+1}u_0 - \sum_{j=0}^{N-1} r^{j+1}p - p$$

(1311) $$= r^{N+1}u_0 - \sum_{j=1}^{N} r^j p - p = r^{N+1}u_0 - \sum_{j=0}^{N} r^j p.$$

Having completed the inductive step, we see that we do indeed have

(1312) $$u_n = r^n u_0 - \sum_{j=1}^{n} r^j p, \quad \text{for } n \geq 0.$$

We now observe that for $n \geq 0$ we have

(1313) $$u_n = r^n u_0 - \sum_{j=0}^{n-1} r^j p = r^n u_0 - p\sum_{j=0}^{n-1} r^j = r^n u_0 - p(\frac{r^n - 1}{r - 1})$$

(1314) $$= (1.0075)^n 100,000 - p(\frac{1.0075^n - 1}{.0075}).$$

Next, we note that paying off the loan in 30 years corresponds to the equation $u_{360} = 0$, from which we see

$$(1.0075)^{360}100,000 - p(\frac{1.0075^{360} - 1}{.0075}) = 0 \tag{1315}$$

$$\rightarrow p = \frac{(1.0075)^{360}100,000}{\frac{1.0075^{360}-1}{.0075}} \approx 804.623. \tag{1316}$$

It follows that a monthly payment of $p = \$804.63$ will allow the homebuyer to pay off the loan in 30 years. Note that the answer is the textbook is $p = \$804.62$, but rounding down does not make sense in the real world. This means that the total amount paid during the term of the loan is approximately

$$360 \cdot \$804.63 = \$289,666.80. \tag{1317}$$

The exact amount paid over the course of the loan is

$$359 \cdot \$804.63 + 1.0075 * u_{359} = \$289,653.283, \tag{1318}$$

with the method of rounding depending on real world conditions.

Finally, we note that paying off the loan in 20 years corresponds to the equation $u_{240} = 0$, from which we see

$$(1.0075)^{240}100,000 - p(\frac{1.0075^{240} - 1}{.0075}) = 0 \tag{1319}$$

$$\rightarrow p = \frac{(1.0075)^{240}100,000}{\frac{1.0075^{240}-1}{.0075}} \approx 899.726. \tag{1320}$$

It follows that a monthly payment of $p = \$899.73$ will allow the homebuyer to pay off the loan in 20 years. This means that the total amount paid during the term of the loan is approximately

$$240 \cdot \$899.73 = \$215,935.20. \tag{1321}$$

The exact amount paid over the course of the loan is

$$239 \cdot \$899.73 + 1.0075 * u_{239} = \$215,932.499, \tag{1322}$$

with the method of rounding depending on real world conditions.

Problem 11.27: Consider the differential equation

$$y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0. \tag{1323}$$

Find all values of α (if any) for which all solutions of equation (1323) tend to zero as $t \to \infty$. Also find all values of α (if any) for which all nonzero solutions become unbounded as $t \to \infty$.

Solution: We see that the characteristic polynomial of equation (1323) is

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1), \tag{1324}$$

which has roots

$$r = \frac{2\alpha - 1 \pm \sqrt{(2\alpha - 1)^2 - 4\alpha(\alpha - 1)}}{2} = \frac{2\alpha - 1 \pm \sqrt{4}}{2} \tag{1325}$$

$$= \alpha - \frac{5}{2}, \alpha + \frac{3}{2}. \tag{1326}$$

Firstly, we note that the characteristic polynomial of equation (1323) never has a double root, so the general solution is

$$y(t) = c_1 e^{(\alpha - \frac{5}{2})t} + c_2 e^{(\alpha + \frac{3}{2})t}. \tag{1327}$$

The nonzero solutions of equation (1323) will become unbounded as $t \to \infty$ if and only if $e^{(\alpha - \frac{5}{2})t}$ and $e^{(\alpha + \frac{3}{2})t}$ each become unbounded as $t \to \infty$. Recalling that $e^{\beta t}$ becomes unbounded as $t \to \infty$ if and only if $\beta > 0$, we see that the nonzero solutions of (1323) become unbounded if and only if $\alpha - \frac{5}{2} > 0$, which occurs precisely when $\alpha > \frac{5}{2}$. Next, we see that the solutions of equations (1323) tend to zero as $t \to \infty$ if and only if $\alpha - \frac{5}{2}$ and $\alpha + \frac{3}{2}$ are both negative, which occurs precisely when $\alpha < -\frac{3}{2}$.

Problem 11.28: Consider the differential equation

$$y'' + (3-\alpha)y' - 2(\alpha-1)y = 0. \tag{1328}$$

Find all values of α (if any) for which all solutions of equation (1328) tend to zero as $t \to \infty$. Also find all values of α (if any) for which all nonzero solutions become unbounded as $t \to \infty$.

Solution: We see that the characteristic polynomial of equation (1328) is

$$r^2 + (3-\alpha)r - 2(\alpha-1), \tag{1329}$$

which has roots

$$r = \frac{-(3-\alpha) \pm \sqrt{(3-\alpha)^2 - 4(-2(\alpha-1))}}{2} \tag{1330}$$

$$= \frac{\alpha - 3 \pm \sqrt{\alpha^2 + 2\alpha + 1}}{2} = \frac{\alpha - 3 \pm (\alpha+1)}{2} = \alpha - 2, -2. \tag{1331}$$

Since $r = -2$ is always a root of the characteristic polynomial, we see that e^{-2t} is always a solution to equation (1328), so there are no values of α for which all nonzero solutions become unbounded as $t \to \infty$. Next, we note that the general solution to equation (1328) is

$$y(t) = \begin{cases} c_1e^{-2t} + c_2e^{(\alpha-2)t} & \text{if } \alpha \neq 0 \\ c_1e^{-2t} + c_2te^{-2t} & \text{if } \alpha = 0 \end{cases}. \tag{1332}$$

In order for all solutions to tend to zero as $t \to \infty$ we need $\alpha - 2$ to be negative, which occurs precisely when $\alpha < 2$.

Problem 11.29: As will be shown in Section 16.4, the equation $y''+py'+qy = f(t)$, where p and q are constants and f is a specified function, is used to model both the mechanical oscillators and electrical circuits. Depending on the values of p and q, the solutions to this equation display a wide variety of behavior. Consider the equation

$$y'' + 9y = 8\sin(t).$$

(a). **(4 points)** Verify that the following equations have the given general solutions

$$y = c_1\sin(3t) + c_2\cos(3t) + \sin t.$$

(b). **(4 points)** Solve the initial value problem with the given initial conditions $y(0) = 0$, $y'(0) = 2$.
(c). **(2 points)** Graph the solutions to the initial value problem, for $t \geq 0$.

Solution to a: We see that for $y(t) = c_1\sin(3t) + c_2\cos(3t) + \sin(t)$ we have

$$y'' + 9y = (\underbrace{c_1\sin(3t) + c_2\cos(3t) + \sin(t)}_{y(t)})'' + 9(\underbrace{c_1\sin(3t) + c_2\cos(3t) + \sin(t)}_{y(t)}) \tag{1333}$$

$$= \underbrace{-9c_1\sin(3t) - 9c_2\cos(3t) - \sin(t)}_{y''(t)} + \underbrace{9c_1\sin(3t) + 9c_2\cos(3t) + 9\sin(t)}_{9y(t)} \tag{1334}$$

$$= 8\sin(t). \tag{1335}$$

Solution to b: Noting that $y'(t) = 3c_1\cos(3t) - 3c_2\sin(3t) + \cos(t)$, we see that

$$\begin{array}{rcccl} 0 & = & y(0) & = & c_2 \\ 2 & = & y'(0) & = & 3c_1 + 1 \end{array} \to (c_1, c_2) = (\frac{1}{3}, 0). \tag{1336}$$

$$\to \boxed{y(t) = \frac{1}{3}\sin(3t) + \sin(t)}. \tag{1337}$$

Solution to c:

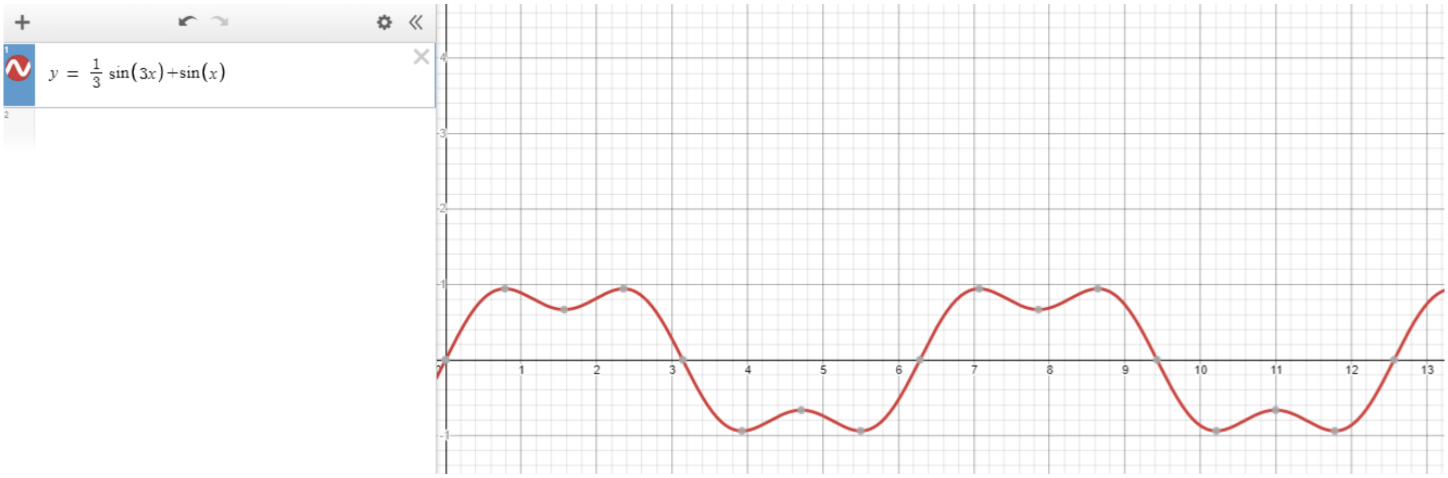


FIGURE 67. A graph of the solution to the initial value problem.

Problem 11.30: Find the Wronskian of the differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0 \tag{1338}$$

without solving the equation.

Solution: Firstly, we will divide both sides of equation (1338) by t^2 to obtain

$$y'' - \left(\frac{t+2}{t}\right)y' + \left(\frac{t+2}{t^2}\right)y = 0. \tag{1339}$$

Since equation (1339) is a second order linear ordinary differential equation of the form

$$y'' + p(t)y' + q(t)y = g(t), \tag{1340}$$

we see that a solution is gaurenteed to exist on $(-\infty, 0)$ or $(0, \infty)$. We also see that the Wronskian is

$$W(t) \stackrel{*}{=} e^{\int -p(t)dt} = e^{\int \frac{t+2}{t}dt} = e^{\int (1+\frac{2}{t})dt} \stackrel{*}{=} e^{t+2\ln(|t|)} = |t|^2e^t = \boxed{t^2e^t}. \tag{1341}$$

We see that the Wronskian is never 0 on $(-\infty, 0)$ or $(0, \infty)$ so equation (1339) has a unique solution for any initial values of the form $y(t_0) = c_1$ and $y'(t_0) = c_2$ with $t_0 \neq 0$ and $c_1, c_2 \in \mathbb{R}$.

Problem 11.31: Given that $y_1(t) = t$ is a solution to equation (1338), use the Wronskian $W(t)$ to find another independent solution $y_2(t)$. (Compare with Problem 11.35)

Solution: We see that

$$t^2e^t \overset{*}{=} W(t) \overset{*}{=} y_1y_2' - y_1'y_2 = ty_2' - y_2 \tag{1342}$$

$$\rightarrow y_2' - \frac{1}{t}y_2 = te^t. \tag{1343}$$

We can solve equation (1343) by multiplying both sides by an integrating factor $I(t)$, which in this case is given by

$$I(t) = e^{\int -\frac{1}{t}dt} \overset{*}{=} e^{-\ln(|t|)} = \frac{1}{|t|}. \tag{1344}$$

Since integrating factors are determined up to a constant, we may simply use $I(t) = \frac{1}{t}$ instead of $I(t) = \frac{1}{|t|}$. Multiplying both sides of (1343) by $\frac{1}{t}$, we see that

$$e^t = \frac{1}{t}y_2' - \frac{1}{t^2}y_2 = (\frac{1}{t}y_2)' \tag{1345}$$

$$\rightarrow \frac{1}{t}y_2 = \int e^t dt \overset{*}{=} e^t \tag{1346}$$

$$\rightarrow \boxed{y_2(t) = te^t}. \tag{1347}$$

Problem 11.32: Solve the initial value problem

$$y'' - 2y' + 5y = 0, \quad y(\frac{\pi}{2}) = 0,\ y'(\frac{\pi}{2}) = 2, \tag{1348}$$

then sketch the graph of the solution and describe the behavior as $t \to \infty$.

Solution: We see that the characteristic polynomial of equation (1348) is

$$r^2 - 2r + 5, \tag{1349}$$

which has roots

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i. \tag{1350}$$

It follows that the general solution to equation (1348) is

$$y(t) = c_1' e^{(1+2i)t} + c_2' e^{(1-2i)t}, \tag{1351}$$

which can also be more conveniently expressed as

$$y(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t). \tag{1352}$$

From the initial condition $y(\frac{\pi}{2}) = 0$ we see that

$$0 = y(\frac{\pi}{2}) = c_1 e^{\frac{\pi}{2}} \cos(\pi) + c_2 e^{\frac{\pi}{2}} \sin(\pi) = -c_1 e^{\frac{\pi}{2}} \to c_1 = 0. \tag{1353}$$

From the initial condition $y'(\frac{\pi}{2}) = 2$ we see that

$$2 = y'(\frac{\pi}{2}) = \frac{d}{dt}(c_2 e^t \sin(2t))\Big|_{t=\frac{\pi}{2}} = (c_2 e^t \sin(2t) + 2c_2 e^t \cos(2t))\Big|_{t=\frac{\pi}{2}} \tag{1354}$$

$$= c_2 e^{\frac{\pi}{2}} \sin(\pi) + 2c_2 e^{\frac{\pi}{2}} \cos(\pi) = -2c_2 e^{\frac{\pi}{2}} \to c_2 = -e^{-\frac{\pi}{2}} \tag{1355}$$

$$(1356) \qquad \rightarrow \boxed{y(t) = -e^{-\frac{\pi}{2}} e^{t} \cos(2t) = -e^{t-\frac{\pi}{2}} \cos(2t)}.$$

We see from the graphs below that the solution $y(t)$ oscillates wildly as $t \to \infty$. Instead of converging to any particular value, the end behavior of $y(t)$ is unbounded and even oscillates between $-\infty$ and ∞.

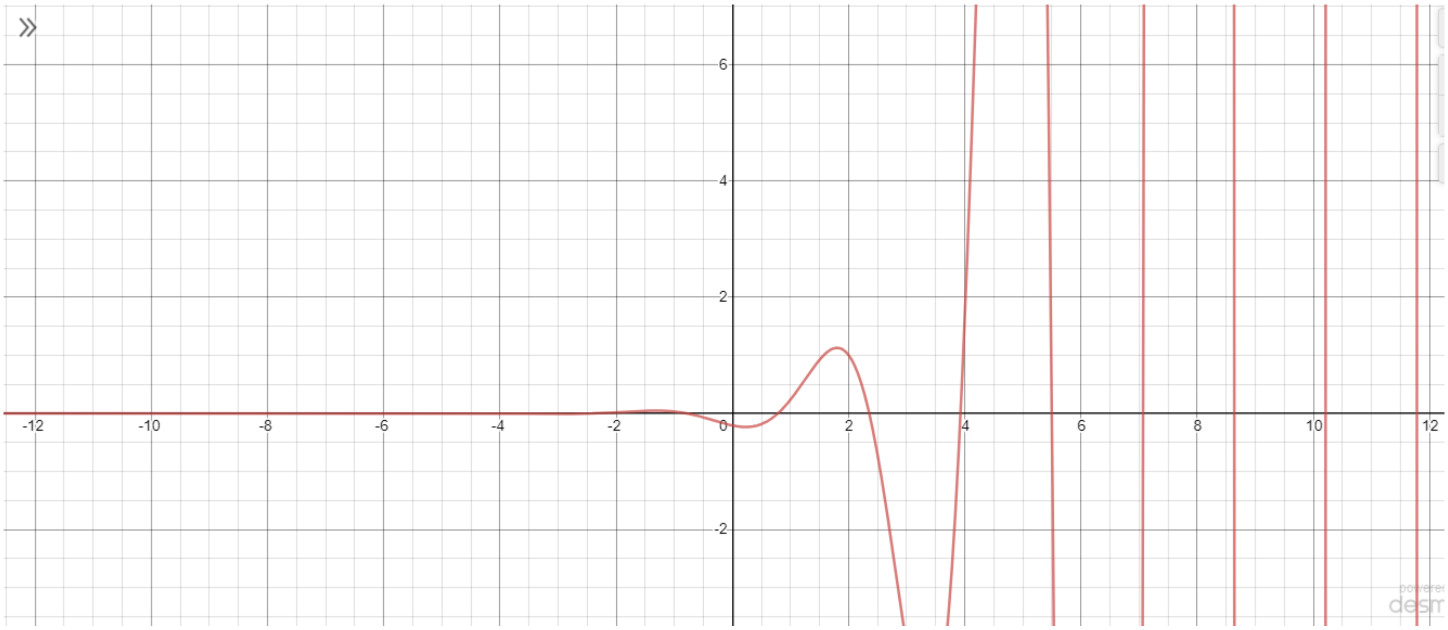


FIGURE 68. The graph of the solution $y(t)$ near the origin.

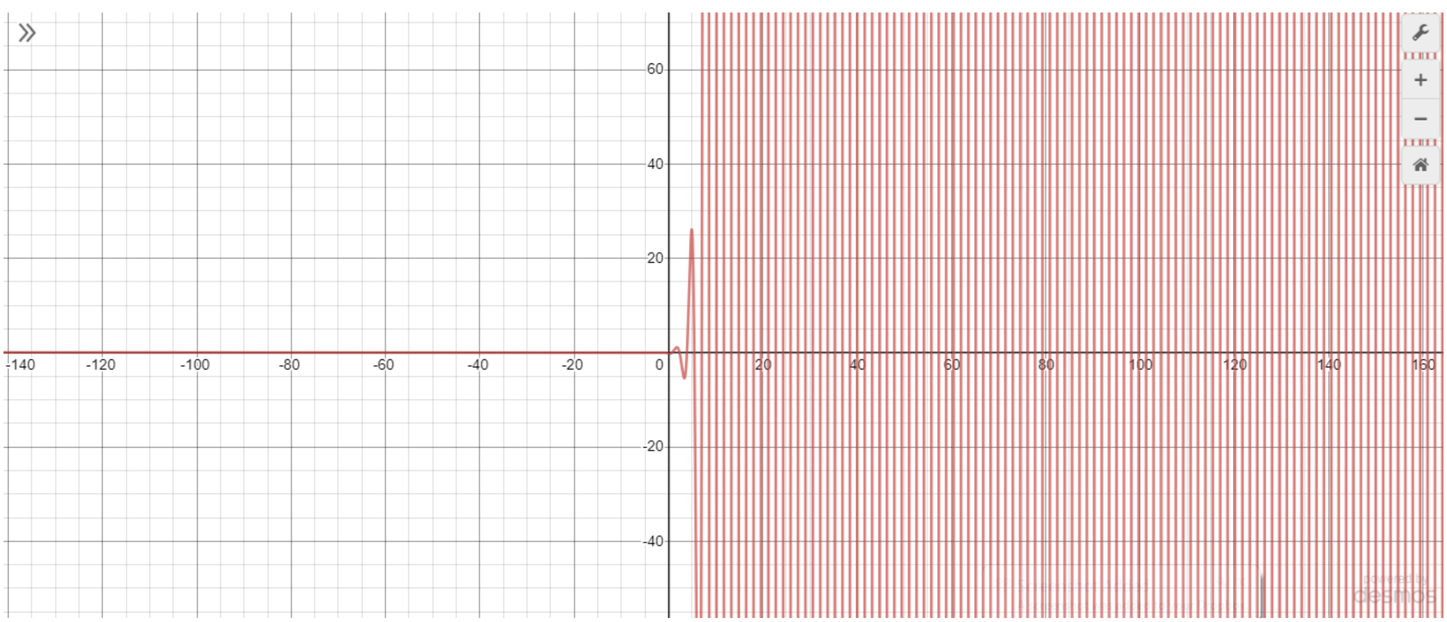


FIGURE 69. The graph of the solution $y(t)$ on a larger domain.

Problem 11.33: Solve the differential equation

$$t^2y'' - ty' + 5y = 0, \quad t > 0. \tag{1357}$$

Solution: Since equation (1357) is an Euler equation, we make the substitution $x = \ln(t)$ and $h(x) = y(e^x) = y(t)$. Since $t = e^x$, we may use the chain rule to see that

$$\frac{dh}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = e^x \frac{dy}{dt} = t\frac{dy}{dt}, \text{ and} \tag{1358}$$

$$\frac{d^2h}{dx^2} = \frac{d}{dx}(\frac{dh}{dx}) = \frac{d}{dx}(e^x\frac{dy}{dt}) \tag{1359}$$

$$= e^x\frac{dy}{dt} + e^x\left(\frac{d}{dx}\frac{dy}{dt}\right) = e^x\frac{dy}{dt} + e^x\left(\frac{d^2y}{dt^2} \cdot \frac{dt}{dx}\right) \tag{1360}$$

$$= e^x\frac{dy}{dt} + e^x\left(e^x\frac{d^2y}{dt^2}\right) = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}. \tag{1361}$$

We now see that substituting $x = \ln(t)$ into equation (1357) yields

$$0 = t^2y'' - ty' + 5y = (t^2y'' + ty') - 2ty' + 5y = h'' - 2h' + 5h. \tag{1362}$$

Since we now have t and x as independent variables, it is important to note that $h' = \frac{dh}{dx}$ and $y' = \frac{dy}{dt}$. This is not the most clear notation, so some people prefer to be more explicit and only write $\frac{dh}{dx}$ and $\frac{dy}{dt}$ without any use of $'$. Regardless of your preferred convention, be careful to avoid the errors that arise when you assume $y' = \frac{dy}{dx}$ and $h' = \frac{dh}{dt}$.

We see that the characteristic polynomial of equation (1362) is

$$r^2 - 2r + 5, \tag{1363}$$

and has roots

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i. \tag{1364}$$

It follows that the general solution to equation (1362) is

$$h(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x). \tag{1365}$$

Finally, we see that

$$y(t) = h(x) = h(\ln(t)) = c_1 e^{\ln(t)} \cos(2\ln(t)) + c_2 e^{\ln(t)} \sin(2\ln(t)) \tag{1366}$$

$$= \boxed{c_1 t \cos(2\ln(t)) + c_2 t \sin(2\ln(t))}. \tag{1367}$$

Problem 11.34: Given $a \in \mathbb{R}$, solve the differential equation

$$y'' + 2ay' + a^2y = 0. \tag{1368}$$

Hint: It helps to consider the Wronskian.

Solution: We see that the characteristic polynomial of equation (1368) is

$$r^2 + 2ar + a^2 = (r + a)^2. \tag{1369}$$

Since the characteristic polynomial has $r = -a$ as a repeated root, we see that one solution to equation (1368) is $y_1(t) = e^{-at}$, but the second solution has yet to be found. To find the second solution, we will proceed as we did in the Bonus to problem 3.2.29. We see that the Wronskian is given by

$$W(t) = e^{\int -2adt} = e^{-2at}. \tag{1370}$$

It follows that the second solution $y_2(t)$ satisfies the differential equation

$$e^{-2at} = W(t) = y_1y_2' - y_1'y_2 = e^{-at}y_2' + ae^{-at}y_2 \tag{1371}$$

$$\rightarrow y_2' + ay_2 = e^{-at}. \tag{1372}$$

We can solve equation (1372) by multiplying both sides by an integrating factor $I(t)$. We see that

$$I(t) = e^{\int adt} = e^{at} \tag{1373}$$

is a suitable choice of integrating factor. After multiplying both sides of equation (1372) by e^{at}, we see that

$$1 = e^{at}y_2' + ae^{at}y_2 = (e^{at}y_2)' \tag{1374}$$

$$\rightarrow e^{at}y_2 = t \rightarrow y_2 = te^{-at}. \tag{1375}$$

Since $y_2(t)$ is indeed an independent solution to $y_1(t)$, we see that the general solution to equation (1368) is

$$\boxed{y(t) = c_1e^{-at} + c_2te^{-at}}. \tag{1376}$$

It is clear that this solution is defined on all of $(-\infty, \infty)$. Furthermore, since the Wronskian $W(t)$ is never 0, we see that for any $t_0, b_1, b_2 \in \mathbb{R}$, there is a unique solution to equation (1368) when we impose the initial conditions $y(t_0) = b_1$ and $y'(t_0) = b_2$.

Problem 11.35: Given that $y_1(t) = t$ is a solution to the differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0, \tag{1377}$$

use the method of reduction of order to find a second solution. (Compare with problem 11.31)

Solution: Let $u(t)$ be such that $y_2(t) = u(t)y_1(t) = tu(t)$ is a second (independent) solution to equation (1377). We see that

$$0 = t^2(tu(t))'' - t(t+2)(tu(t))' + (t+2)tu(t) \tag{1378}$$

$$= t^2(tu''(t) + 2u'(t)) - t(t+2)(tu'(t) + u(t)) + (t+2)tu(t) \tag{1379}$$

$$= t^3u''(t) + 2t^2u'(t) - t^3u'(t) - 2t^2u'(t) - t^2u(t) - 2tu(t) + t^2u(t) + 2tu(t) \tag{1380}$$

$$= t^3u''(t) - t^3u'(t) \to 0 = u''(t) - u'(t) \tag{1381}$$

$$\to u'(t) = u''(t) = \frac{du'(t)}{dt} \to dt = \frac{du'(t)}{u'(t)} \tag{1382}$$

$$\to \int dt = \int \frac{du'(t)}{u'(t)} \tag{1383}$$

$$\to t \overset{*}{=} \ln(u'(t)) \to u'(t) = e^t \tag{1384}$$

$$\to u(t) = \int e^t dt \overset{*}{=} e^t. \tag{1385}$$

It follows that a second solution to equation (1377) is $y_2(t) = tu(t) = te^t$. After plugging te^t back into equation (1377) to check our work, we see that $y_2(t) = te^t$ is indeed a second solution to equation (1377) that is independent from $y_1(t) = t$.

Problem 11.36: Use the method of variation of parameters to find the general solution to the differential equation

(1386) $$(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1,$$

given that $y_1(t) = e^t$ and $y_2(t) = t$ are solutions to the corresponding homogeneous equation.

Solution: We begin by considering solutions to equation (1386) of the form

(1387) $$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = e^tu_1(t) + tu_2(t),$$

where $u_1(t)$ and $u_2(t)$ are functions that are yet to be determined. We see that

(1388) $$y'(t) = e^tu_1'(t) + e^tu(t) + tu_2'(t) + u_2(t).$$

Viewing $u_1(t)$ and $u_2(t)$ as free variables, we see that we have 2 degrees of freedom, but we currently only have 1 constraint, which is that $y(t)$ satisfy equation (1386). It follows that we can impose a second constraint, so we impose

(1389) $$e^tu_1'(t) + tu_2'(t) = 0,$$

from which we see that

(1390) $$y'(t) = e^tu_1(t) + u_2(t).$$

We now see that

(1391) $$y''(t) = e^tu_1'(t) + e^tu_1(t) + u_2'(t), \text{ so}$$

(1392) $$2(t-1)^2e^{-t} = (1-t)y'' + ty' - y$$

(1393) $$= (1-t)(e^tu_1'(t) + e^tu_1(t) + u_2'(t)) + t(e^tu_1(t) + u_2(t)) - (e^tu_1(t) + tu_2(t))$$

(1394) $$= e^t u_1'(t) - te^t u_1'(t) + u_2'(t) - tu_2'(t)$$

(1395) $$\overset{\text{by } (1389)}{=} -tu_2'(t) + t^2 u_2'(t) + u_2'(t) - tu_2'(t) = (t-1)^2 u_2'$$

(1396) $$\to u_2'(t) = 2e^{-t} \overset{\text{by } (1389)}{\to} u_1'(t) = -2te^{-2t}$$

(1397) $$\to u_1(t) \overset{*}{=} te^{-2t} + \frac{1}{2}e^{-2t} \text{ and } u_2(t) \overset{*}{=} -2e^{-t}$$

(1398) $$\to y(t) = te^{-t} + \frac{1}{2}e^{-t} - 2te^{-t} = \boxed{(\frac{1}{2} - t)e^{-t}}.$$

Problem 11.37: Use the method of reduction of order to find the general solution to the differential equation

$$(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1, \tag{1399}$$

given that $y_1(t) = e^t$ is a solution to the corresponding homogeneous equation.

Solution: We search for solutions of the form $y(t) = v(t)y_1(t) = e^tv(t)$. Noting that

$$y'(t) = e^tv(t) + e^tv'(t), \text{ and} \tag{1400}$$

$$y''(t) = e^tv(t) + 2e^tv'(t) + e^tv''(t), \tag{1401}$$

we see that

$$2(t-1)^2e^{-t} = (1-t)y'' + ty' - y \tag{1402}$$

$$= (1-t)(e^tv(t) + 2e^tv'(t) + e^tv''(t)) + t(e^tv(t) + e^tv'(t)) - e^tv(t) \tag{1403}$$

$$= \underbrace{((1-t)e^t + te^t - e^t)}_{\text{This part will always be 0.}} v(t) + (2(1-t)e^t + te^t)v'(t) + e^tv''(t) \tag{1404}$$

$$= (2e^t - te^t)v'(t) + (1-t)e^tv''(t). \tag{1405}$$

$$\to v''(t) + \left(\frac{2-t}{1-t}\right)v'(t) = 2(1-t)e^{-2t}. \tag{1406}$$

Since equation (1406) is a first order linear differential equation with respect to $v'(t)$ (instead of $v(t)$) and it is in standard form, we can solve it by using an integrating factor. We see that the integrating factor $I(t)$ is given by

$$I(t) = e^{\int p(t)dt} = e^{\int \frac{2-t}{1-t}dt} = e^{\int (\frac{1}{1-t}+1)dt} \overset{**}{=} e^{-\ln(1-t)+t} = \frac{e^t}{1-t}. \tag{1407}$$

Multiplying both sides of equation (1406) by $I(t)$ yields

(1408) $$2e^{-t} = \frac{e^t}{1-t}v''(t) + \frac{(2-t)e^t}{(1-t)^2}v'(t) = (\frac{e^t}{1-t}v'(t))'$$

(1409) $$\rightarrow \frac{e^t}{1-t}v'(t) = -2e^{-t} + c_1 \rightarrow v'(t) = -2(1-t)e^{-2t} + c_1(1-t)e^{-t}$$

(1410) $$\rightarrow v(t) = (1-t)e^{-2t} - \frac{1}{2}e^{-2t} + c_1te^{-t} + c_2$$

(1411) $$= (\frac{1}{2} - t)e^{-2t} + c_1te^{-t} + c_2$$

(1412) $$\rightarrow y(t) = e^tv(t) = \boxed{(\frac{1}{2} - t)e^{-t} + c_1t + c_2e^t}.$$

Remark: Observe that the c_1t corresponds to the fact that $y_2(t) = t$ is the second solution to the homogeneous equation corresponding to (1399). So in this case the method of reduction of order has given us more than just a particular solution to equation (1399)!

Problem 11.38: A spring–mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos(3t) - 2\sin(3t))$ N, determine the steady state response. Express your answer in the form $R\cos(\omega t - \delta)$.

Solution: We know that the general equation governing the motion of a spring is

$$mu'' + \gamma u' + ku = F(t), \tag{1413}$$

and we are given that $m = 2$, $k = 3$, and $F(t) = 3\cos(3t) - 2\sin(3t)$. To find the damping constant γ, we recall that the resistance offered by a viscous fluid with damping constant γ is $\gamma u'$, and we are told in this case that $\gamma u' = u'$ so $\gamma = 1$. It follows that we want to solve the differential equation

$$2u'' + u' + 3u = 3\cos(3t) - 2\sin(3t). \tag{1414}$$

We would like to use the method of undetermined coefficients to proceed, but we should first solve the corresponding homogeneous equation in order to determine the general form of the solution. We see that the differential equation

$$2u'' + u' + 3u = 0 \tag{1415}$$

has characteristic polynomial

$$2r^2 + r + 3, \tag{1416}$$

which has roots

$$r = \frac{-1 \pm \sqrt{1^2 - 4\cdot 2\cdot 3}}{2\cdot} = \frac{-1 \pm \sqrt{-23}}{4}. \tag{1417}$$

Since $\sin(3t)$ and $\cos(3t)$ are not solutions to equation (1415), we see that the general form of a particular solution to equation (1414) is

$$y(t) = A\sin(3t) + B\cos(3t). \tag{1418}$$

Observing that

$$y'(t) = 3A\cos(3t) - 3B\sin(3t) \text{ and} \tag{1419}$$

$$y''(t) = -9A\sin(3t) - 9B\cos(3t), \tag{1420}$$

we see that

$$3\cos(3t) - 2\sin(3t) = 2u'' + u' + 3u \tag{1421}$$

$$= 2(-9A\sin(3t) - 9B\cos(3t)) + (3A\cos(3t) - 3B\sin(3t)) + 3(A\sin(3t) + B\cos(3t)) \tag{1422}$$

$$= (3A - 15B)\cos(3t) + (-15A - 3B)\sin(3t) \tag{1423}$$

$$\rightarrow \begin{array}{rcrcr} 3A & - & 15B & = & 3 \\ -15A & - & 3B & = & -2 \end{array} \rightarrow (A, B) = (\frac{1}{6}, -\frac{1}{6}). \tag{1424}$$

Since we are searching for the steady state solution, we ignore any potential contribution from equation (1415), as our current solution of

$$y(t) = A\sin(3t) + B\cos(3t) = \frac{1}{6}\sin(3t) - \frac{1}{6}\cos(3t) \tag{1425}$$

is already a periodic solution, and any contributions from equation (1415) would result in a non-periodic solution. All that remains is to express the answer in the form $y(t) = R\cos(\omega_0 t - \delta)$. To do this, we proceed as we did in problem 3.7.4. We see that $\omega_0 = 3$, and that R is given by

$$R = \sqrt{(\frac{1}{6})^2 + (-\frac{1}{6})^2} = \frac{\sqrt{2}}{6}. \tag{1426}$$

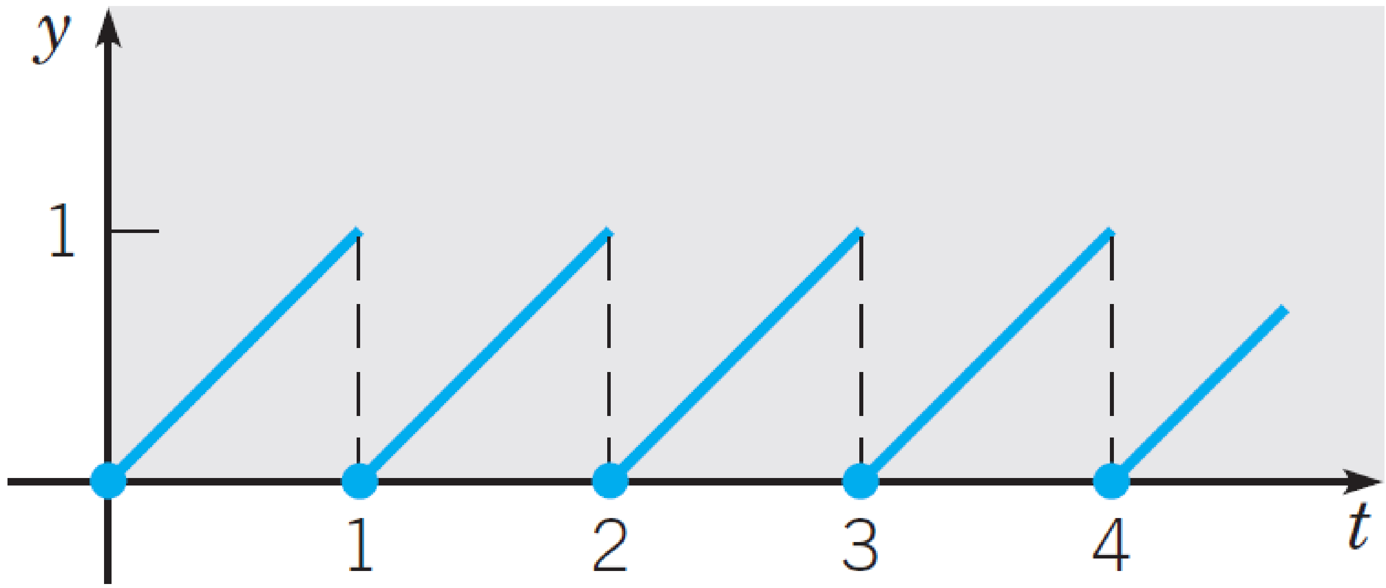
Since

$$\cos(\delta) = -\frac{1}{\sqrt{2}} \text{ and } \sin(\delta) = \frac{1}{\sqrt{2}}, \tag{1427}$$

we see that $\delta = \frac{3\pi}{4}$. We were lucky enough to have δ be a special angle, so we did not have to work with inverse trig functions this time! In conclusion, the steady state solution is given by

$$\boxed{y(t) = \frac{\sqrt{2}}{6}\cos(3t - \frac{3\pi}{4})}. \tag{1428}$$

Problem 11.39: Find the Laplace transform of the function $f : [0, \infty) \to [0, 1)$ that is defined by $f(t) = t$ when $0 \le t < 1$ and $f(t+1) = f(t)$.



Solution: Firstly, we note that $0 \le f(t) < 1$ for every $t \in [0, \infty)$, we see that $\mathcal{L}\{f(t)\} = F(s)$ is defined for every $s > 0$. Using the same notation as the course textbook we recall that for $c \in \mathbb{R}$ we have

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \ge c \end{cases}. \tag{1429}$$

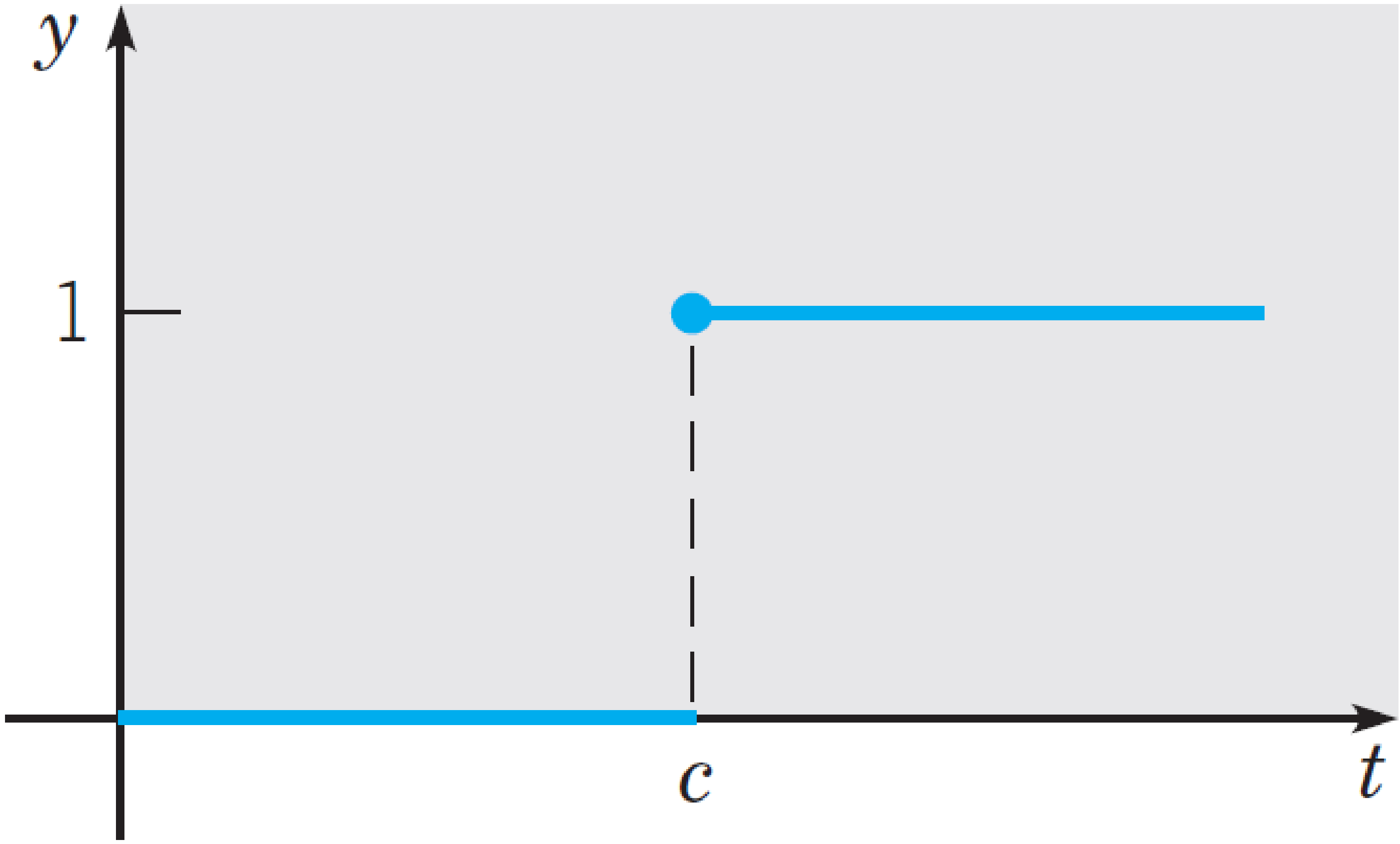


FIGURE 6.3.1 Graph of $y = u_c(t)$.

We now observe that for any $j \geq 0$ we have

(1430) $$u_j(t) - u_{j+1}(t) = \begin{cases} 0 & \text{if } t < j \\ 1 & \text{if } j \leq t < j+1 \\ 0 & \text{if } j+1 \leq t \end{cases}.$$

It follows that we can write

(1431) $$f(t) = \sum_{j=0}^{\infty}(u_j(t) - u_{j+1}(t))(t-j),\text{ so for } s > 0 \text{ we have}$$

(1432) $$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty}\sum_{j=0}^{\infty}(u_j(t) - u_{j+1}(t))(t-j)e^{-st}dt.$$

$$\stackrel{*}{=} \sum_{j=0}^{\infty}\int_0^{\infty}(u_j(t)-u_{j+1}(t))(t-j)e^{-st}dt \tag{1433}$$

We will now calculate each integral in equation (1433). Firstly, we note that

$$\int_0^{\infty}u_j(t)(t-j)e^{-st}dt = e^{-sj}\int_0^{\infty}u_j(t)(t-j)e^{-s(t-j)}dt \tag{1434}$$

$$= e^{-sj}\int_0^{\infty}te^{-st}dt = e^{-sj}\mathcal{L}\{t\} = \frac{e^{-sj}}{s^2}. \tag{1435}$$

We can also deduce the results of equations (1434) and (1435) directly from Theorem 6.3.1 of the textbook. Next, we note that

$$\int_0^{\infty}-u_{j+1}(t)(t-j)e^{-st}dt = -\int_0^{\infty}u_{j+1}(t)(t-(j+1)+1)e^{-st}dt \tag{1436}$$

$$= -\int_0^{\infty}u_{j+1}(t)(t-(j+1))e^{-st}dt - \int_0^{\infty}u_{j+1}(t)e^{-st}dt \tag{1437}$$

$$= -\mathcal{L}\{u_{j+1}(t)(t-(j+1))\} - \mathcal{L}\{u_{j+1}(t)\cdot 1\} \tag{1438}$$

$$\stackrel{\text{by Thm. 6.3.1}}{=} -\frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}. \tag{1439}$$

Putting together the results of equations (1434)-(1439) we see that

$$\int_0^{\infty}(u_j(t)-u_{j+1}(t))(t-j)e^{-st} = \frac{e^{-sj}}{s^2} - \frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}. \tag{1440}$$

Plugging in the results of equation (1440) back into equation (1433) we see that

$$\sum_{j=0}^{\infty}\int_0^{\infty}(u_j(t)-u_{j+1}(t))(t-j)e^{-st} \tag{1441}$$

$$= \sum_{j=0}^{\infty}\left(\frac{e^{-sj}}{s^2} - \frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}\right) \tag{1442}$$

$$= \frac{1}{s^2} + \sum_{j=0}^{\infty} -\frac{e^{-s(j+1)}}{s} = \frac{1}{s^2} - \frac{1}{s}\sum_{j=0}^{\infty} e^{-s(j+1)} \tag{1443}$$

$$= \frac{1}{s^2} - \frac{1}{s}\sum_{j=1}^{\infty} e^{-sj} = \frac{1}{s^2} - \frac{1}{s}\left(\frac{e^{-s}}{1-e^{-s}}\right) = \boxed{\frac{1}{s^2} - \frac{1}{s(e^s-1)}}. \tag{1444}$$

Problem 11.40: Solve the initial value problem

$$y'' + 4y = \sin(t) - u_{2\pi}(t)\sin(t-2\pi); \quad y(0) = 0,\ y'(0) = 0. \tag{1445}$$

Solution: From Corollary 6.2.2 of the textbook we see that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\}, \text{ and} \tag{1446}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\}, \text{ so} \tag{1447}$$

$$\mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = s^2\mathcal{L}\{y(t)\} + 4\mathcal{L}\{y(t)\} \tag{1448}$$

$$= (s^2+4)\mathcal{L}\{y(t)\}. \tag{1449}$$

We also see that

$$\mathcal{L}\{\sin(t) - u_{2\pi}(t)\sin(t-2\pi)\} = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\} \tag{1450}$$

$$= \mathcal{L}\{\sin(t)\} - e^{-2\pi s}\mathcal{L}\{\sin(t)\} = (1-e^{-2\pi s})\mathcal{L}\{\sin(t)\} = \frac{1-e^{-2\pi s}}{s^2+1}. \tag{1451}$$

We now see that taking the Laplace transform of both sides of equation (1445) yields

$$\mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{\sin(t) - u_{2\pi}(t)\sin(t-2\pi)\} \tag{1452}$$

$$\to (s^2+4)\mathcal{L}\{y(t)\} = \frac{1-e^{-2\pi s}}{s^2+1} \tag{1453}$$

$$\to \mathcal{L}\{y(t)\} = \frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)}. \tag{1454}$$

Now that we have calculated $\mathcal{L}(y(t))$, we want to determine $y(t)$. We first require preliminary calculations with partial fractions before we can attempt to calculate the inverse Laplace transform. We see that

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \tag{1455}$$

$$\rightarrow \frac{1}{(s^2+1)(s^2+4)} = \frac{(A+C)s^3+(B+D)s^2+(4A+C)s+(4B+D)}{(s^2+1)(s^2+4)} \tag{1456}$$

$$\rightarrow \begin{array}{rcrcl} A & + & C & = & 0 \\ B & + & D & = & 0 \\ 4A & + & C & = & 0 \\ 4B & + & D & = & 1 \end{array} \rightarrow (A,B,C,D) = (0,\frac{1}{3},0,-\frac{1}{3}) \tag{1457}$$

$$\rightarrow \frac{1}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4}. \tag{1458}$$

$$\rightarrow \mathcal{L}\{y(t)\} = \frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1-e^{-2\pi s}}{s^2+1} - \frac{1-e^{-2\pi s}}{s^2+4}\right). \tag{1459}$$

Now we observe that

$$\mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{1}{s^2+1}\right)\} = \frac{1}{3}\sin(t), \tag{1460}$$

$$\mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{-e^{-2\pi s}}{s^2+1}\right)\} \overset{\text{by Thm. 6.3.1}}{=} -\frac{1}{3}u_{2\pi}(t)\sin(t-2\pi) = -\frac{1}{3}u_{2\pi}(t)\sin(t), \tag{1461}$$

$$\mathcal{L}^{-1}\{-\frac{1}{3}\left(\frac{1}{s^2+4}\right)\} = -\frac{1}{6}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\} = -\frac{1}{6}\sin(2t), \text{and} \tag{1462}$$

$$\mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{e^{-2\pi s}}{s^2+4}\right)\} \overset{\text{by Thm. 6.3.1}}{=} \frac{1}{6}\mathcal{L}^{-1}\{\frac{2e^{-2\pi s}}{s^2+4}\} = \frac{1}{6}u_{2\pi}(t)\sin(2(t-2\pi)) \tag{1463}$$

$$= \frac{1}{6}u_{2\pi}(t)\sin(2t-4\pi) = \frac{1}{6}u_{2\pi}(t)\sin(2t). \tag{1464}$$

It follows that

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{1-e^{-2\pi s}}{s^2+1} - \frac{1-e^{-2\pi s}}{s^2+4}\right)\} \tag{1465}$$

$$= \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{1}{s^2+1}\right)\} + \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{-e^{-2\pi s}}{s^2+1}\right)\} + \mathcal{L}^{-1}\{-\frac{1}{3}\left(\frac{1}{s^2+4}\right)\} + \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{e^{-2\pi s}}{s^2+4}\right)\} \tag{1466}$$

$$= \frac{1}{3}\sin(t) - \frac{1}{3}u_{2\pi}(t)\sin(t) - \frac{1}{6}\sin(2t) + \frac{1}{6}u_{2\pi}(t)\sin(2t) \tag{1467}$$

$$= \frac{1}{3}\sin(t)(1-u_{2\pi}(t)) - \frac{1}{6}\sin(2t)(1-u_{2\pi}(t)) \tag{1468}$$

$$= \boxed{\frac{1}{6}(1-u_{2\pi}(t))(2\sin(t)-\sin(2t))}. \tag{1469}$$

Problem 11.41: Solve the initial value problem

$$y^{(4)}+5y''+4y = 1-u_{2\pi}(t); y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 0. \tag{1470}$$

Hint: It may help to do Problem 11.40 first.

Solution: From Corollary 6.2.2 of the textbook we see that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\}, \tag{1471}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\}, \tag{1472}$$

$$\mathcal{L}\{y'''(t)\} = s^3\mathcal{L}\{y(t)\} - s^2y(0) - sy'(0) - y''(0) = s^3\mathcal{L}\{y(t)\},\ \text{and} \tag{1473}$$

$$\mathcal{L}\{y^{(4)}(t)\} = s^4\mathcal{L}\{y(t)\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) \\ = s^4\mathcal{L}\{y(t)\}. \tag{1474}$$

We now take the Laplace transform of both sides of equation (1470) and see that

$$\mathcal{L}\{y^{(4)}(t) + 5y''(t) + 4y(t)\} = \mathcal{L}\{1 - u_{2\pi}(t)\} \tag{1475}$$

$$\rightarrow \mathcal{L}\{y^{(4)}(t)\} + 5\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u_{2\pi}(t)\} \tag{1476}$$

$$\rightarrow s^4\mathcal{L}\{y(t)\} + 5s^2\mathcal{L}\{y(t)\} + 4\mathcal{L}\{y(t)\} = \frac{1}{s} - \frac{e^{-2\pi s}}{s} \tag{1477}$$

$$\rightarrow (s^4 + 5s^2 + 4)\mathcal{L}\{y(t)\} = \frac{1 - e^{-2\pi s}}{s} \tag{1478}$$

$$\rightarrow \mathcal{L}\{y(t)\} = \frac{1 - e^{-2\pi s}}{s(s^4 + 5s^2 + 4)} = \frac{1 - e^{-2\pi s}}{s(s^2 + 4)(s^2 + 1)}. \tag{1479}$$

Now under normal circumstances we would attempt to use partial fractions and obtain a decomposition of the form

$$\frac{1}{s(s^2+4)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{E}{s}, \tag{1480}$$

but let us recall that when solving Problem 11.40 we showed that

$$\mathcal{L}^{-1}\{\frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)}\} = \frac{1}{6}(1-u_{2\pi}(t))(2\sin(t)-\sin(2t)). \tag{1481}$$

We now see that

$$\frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)} = s\mathcal{L}\{y(t)\} = \mathcal{L}\{y'(t)\} \tag{1482}$$

$$\to y'(t) = \mathcal{L}^{-1}\{\frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)}\} = \frac{1}{6}(1-u_{2\pi}(t))(2\sin(t)-\sin(2t)) \tag{1483}$$

$$\to y(t) = \int_0^t \frac{1}{6}(1-u_{2\pi}(u))(2\sin(u)-\sin(2u))du + c \tag{1484}$$

$$= \frac{1}{6}(1-u_{2\pi}(t))\int_0^t (2\sin(u)-\sin(2u))du + \frac{1}{6}u_{2\pi}(t)\int_0^{2\pi}(2\sin(u)-\sin(2u))du + c \tag{1485}$$

$$= \frac{1}{6}(1-u_{2\pi}(t))(-2\cos(u)+\frac{1}{2}\cos(2u))\Big|_{u=0}^{t} + \frac{1}{6}u_{2\pi}(t)(-2\cos(u)+\frac{1}{2}\cos(2u))\Big|_{u=0}^{2\pi} + c \tag{1486}$$

$$= \frac{1}{6}(1-u_{2\pi}(t))(-2\cos(t)+\frac{1}{2}\cos(2t)+\frac{3}{2}) + c. \tag{1487}$$

Recalling that $y(0)=0$, we see that $c=0$, so our final answer is

$$y(t) = \frac{1}{6}(1 - u_{2\pi}(t))(-2\cos(t) + \frac{1}{2}\cos(2t) + \frac{3}{2}). \tag{1488}$$

Problem 11.42: Solve the initial value problem

$$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t); \quad y(0) = 0, y'(0) = 0. \tag{1489}$$

Solution: We recall that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\}, \text{ and} \tag{1490}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\}. \tag{1491}$$

We now take the Laplace transform of both sides of equation (1489) and see that

$$\mathcal{L}\{y''(t) + 3y'(t) + 2y(t)\} = \mathcal{L}\{\delta(t-5) + u_{10}(t)\} \tag{1492}$$

$$\to \mathcal{L}\{y''(t)\} + 3\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t-5)\} + \mathcal{L}\{u_{10}(t) \cdot 1\} \tag{1493}$$

$$\to s^2\mathcal{L}\{y(t)\} + 3s\mathcal{L}\{y(t)\} + 2\mathcal{L}\{y(t)\} = e^{-5s} + \frac{e^{-10s}}{s} \tag{1494}$$

$$\to \mathcal{L}\{y(t)\} = \frac{e^{-5s} + \frac{e^{-10s}}{s}}{s^2 + 3s + 2} = \frac{se^{-5s} + e^{-10s}}{s(s+1)(s+2)}. \tag{1495}$$

We will now use the method of partial fractions in order to break up the final expression in equation (1495) into simpler components. We see that

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \tag{1496}$$

$$= \frac{A(s+1)(s+2) + Bs(s+2) + Cs(s+1)}{s(s+1)(s+2)} \tag{1497}$$

$$= \frac{(A+B+C)s^2 + (3A+2B+C)s + 2A}{s(s+1)(s+2)} \tag{1498}$$

$$\begin{array}{ccccccc} A & + & B & + & C & = & 0 \\ \to 3A & + & 2B & + & C & = & 0 \\ 2A & & & & & = & 1 \end{array} \to A = \frac{1}{2} \tag{1499}$$

$$\to \begin{array}{ccccc} B & + & C & = & -\frac{1}{2} \\ 2B & + & C & = & -\frac{3}{2} \end{array} \to B = (2B + C) - (B + C) = -1 \tag{1500}$$

$$\to C = -\frac{1}{2} - B = \frac{1}{2} \to (A, B, C) = (\frac{1}{2}, -1, \frac{1}{2}). \tag{1501}$$

$$\to \frac{1}{s(s+1)(s+2)} = \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2}. \tag{1502}$$

We also see that

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}. \tag{1503}$$

It follows that

$$\mathcal{L}\{y(t)\} = \frac{se^{-5s} + e^{-10s}}{s(s+1)(s+2)} = \frac{e^{-5s}}{(s+1)(s+2)} + \frac{e^{-10s}}{s(s+1)(s+2)} \tag{1504}$$

$$= \frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s+2} + \frac{\frac{1}{2}e^{-10s}}{s} + \frac{-e^{-10s}}{s+1} + \frac{\frac{1}{2}e^{-10s}}{s+2} \tag{1505}$$

$$\begin{aligned} \to y(t) = \mathcal{L}^{-1}\{\frac{e^{-5s}}{s+1}\} - \mathcal{L}^{-1}\{\frac{e^{-5s}}{s+2}\} \\ + \frac{1}{2}\mathcal{L}^{-1}\{\frac{e^{-10s}}{s}\} - \mathcal{L}^{-1}\{\frac{e^{-10s}}{s+1}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{e^{-10s}}{s+2}\} \end{aligned} \tag{1506}$$

$$\begin{aligned} = u_5(t)e^{-(t-5)}(t-5) - u_5(t)e^{-2(t-5)}(t-5) + \frac{1}{2}u_{10}(t)(t-10) \\ - u_{10}(t)e^{-(t-10)}(t-10) + \frac{1}{2}u_{10}(t)e^{-2(t-10)}(t-10) \end{aligned} \tag{1507}$$

(1508)

$$= \boxed{(e^{-(t-5)} - e^{-2(t-5)})u_5(t)(t-5) + (\frac{1}{2} - e^{-(t-10)} + \frac{1}{2}e^{-2(t-10)})u_{10}(t)(t-10)}.$$

Problem 11.43: Solve the initial value problem

$$y'' + 3y' + 2y = \cos(\alpha t); \quad y(0) = 1, y'(0) = 0 \tag{1509}$$

by using the Laplace transform and convolution integrals.

Solution: We recall that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\} - 1, \text{ and} \tag{1510}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\} - s. \tag{1511}$$

We now take the Laplace transform of both sides of equation (1509) and see that

$$\mathcal{L}\{y''(t) + 3y'(t) + 2y(t)\} = \mathcal{L}\{\cos(\alpha t)\} \tag{1512}$$

$$\to \mathcal{L}\{y''(t)\} + 3\mathcal{L}\{y'(t)\} + 2\mathcal{L}\{y(t)\} = \mathcal{L}\{\cos(\alpha t)\} \tag{1513}$$

$$\to (s^2\mathcal{L}\{y(t)\} - s) + 3(s\mathcal{L}\{y(t)\} - 1) + 2\mathcal{L}\{y(t)\} = \frac{s}{s^2 + \alpha^2} \tag{1514}$$

$$\to (s^2 + 3s + 2)\mathcal{L}\{y(t)\} = \frac{s}{s^2 + \alpha^2} + s + 3 \tag{1515}$$

$$\to \mathcal{L}\{y(t)\} = \frac{1}{(s+1)(s+2)}\left(\frac{s}{s^2 + \alpha^2} + s + 3\right) \tag{1516}$$

$$= \frac{s}{(s^2 + \alpha^2)(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}. \tag{1517}$$

As in the previous problem, we observe that

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2} \tag{1518}$$

$$\rightarrow \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s+1} - \frac{s+3}{s+2} = (1+\frac{2}{s+1}) - (1+\frac{1}{s+2}) = \frac{2}{s+1} - \frac{1}{s+2}. \tag{1519}$$

$$\rightarrow \mathcal{L}^{-1}\{\frac{s+3}{(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{2}{s+1} - \frac{1}{s+2}\} = 2\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}^{-1}\{\frac{1}{s+2}\} = 2e^{-t} - e^{-2t}. \tag{1520}$$

Another consequence of equation (1518) is that

$$\mathcal{L}^{-1}\{\frac{1}{(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{1}{s+1}\} - \mathcal{L}\{\frac{1}{s+2}\} = e^{-t} - e^{-2t} \tag{1521}$$

Since

$$\mathcal{L}^{-1}\{\frac{s}{s^2+\alpha^2}\} = \cos(\alpha t), \text{ we see that} \tag{1522}$$

$$\mathcal{L}^{-1}\{\frac{s}{(s^2+\alpha^2)(s+1)(s+2)}\} = \mathcal{L}^{-1}\{\frac{s}{s^2+\alpha^2} \cdot \frac{1}{(s+1)(s+2)}\} \tag{1523}$$

$$= \mathcal{L}^{-1}\{\frac{s}{s^2+\alpha^2}\} * \mathcal{L}^{-1}\{\frac{1}{(s+1)(s+2)}\} \tag{1524}$$

$$\stackrel{*}{=} (\cos(\alpha t)) * (e^{-t} - e^{-2t}) \tag{1525}$$

$$= \int_0^t \cos(\alpha u)\left(e^{t-u} - e^{2(t-u)}\right) du. \tag{1526}$$

We now see that

$$y(t) = \mathcal{L}^{-1}\{\frac{s}{(s^2+\alpha^2)(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)}\} \tag{1527}$$

$$= \mathcal{L}^{-1}\{\frac{s}{(s^2+\alpha^2)(s+1)(s+2)}\} + \mathcal{L}^{-1}\{\frac{s+3}{(s+1)(s+2)}\} \tag{1528}$$

$$= \boxed{\int_0^t \cos(\alpha u)\left(e^{t-u} - e^{2(t-u)}\right) du + 2e^{-t} - e^{-2t}}. \tag{1529}$$

Problem 11.44: Show that $W(5, \sin^2(t), \cos(2t)) = 0$. Can this also be shown without directly computing the Wronskian?

Solution: We first proceed by direct calculation. Let $f(t) = 5$, $g(t) = \sin^2(t)$, and $h(t) = \cos(2t)$. We see that

$$f'(t) = f''(t) = 0, \tag{1530}$$

$$g'(t) = 2\sin(t)\cos(t) = \sin(2t) \to g''(t) = 2\cos(2t), \text{ and} \tag{1531}$$

$$h'(t) = -2\sin(2t) \to h''(t) = -4\cos(2t), \text{ so} \tag{1532}$$

$$W(5, \sin^2(t), \cos(2t)) = W(f, g, h) = \begin{vmatrix} f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \end{vmatrix} \tag{1533}$$

$$= \begin{vmatrix} 5 & \sin^2(t) & \cos(2t) \\ 0 & \sin(2t) & -2\sin(2t) \\ 0 & 2\cos(2t) & -4\cos(2t) \end{vmatrix} \tag{1534}$$

$$= 5\begin{vmatrix} \sin(2t) & -2\sin(2t) \\ 2\cos(2t) & -4\cos(2t) \end{vmatrix} - 0 \cdot \begin{vmatrix} \sin^2(t) & \cos(2t) \\ 2\cos(2t) & -4\cos(2t) \end{vmatrix} + 0 \cdot \begin{vmatrix} \sin^2(t) & \cos(2t) \\ \sin(2t) & -2\sin(2t) \end{vmatrix} \tag{1535}$$

$$= 5\begin{vmatrix} \sin(2t) & -2\sin(2t) \\ 2\cos(2t) & -4\cos(2t) \end{vmatrix} \tag{1536}$$

$$= 5\left((\sin(2t)) \cdot (-4\cos(2t)) - (-2\sin(2t)) \cdot (2\cos(2t)\right) = 0. \tag{1537}$$

Since $W(5, \sin^2(t), \cos(2t)) = 0$, we see that $5, \sin^2(t)$, and $\cos(2t)$ are linearly dependent. To find the linear dependence relation, we recall that $\cos(2t) = 1 - 2\sin^2(t)$, so

$$-\frac{1}{5}\cdot(5)+2(\sin^2(t))+(\cos(2t)) = -1+2\sin^2(t)+(1-2\sin^2(t)) = 0. \tag{1538}$$

The linear dependence relation that is shown between $5, \sin^2(t)$, and $\cos(2t)$ in equation (1538) is also sufficient for deducing that $W(5, \sin^2(t), \cos(2t)) = 0$.

Problem 11.45: Find the general solution to the differential equation

(1539) $$y''' + y' = \sec(t).$$

Solution: We see that $1, \sin(t)$, and $\cos(t)$ are 3 linearly independent solutions to the homogeneous equation corresponding to equation (1539). Letting $Y(t)$ denote the general solution to equation (1539), we recall that

(1540) $$Y(t) = y_1(t)\int_0^t \frac{W_1(t)g(t)}{W(t)}dt + y_2(t)\int_0^t \frac{W_2(t)g(t)}{W(t)}dt + y_3(t)\int_0^t \frac{W_3(t)g(t)}{W(t)}dt$$

(1541) $$= 1\cdot\int_0^t \frac{W_1(t)\sec(t)}{W(t)}dt + \sin(t)\int_0^t \frac{W_2(t)\sec(t)}{W(t)}dt + \cos(t)\int_0^t \frac{W_3(t)\sec(t)}{W(t)}dt.$$

Noting that

(1542) $$W(t) = W(1,\sin(t),\cos(t)) = \begin{vmatrix} 1 & \sin(t) & \cos(t) \\ 0 & \cos(t) & -\sin(t) \\ 0 & -\sin(t) & -\cos(t) \end{vmatrix}$$

(1543) $$= 1\cdot\begin{vmatrix} \cos(t) & -\sin(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} - 0\cdot\begin{vmatrix} \sin(t) & \cos(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} + 0\cdot\begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix}$$

(1544) $$= \begin{vmatrix} \cos(t) & -\sin(t) \\ -\sin(t) & -\cos(t) \end{vmatrix} = \cos(t)(-\cos(t)) - (-\sin(t))(-\sin(t)) = -1,$$

(1545) $$W_1(t) = W_1(1,\sin(t),\cos(t))(t) = \begin{vmatrix} 0 & \sin(t) & \cos(t) \\ 0 & \cos(t) & -\sin(t) \\ 1 & -\sin(t) & -\cos(t) \end{vmatrix}$$

(1546) $$= \begin{vmatrix} \sin(t) & \cos(t) \\ \cos(t) & -\sin(t) \end{vmatrix} = \sin(t)(-\sin(t)) - \cos(t)\cos(t) = -1,$$

(1547) $$W_2(t) = W_2(1, \sin(t), \cos(t))(t) = \begin{vmatrix} 1 & 0 & \cos(t) \\ 0 & 0 & -\sin(t) \\ 0 & 1 & -\cos(t) \end{vmatrix}$$

(1548) $$= -\begin{vmatrix} 1 & \cos(t) \\ 0 & -\sin(t) \end{vmatrix} = -\left(1 \cdot (-\sin(t)) - 0 \cdot \cos(t)\right) = \sin(t), \text{ and}$$

(1549) $$W_3(t) = W_3(1, \sin(t), \cos(t)) = \begin{vmatrix} 1 & \sin(t) & 0 \\ 0 & \cos(t) & 0 \\ 0 & -\sin(t) & 1 \end{vmatrix}$$

(1550) $$= \begin{vmatrix} 1 & \sin(t) \\ 0 & \cos(t) \end{vmatrix} = 1 \cdot \cos(t) - 0 \cdot \sin(t) = \cos(t).$$

We now see that

(1551) $$Y(t) = 1 \cdot \int_0^t \frac{W_1(t)\sec(t)}{W(t)}dt + \sin(t)\int_0^t \frac{W_2(t)\sec(t)}{W(t)}dt + \cos(t)\int_0^t \frac{W_3(t)\sec(t)}{W(t)}dt$$

(1552) $$= \int_0^t \frac{-1 \cdot \sec(t)}{-1}dt + \sin(t)\int_0^t \frac{\sin(t)\sec(t)}{-1}dt + \cos(t)\int_0^t \frac{\cos(t)\sec(t)}{-1}dt$$

(1553) $$= \int_0^t \sec(t)dt - \sin(t)\int_0^t \tan(t)dt - \cos(t)\int_0^t 1dt$$

(1554) $$= \ln|\sec(t) + \tan(t)| + c_1 - \sin(t)(-\ln|\cos(t)| + c_2) - \cos(t)(t + c_3)$$

(1555) $$= \underbrace{(\ln|\sec(t) + \tan(t)| + \sin(t)\ln|\cos(t)| - t\cos(t))}_{y_p(t)} + \underbrace{(c_1 - c_2\sin(t) - c_3\cos(t))}_{y_c(t)}.$$

Problem 11.46: Let $y = \phi(x)$ be a solution to the initial value problem

$$y'' + x^2y' + \sin(x)y = 0; \quad y(0) = a_0, y'(0) = a_1. \tag{1556}$$

Find $\phi''(0), \phi'''(0)$, and $\phi^{(4)}(0)$.

Solution: We proceed by trying to find a series solutions to equation (1556) centered at $x = 0$. Letting

$$y(x) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots, \tag{1557}$$

we see that $\phi^{(n)}(0) = n!a_n$, so we only need to determine a_2, a_3, and a_4. We also note that

$$y'(x) = \sum_{n=0}^{\infty} na_nx^{n-1} \overset{m=n-1}{=} \sum_{m=-1}^{\infty} (m+1)a_{m+1}x^m$$
$$= \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots, \tag{1558}$$

$$x^2y'(x) = \sum_{m=0}^{\infty}(m+1)a_{m+1}x^{m+2} \overset{k=m+2}{=} \sum_{k=2}^{\infty}(k-1)a_{k-1}x^k$$
$$= a_1x^2 + 2a_2x^3 + 3a_3x^4 + \cdots, \tag{1559}$$

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} \overset{j=n-2}{=} \sum_{j=-2}^{\infty}(j+2)(j+1)a_{j+2}x^j$$
$$= \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j = 2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+\cdots, \text{ and} \tag{1560}$$

$$\sin(x)y(x) = (x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) \tag{1561}$$

$$= x(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) - \frac{x^3}{6}(a_0 + a_1x + \cdots) + \cdots \tag{1562}$$

$$= a_0x + a_1x^2 + (a_2 - \frac{a_0}{6})x^3 + (a_3 - \frac{a_1}{6})x^4 + \cdots . \tag{1563}$$

Combining the results of the previous calculations, we see that

$$\begin{aligned} 0 &= y'' + x^2y' + \sin(x)y \tag{1564} \\ &= \left(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \cdots\right) + \left(a_1x^2 + 2a_2x^3 + 3a_3x^4 + \cdots\right) \\ &\quad + \left(a_0x + a_1x^2 + (a_2 - \frac{a_0}{6})x^3 + (a_3 - \frac{a_1}{6})x^4 + \cdots\right) \end{aligned}$$

$$\begin{aligned} &= (2a_2) + (6a_3 + a_0)x + (12a_4 + 2a_1)x^2 + (20a_5 + 3a_2 - \frac{a_0}{6})x^3 \\ &\quad + (30a_6 + 4a_3 - \frac{a_1}{6})x^4 + \cdots \end{aligned} \tag{1565}$$

$$\rightarrow \begin{array}{ccccccc} 2a_2 & & & & & = & 0 \\ a_0 & + & 6a_3 & & & = & 0 \\ 2a_1 & + & 12a_4 & & & = & 0 \\ -\frac{a_0}{6} & + & 3a_2 & + & 20a_5 & = & 0 \\ -\frac{a_1}{6} & + & 4a_3 & + & 30a_6 & = & 0 \end{array} \rightarrow (a_2, a_3, a_4) = (0, -\frac{a_0}{6}, -\frac{a_1}{6}) \tag{1566}$$

$$\rightarrow \boxed{(\phi''(0), \phi'''(0), \phi^{(4)}(0)) = (0, -a_0, -4a_1)}. \tag{1567}$$

Problem 11.47: Solve the differential equation

$$y' + (x+1)y = x+1 \tag{1568}$$

by finding a series solution and by using an integrating factor, then compare your answers.

Solution: We will first solve equation (1568) by finding a series solution. We choose to find a series solution centered at $x = -1$ for convenience. Letting

$$y(x) = \sum_{n=0}^{\infty} a_n(x-(-1))^n = \sum_{n=0}^{\infty} a_n(x+1)^n = a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots \tag{1569}$$

we see that

$$y'(x) = \sum_{n=0}^{\infty} na_n(x+1)^{n-1} \overset{m=n-1}{=} \sum_{m=0}^{\infty}(m+1)a_{m+1}(x+1)^m, \text{ and} \tag{1570}$$

$$(x+1)y(x) = \sum_{n=0}^{\infty} a_n(x+1)^{n+1} \overset{k=n+1}{=} \sum_{k=1}^{\infty} a_{k-1}(x+1)^k. \tag{1571}$$

Since

$$1\cdot(x+1) = y' + (x+1)y = \sum_{m=0}^{\infty}(m+1)a_{m+1}(x+1)^m + \sum_{k=1}^{\infty} a_{k-1}(x+1)^k \tag{1572}$$

$$\overset{*}{=} a_1 + \sum_{n=1}^{\infty}\left((n+1)a_{n+1} + a_{n-1}\right)(x+1)^n, \tag{1573}$$

we see that

$$\begin{array}{rcrcl} & & a_1 & = & 0 \\ 2a_2 & + & a_0 & = & 1 \\ (n+1)a_{n+1} & + & a_{n-1} & = & 0 \text{ for } n \geq 2 \end{array} \tag{1574}$$

(1575) $$\to a_2 = \frac{1-a_0}{2}, a_{n+1} = -\frac{1}{n+1}a_{n-1} \text{ for } n \geq 2$$

(1576) $$\to a_4 = -\frac{a_2}{4} = -\frac{1-a_0}{4\cdot 2}, a_6 = -\frac{a_4}{6} = \frac{1-a_0}{6\cdot 4\cdot 2}, a_8 = \cdots$$

(1577) $$\to a_n = \begin{cases} 0 & \text{if } n \text{ is odd.} \\ \frac{a_0-1}{(-2)^{\frac{n}{2}}(\frac{n}{2}!)} & \text{if } n \text{ is even and } n \geq 2 \end{cases}$$

It follows that the series solutions to equation (1568) is

(1578) $$y(x) \overset{m=\frac{n}{2}}{=} a_0 + (a_0-1)\sum_{m=1}^{\infty}\frac{(x+1)^{2m}}{(-2)^m m!} = \boxed{1 + (a_0-1)\sum_{m=0}^{\infty}\frac{(x+1)^{2m}}{(-2)^m m!}},$$

where a_0 can be determined by an initial condition if one is given.

We will now solve equation (1568) by using an integrating factor. For convenience, we recall that equation (1568) is

(1579) $$y' + (x+1)y = x+1.$$

Since the coefficient of y' is already 1, we see that the integrating factor $I(x)$ is given by

(1580) $$I(x) = e^{\int p(x)dx} = e^{\int (x+1)dx} \overset{*}{=} e^{\frac{(x+1)^2}{2}}.$$

Multiplying both sides of equation (1579) by $I(x)$ yields

(1581) $$(x+1)e^{\frac{(x+1)^2}{2}} = e^{\frac{(x+1)^2}{2}}y' + (x+1)e^{\frac{(x+1)^2}{2}}y = (e^{\frac{(x+1)^2}{2}}y)'$$

(1582) $$e^{\frac{(x+1)^2}{2}}y = \int (x+1)e^{\frac{(x+1)^2}{2}}dx = e^{\frac{(x+1)^2}{2}} + c$$

(1583) $$\to y(x) = \boxed{1 + ce^{-\frac{(x+1)^2}{2}}}.$$

Recalling that

(1584) $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ we see that}$$

(1585) $$y(x) = 1 + ce^{-\frac{(x+1)^2}{2}} = 1 + c\sum_{n=0}^{\infty} \frac{(-\frac{(x+1)^2}{2})^n}{n!} = 1 + c\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{(-2)^n n!}.$$

By identifying m with n and identifying c with $a_0 - 1$, we see that both methods of solution yield the same answer.

Problem 11.48: Determine a lower bound for the radii of convergence r_1 and r_2 of the series solution to the differential equation

$$(1 + x^3)y'' + 4xy' + y = 0, \tag{1586}$$

centered at $x_1 = 0$ and $x_2 = 2$. Then find the series solution to equation (1586) centered at $x_2 = 2$.

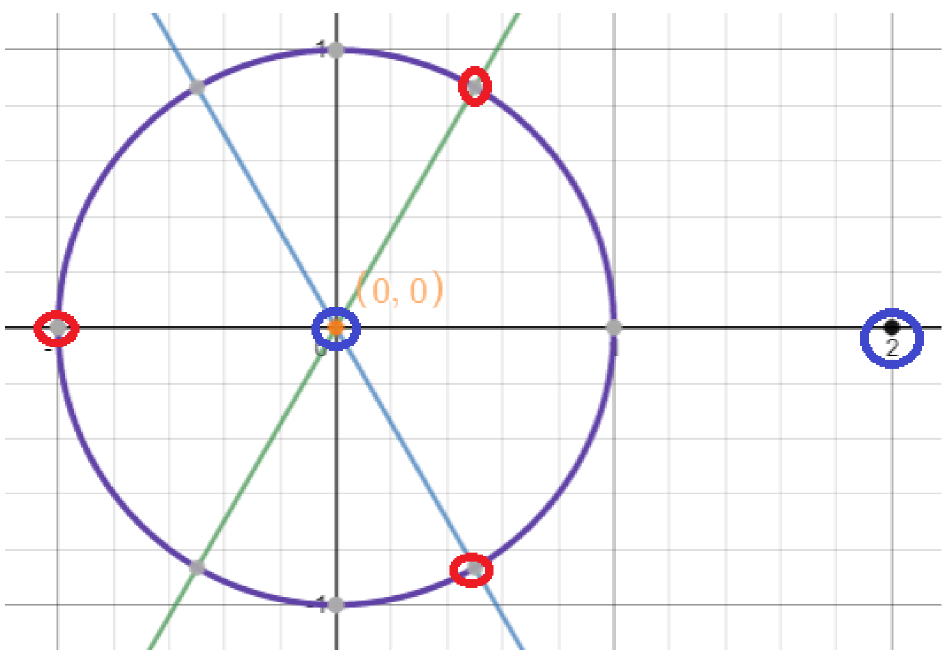
Solution: Firstly, we rewrite equation (1586) in standard form to obtain

$$y'' + \frac{4x}{1+x^3}y' + \frac{1}{1+x^3}y = 0. \tag{1587}$$

We see that as long as $1 + x^3 \neq 0$, then all coefficient functions of equation (1587) are continuous. We see that for

$$x \in \{e^{\frac{\pi}{3}i}, e^{\pi i}, e^{\frac{5\pi}{3}i}\} = \{\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i\}, \text{ we have} \tag{1588}$$

$$1 + x^3 = 1 + e^{\pi i} = 0. \tag{1589}$$



We now see that all coefficient functions in equation (1587) are continuous in a ball of radius 1 (in the complex plane) centered at the origin, so a series solution to equation (1586) centered at $x = 0$ has a radius of convergence of at least 1. Similarly, we note that

$$|2-(-1)| = 3, \tag{1590}$$

$$|2-(\frac{1}{2}+\frac{\sqrt{3}}{2}i)| = |\frac{3}{2}-\frac{\sqrt{3}}{2}i| = \sqrt{(\frac{3}{2})^2+(\frac{\sqrt{3}}{2})^2} = \sqrt{3}, \text{ and} \tag{1591}$$

$$|2-(\frac{1}{2}-\frac{\sqrt{3}}{2}i)| = |\frac{3}{2}+\frac{\sqrt{3}}{2}| = \sqrt{(\frac{3}{2})^2+(\frac{\sqrt{3}}{2})^2} = \sqrt{3}, \tag{1592}$$

so the coefficient functions in equation (1587) are continuous in a ball of radius $\sqrt{3}$ (in the complex plane) centered at 2, so the series solution to equation (1586) centered at $x = 2$ has a radius of convergence of at least $\sqrt{3}$.

We will now begin finding the series solution to equation (1586) centered at $x = 2$. Firstly, we note that we can rewrite equation (1586) as follows.

$$0 = (1+x^3)y''+4xy'+y = (1+(x-2+2)^3)y''+4(x-2+2)y'+y \tag{1593}$$

$$= (1+(x-2)^3+6(x-2)^2+12(x-2)+8)y''+4(x-2+2)y'+y \tag{1594}$$

$$= (x-2)^3y''+6(x-2)^2y''+12(x-2)y''+9y''+4(x-2)y'+2y'+y. \tag{1595}$$

Since we are working with the series solution for $y = y(x)$ centered at $x = 2$, we have

$$y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n, \tag{1596}$$

$$y'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n, \tag{1597}$$

(1598) $$(x-2)y'(x) = \sum_{n=1}^{\infty} na_n(x-2)^n,$$

(1599) $$y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-2)^n,$$

(1600) $$(x-2)y''(x) = \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-2)^n$$

(1601) $$(x-2)^2y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^n$$

(1602) $$(x-2)^3y''(x) = \sum_{n=3}^{\infty}(n-1)(n-2)a_{n-1}(x-2)^n, \text{ so}$$

(1603) $$0 = (x-2)^3y''+6(x-2)^2y''+12(x-2)y''+9y''+4(x-2)y'+2y'+y.$$

(1604) $$\begin{aligned} &= \sum_{n=3}^{\infty}(n-1)(n-2)a_{n-1}(x-2)^n + 6\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^n \\ &+ 12\sum_{n=1}^{\infty}(n+1)na_{n+1}(x-2)^n + 9\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-2)^n \\ &+ 4\sum_{n=1}^{\infty} na_n(x-2)^n + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-2)^n + \sum_{n=0}^{\infty} a_n(x-2)^n \end{aligned}$$

(1605)
$$\begin{aligned} &= (a_0+2a_1+18a_2)+(5a_1+28a_2+54a_3)(x-2)+(21a_2+82a_3+108a_4)(x-2)^2 \\ &+ \sum_{n=3}^{\infty}\Big((n-1)(n-2)a_{n-1} + 6n(n-1)a_n + 12(n+1)na_{n+1} \\ &+ 9(n+2)(n+1)a_{n+2} + 4na_n + 2(n+1)a_{n+1} + a_n\Big)(x-2)^n \end{aligned}$$

(1606)
$$= (a_0+2a_1+18a_2)+(5a_1+28a_2+54a_3)(x-2)+(21a_2+82a_3+108a_4)(x-2)^2 + \sum_{n=3}^{\infty}\Big((n-1)(n-2)a_{n-1}+(6n^2-2n+1)a_n+(12n^2+14n+2)a_{n+1} + 9(n+2)(n+1)a_{n+2}\Big)(x-2)^n$$

(1607)
$$\rightarrow \begin{array}{l} a_0 = y(2) \\ a_1 = y'(2) \\ a_0+2a_1+18a_2 = 0 \\ 5a_1+28a_2+54a_3 = 0 \\ 21a_2+82a_3+108a_4 = 0 \\ (n-1)(n-2)a_{n-1}+(6n^2-2n+1)a_n \\ +(12n^2+14n+2)a_{n+1}+9(n+2)(n+1)a_{n+2} = 0 \text{ for } n \geq 3 \end{array}$$

(1608)
$$\rightarrow \begin{array}{l} a_0 = y(2) \\ a_1 = y'(2) \\ a_2 = -\frac{1}{9}a_1 - \frac{1}{18}a_0 \\ a_3 = -\frac{28}{54}a_2 - \frac{5}{54}a_1 \\ a_4 = -\frac{82}{108}a_3 - \frac{21}{108}a_2 \\ a_{n+2} = \frac{1}{9(n+2)(n+1)}\Big((n-1)(n-2)a_{n-1} \\ +(6n^2-2n+1)a_n+(12n^2+14n+2)a_{n+1}\Big) \text{ for } n \geq 3 \end{array} .$$

Once the recurrence in equations (1608) is solved, our solution will be

(1609)
$$y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n.$$

Problem 12.1: Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{1610}$$

Show that for a solution $u(r,\theta) = R(r)\Theta(\theta)$ having separated variables, we must have

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \text{ and} \tag{1611}$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \tag{1612}$$

where λ is some constant.

Solution: We begin by plugging $u(r,\theta) = R(r)\Theta(\theta)$ into equation (1610) to see that

$$0 = \frac{\partial^2}{\partial r^2}(R(r)\Theta(\theta)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)\Theta(\theta)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)\Theta(\theta)) \tag{1613}$$

$$= R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) \tag{1614}$$

$$\to -\frac{1}{r^2}R(r)\Theta''(\theta) = R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) \tag{1615}$$

$$\to \frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} \stackrel{*}{=} \gamma. \tag{1616}$$

To derive equation (1611) we use equation (1616) to see that

$$\frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} = \gamma \to R''(r) + \frac{1}{r}R(r) = -\frac{\gamma}{r^2}R(r) \tag{1617}$$

$$\to R''(r) + \frac{1}{r}R'(r) + \frac{\gamma}{r^2}R(r) = 0 \to r^2 R''(r) + rR'(r) + \gamma R(r) = 0. \tag{1618}$$

To derive equation (1612) we use equation (1616) to see that

(1619) $$\frac{\Theta''(\theta)}{\Theta(\theta)} = \gamma \to \Theta''(\theta) = \gamma\Theta(\theta) \to \Theta''(\theta) - \gamma\Theta(\theta) = 0.$$

We now see that we can pick our constant λ as $\lambda = -\gamma$.

Problem 12.2: Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1620}$$

Show that for a solution $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ having separated variables, we must have

$$\Theta''(\theta) + \mu\Theta(\theta) = 0, \tag{1621}$$

$$Z''(z) + \lambda Z(z) = 0, \text{ and} \tag{1622}$$

$$r^2R''(r) + rR'(r) - (r^2\lambda + \mu)R(r) = 0, \tag{1623}$$

where μ and λ are constants.

Solution: We proceed as in problem 6.2.27 and plug $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$ into equation (1620) to see that

$$\frac{\partial^2}{\partial r^2}(R(r)\Theta(\theta)Z(z)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)\Theta(\theta)Z(z)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)\Theta(\theta)Z(z)) + \frac{\partial^2}{\partial z^2}(R(r)\Theta(\theta)Z(z)) = 0 \tag{1624}$$

$$\rightarrow R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) = 0. \tag{1625}$$

We will now try to derive equation (1622) from equation (1625). Beginning with equation (1625) we see that

$$R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) = 0. \tag{1626}$$

$$-R(r)\Theta(\theta)Z''(z) = R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) \tag{1627}$$

$$\rightarrow \frac{Z''(z)}{Z(z)} = \frac{R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta)}{-R(r)\Theta(\theta)} \stackrel{*}{=} -\lambda \tag{1628}$$

$$\rightarrow Z''(z) = -\lambda Z(z) \rightarrow Z''(z) + \lambda Z(z) = 0. \tag{1629}$$

We will now derive equation (1621) from equation (1625). Beginning with equation (1625) we see that

(1630) $$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)=0.$$

(1631) $$-\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)=R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)$$

(1632) $$\to\frac{\Theta''(\theta)}{\Theta(\theta)}=\frac{R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)}{-\frac{1}{r^2}R(r)Z(z)}\overset{*}{=}-\mu$$

(1633) $$\to\Theta''(\theta)=-\mu\Theta(\theta)\to\Theta''(\theta)+\mu\Theta(\theta)=0.$$

Lastly, we will derive equation (1623) from equation (1625). Beginning with equation (1625) we see that

(1634) $$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)=0.$$

(1635) $$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)=-\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)-R(r)\Theta(\theta)Z''(z)$$

(1636) $$\to\frac{R''(r)+\frac{1}{r}R'(r)}{R(r)}=\frac{-\frac{1}{r^2}\Theta''(\theta)Z(z)-\Theta(\theta)Z''(z)}{\Theta(\theta)Z(z)}=-\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}+\frac{-Z''(z)}{Z(z)}=\frac{\mu}{r^2}+\lambda$$

(1637) $$\to R''(r)+\frac{1}{r}R'(r)=(\frac{\mu}{r^2}+\lambda)R(r)\to R''(r)+\frac{1}{r}R'(r)-(\frac{\mu}{r^2}+\lambda)R(r)=0$$

(1638) $$\to r^2R''(r)+rR'(r)-(\mu+r^2\lambda)R(r)=0.$$

Problem 12.3: Find the values of λ (eigenvalues) for which the following problem has a nontrivial solution. Also determine the corresponding nontrivial solutions (eigenfunctions).

$$y'' + \lambda y = 0; \quad 0 < x < \pi, \quad y(0) - y'(0) = 0,\ y(\pi) = 0. \tag{1639}$$

Solution: We begin by examining the characteristic equation for equation (1639) and see that

$$r^2 + \lambda = 0 \to r = \pm\sqrt{-\lambda}. \tag{1640}$$

We now consider 3 separate cases based on the sign of λ.

Case 1: $\lambda = 0$.

In this case we see that $r = 0$ is a double root of the characteristic equation, so the general solution to equation (1639) is

$$y(t) = c_1e^{0\cdot t} + c_2te^{0\cdot t} = c_1 + c_2t. \tag{1641}$$

Noting that

$$y'(t) = c_2, \tag{1642}$$

we proceed to make use of the initial conditions to see that

$$\begin{array}{rcllcrcl} 0 & = & y(0) - y'(0) & = c_1 - c_2 & \multirow{2}{*}{$\to$} & c_1 & = & c_2 \\ 0 & = & y(\pi) & = c_1 + \pi c_2 & & c_1 & = & -\pi c_2 \end{array} \to (c_1, c_2) = (0, 0), \tag{1643}$$

so we only have trivial solutions in this case.

Case 2: $\lambda < 0$.

In this case we see that $r = \sqrt{-\lambda}$ and $r = -\sqrt{-\lambda}$ are distinct real roots of the characteristic equation, so the general solution to equation (1639) is

(1644) $$y(t) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}.$$

Noting that

(1645) $$y'(t) = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}t} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}t},$$

we proceed to make use of the initial conditions to see that

(1646) $$\begin{aligned} 0 &= y(0) - y'(0) &= c_1(1-\sqrt{-\lambda}) + c_2(1+\sqrt{-\lambda}) \\ 0 &= y(\pi) &= c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} \end{aligned}$$

(1647) $$\rightarrow \underbrace{\begin{bmatrix} 1-\sqrt{-\lambda} & 1+\sqrt{-\lambda} \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \text{ Since}$$

(1648) $$\det(A) = e^{-\sqrt{-\lambda}\pi}(1-\sqrt{-\lambda}) - e^{\sqrt{-\lambda}\pi}(1+\sqrt{-\lambda}) < 0,$$

we see that $\det(A) \neq 0$, so A is a nonsingular matrix. It follows that equation (1647) only has the trivial solution of $(c_1, c_2) = (0, 0)$, so we only have trivial solutions to equation (1639) in this case as well.

Case 3: $\lambda > 0$.

In this case we see that $r = \sqrt{-\lambda}$ and $r = -\sqrt{-\lambda}$ are distinct complex roots of the characteristic equation, so the general solution to equation (1639) is

(1649) $$y(t) = c_1' e^{\sqrt{-\lambda}t} + c_2' e^{-\sqrt{-\lambda}t} = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t).$$

Noting that

(1650) $$y'(t) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}t) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}t),$$

we proceed to make use of the initial conditions to see that

$$0 = y(0) - y'(0) = c_1 - c_2\sqrt{\lambda} \qquad (1651)$$
$$0 = y(\pi) = c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi)$$

$$\rightarrow \begin{array}{rcl} c_1 & = & c_2\sqrt{\lambda} \\ 0 & = & c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) \end{array} \qquad (1652)$$

$$\rightarrow \begin{array}{rcl} c_1 & = & c_2\sqrt{\lambda} \\ 0 & = & c_2\left(\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi)\right) \end{array}. \qquad (1653)$$

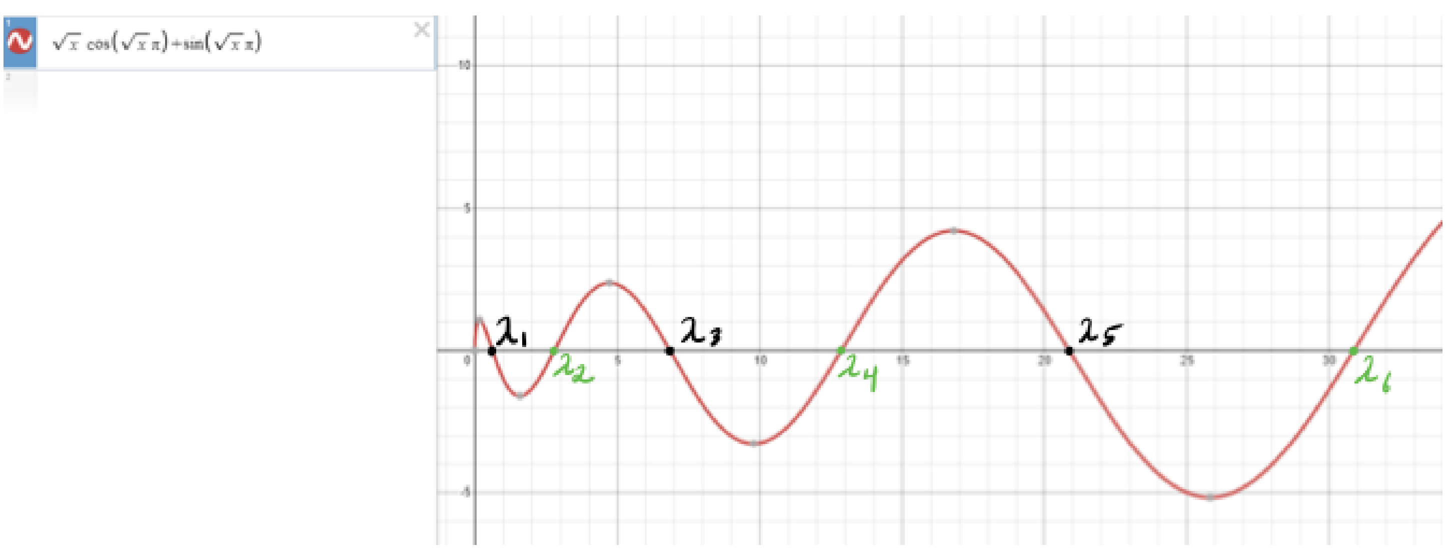
In order to have nontrivial solutions to equation (1639) we need to have nontrivial solutions to system of equations in (1653). We see that $c_1 = 0$ if and only if $c_2 = 0$, and that c_2 will be 0 if

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) \neq 0. \qquad (1654)$$

It follows that we want to find the values of λ for which

$$\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) = 0, \qquad (1655)$$

so that we can find a corresponding $c_2 \neq 0$. Sadly, equation (1655) is not something that can be explicitly solved by hand. Therefore, we let $\{\lambda_n\}_{n=1}^{\infty}$ denote the solutions to equation (1655) as shown in the picture below.



To be precise, we know that the solutions to equation (1655) exist even though we cannot write down exactly what they are, so we talk about them by enumerating them as $\{\lambda_n\}_{n=1}^{\infty}$.

We note that for any $n \geq 1$, if $\lambda = \lambda_n$, then the second equation in (1653) holds for any value of c_2, so we will have $(c_1, c_2) = (c_2\sqrt{\lambda_n}, c_2)$ is a nontrivial solution to equation (1639). In conclusion, the eigenvalues of (1639) are $\{\lambda_n\}_{n=1}^{\infty}$ and the eigen functions corresponding to any given λ_n are

$$y(t) = c\left(\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}t) + \sin(\sqrt{\lambda_n}t)\right); c \in \mathbb{R}. \tag{1656}$$

Problem 12.4: Find the values of λ for which the initial value problem given by

$$y'' - 2y' + \lambda y = 0; \quad 0 < x < \pi \tag{1657}$$

$$y(0) = y(\pi) = 0 \tag{1658}$$

has nontrivial solutions. Then, for each such λ, find the nontrivial solutions.

Solution: We see that the characteristic polynomial of this equation is $r^2 - 2r + \lambda$ and has roots

$$r = \frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}. \tag{1659}$$

We now consider 3 separate cases depending on the sign of $(1 - \lambda)$.

Case 1: $1 - \lambda = 0$.

In this case, $\lambda = 1$ and $r = 1$ is a double root of the characteristic polynomial, so the general solution to equation 1657 is

$$y(t) = c_1 e^t + c_2 t e^t. \tag{1660}$$

We see that

$$0 = y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1, \text{ and} \tag{1661}$$

$$0 = y(\pi) = c_2 \cdot \pi \cdot e^\pi \to c_2 = 0. \tag{1662}$$

Since $(c_1, c_2) = (0, 0)$, we see that in this case we only have the trivial solution.

Case 2: $1 - \lambda > 0$.

In this case, we see that the general solution to equation 1657 is

$$y(t) = c_1e^{(1+\sqrt{1-\lambda})t} + c_2e^{(1-\sqrt{1-\lambda})t}. \tag{1663}$$

We see that

$$0 = y(0) = c_1e^{(1+\sqrt{1-\lambda})\cdot 0} + c_2e^{(1-\sqrt{1-\lambda})\cdot 0} = c_1 + c_2, \text{ and} \tag{1664}$$

$$0 = y(\pi) = c_1e^{(1+\sqrt{1-\lambda})\pi} + c_2e^{(1-\sqrt{1-\lambda})\pi}. \tag{1665}$$

Solving the system of equations given by (1664) and (1665), we see that

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ e^{(1+\sqrt{1-\lambda})\pi} & e^{(1-\sqrt{1-\lambda})\pi} & 0 \end{array}\right] \tag{1666}$$

$$\xrightarrow{R_2-e^{(1+\sqrt{1-\lambda})\pi}R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & e^{(1-\sqrt{1-\lambda})\pi} - e^{(1+\sqrt{1-\lambda})\pi} & 0 \end{array}\right] \tag{1667}$$

$$\xrightarrow{\frac{1}{e^{(1-\sqrt{1-\lambda})\pi}-e^{(1+\sqrt{1-\lambda})\pi}}R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \tag{1668}$$

so $(c_1, c_2) = (0, 0)$. We once again see that we only have the trivial solution.

Case 3: $1 - \lambda < 0$.

In this case, we see that

$$\text{Re}(1 \pm \sqrt{1-\lambda}) = 1 \text{ and } \text{Im}(1 \pm \sqrt{1-\lambda}) = \pm\sqrt{\lambda - 1}, \tag{1669}$$

so the general solution to equation (1657) is

$$y(t) = c_1e^t\cos(\sqrt{\lambda-1}t) + c_2e^t\sin(\sqrt{\lambda-1}t). \tag{1670}$$

We see that

(1671) $$0 = y(0) = c_1e^0\cos(\sqrt{\lambda-1}\cdot 0) + c_2e^0\sin(\sqrt{\lambda-1}\cdot 0) = c_1, \text{ and}$$

(1672) $$0 = y(\pi) = c_2e^{\pi}\sin(\sqrt{\lambda-1}\pi).$$

If $e^{\pi}\sin(\sqrt{\lambda-1}\pi) \neq 0$, then we will have that $(c_1, c_2) = (0, 0)$. Since we are looking for nontrivial solutions, we want the values of λ for which $e^{\pi}\sin(\sqrt{\lambda-1}\pi) = 0$, which is the same as the values of λ for which

(1673) $$\sin(\sqrt{\lambda-1}\pi) = 0.$$

Note: The equation for some other problems of this type (such as problem 6.2.13 from the second edition of the textbook) that corresponds to equation (1673) is not solvable by hand. In such a situation, it is perfectly acceptable to say 'Let $(\lambda_n)_{n=1}^{\infty}$ be the solutions to equation (1673).' From then on, you may work with $(\lambda_n)_{n=1}^{\infty}$ as known values. Luckily, equation (1673) is solvable by hand, so we will just go ahead and solve it.

We recall that the $0's$ of $\sin(x)$ occur exactly at the integer multiples of π. Given $n \in \mathbb{Z}$, we see that

(1674) $$n = \sqrt{\lambda-1} \Leftrightarrow \lambda = n^2 + 1,$$

so $(n^2+1)_{n\in\mathbb{Z}}$ is all of the solutions of equation (1673). We now see that for each integer n, equation (1672) is satisfied by any $c_2 \in \mathbb{R}$.

Putting together the results of all 3 cases, we see that the initial value problem given by equations (1657) and (1658) has nontrivial solutions if and only if $\lambda = n^2 + 1$ for some integer n. Furthermore, for any such $\lambda = n^2 + 1$, the solution to the initial value problem is

(1675) $$y(t) = ce^t\sin(nt),$$

where c can be any real number.

Problem 12.5: Find the fourier series of the function

(1676) $$f(x) = \begin{cases} 1 & \text{if } -2 < x < 0 \\ x & \text{if } 0 < x < 2 \end{cases},$$

over the interval $[-2, 2]$.

Solution: Since our interval has a radius of $L = 2$, we see that the basis we will work with is $(\sin(\frac{2\pi nx}{2L}))_{n=1}^{\infty} \cup (\cos(\frac{2\pi mx}{2L}))_{m=1}^{\infty}$ which simplifies to $(\sin(\frac{\pi nx}{2}))_{n=1}^{\infty} \cup (\cos(\frac{\pi mx}{2}))_{m=1}^{\infty}$. We may now let a_0, $(a_n)_{n=1}^{\infty}$, and $(b_n)_{n=1}^{\infty}$ be such that

(1677) $$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi nx}{2}) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi nx}{2}).$$

First let us determine the sequence $(b_n)_{n=1}^{\infty}$. We note that for each $n \geq 1$ we have

(1678) $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{2\pi nx}{2L})dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{\pi nx}{2})dx$$

$$=\frac{1}{2}\int_{-2}^{2}f(x)\sin(\frac{\pi nx}{2})dx=\frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi nx}{2})dx+\frac{1}{2}\int_{0}^{2}x\sin(\frac{\pi nx}{2})dx. \tag{1679}$$

We see that

$$\frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi nx}{2})dx=-\frac{1}{\pi n}\cos(\frac{\pi nx}{2})\Big|_{x=-2}^{0}=-\frac{2}{\pi n}+\frac{2}{\pi n}\cos(-\pi n) \tag{1680}$$

$$=\begin{cases}0 & \text{if n is even}\\ -\frac{2}{\pi n} & \text{if n is odd}\end{cases}. \tag{1681}$$

Using integration by parts, we also see that

$$\frac{1}{2}\int_{0}^{2}x\sin(\frac{\pi nx}{2})dx=-\frac{1}{\pi n}x\cos(\frac{\pi nx}{2})\Big|_{x=0}^{2}-\int_{0}^{2}-\frac{2}{\pi n}\cos(\frac{\pi nx}{2})dx \tag{1682}$$

$$=-\frac{2}{\pi n}\cos(\pi n)+\left(\frac{2}{\pi^2n^2}\sin(\frac{\pi nx}{2})\Big|_{x=0}^{2}\right)=-\frac{2}{\pi n}\cos(\pi n) \tag{1683}$$

$$=\begin{cases}-\frac{2}{\pi n} & \text{if n is even}\\ \frac{2}{\pi n} & \text{if n is odd}\end{cases}. \tag{1684}$$

Putting all of this together, we see that for $n\ge 1$ we have

$$b_n=\frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi nx}{2})dx+\frac{1}{2}\int_{0}^{2}x\sin(\frac{\pi nx}{2})dx=\begin{cases}-\frac{2}{\pi n} & \text{if n is even}\\ 0 & \text{if n is odd}\end{cases}. \tag{1685}$$

Now let us determine the sequence $(a_n)_{n=1}^{\infty}$. We note that for $n\ge 1$ we have

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos(\frac{2\pi nx}{2L})dx=\frac{1}{2}\int_{-2}^{2}f(x)\cos(\frac{\pi nx}{2})dx \tag{1686}$$

$$=\frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi nx}{2})dx+\frac{1}{2}\int_{0}^{2}x\cos(\frac{\pi nx}{2})dx. \tag{1687}$$

We see that

(1688) $$\frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi nx}{2})dx = \frac{1}{\pi n}\sin(\frac{\pi nx}{2})\Big|_{x=-2}^{0} = 0.$$

Using integration by parts, we also see that

(1689) $$\frac{1}{2}\int_{0}^{2}x\cos(\frac{\pi nx}{2})dx = \frac{1}{\pi n}x\sin(\frac{\pi nx}{2})\Big|_{x=0}^{2} - \int_{0}^{2}\frac{2}{\pi n}\sin(\frac{\pi n}{2})dx$$

(1690) $$= -\frac{1}{\pi n}\int_{0}^{2}\sin(\frac{\pi nx}{2})dx = \frac{2}{\pi^2 n^2}\cos(\frac{\pi nx}{2})\Big|_{x=0}^{2}$$

(1691) $$= \frac{2}{\pi^2 n^2}(\cos(\pi n) - 1) = \begin{cases} 0 & \text{if n is even} \\ \frac{-4}{\pi^2 n^2} & \text{if n is odd} \end{cases}.$$

Putting all of this together, we see that for $n \geq 1$ we have

(1692)
$$a_n = \frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi nx}{2})dx + \frac{1}{2}\int_{0}^{2}x\cos(\frac{\pi nx}{2})dx = \begin{cases} 0 & \text{if n is even} \\ -\frac{4}{\pi^2 n^2} & \text{if n is odd} \end{cases}.$$

Lastly, we see that

(1693) $$a_0 = \frac{1}{2L}\int_{-L}^{L}f(x)dx = \frac{1}{4}\int_{-2}^{2}f(x)dx = \frac{1}{4}\int_{-2}^{0}1dx + \frac{1}{4}\int_{0}^{2}xdx$$

(1694) $$\frac{1}{2} + \left(\frac{x^2}{8}\Big|_{x=0}^{2}\right) = 1.$$

Finally, we see that

(1695) $$f(x) \sim 1 + \left(\sum_{n=1}^{\infty}\frac{2}{\pi^2 n^2}((-1)^n - 1)\cos(\frac{\pi n}{2}x)\right) + \left(\sum_{n=1}^{\infty}\frac{1}{\pi n}((-1)^{n+1} - 1)\sin(\frac{\pi nx}{2})\right)$$

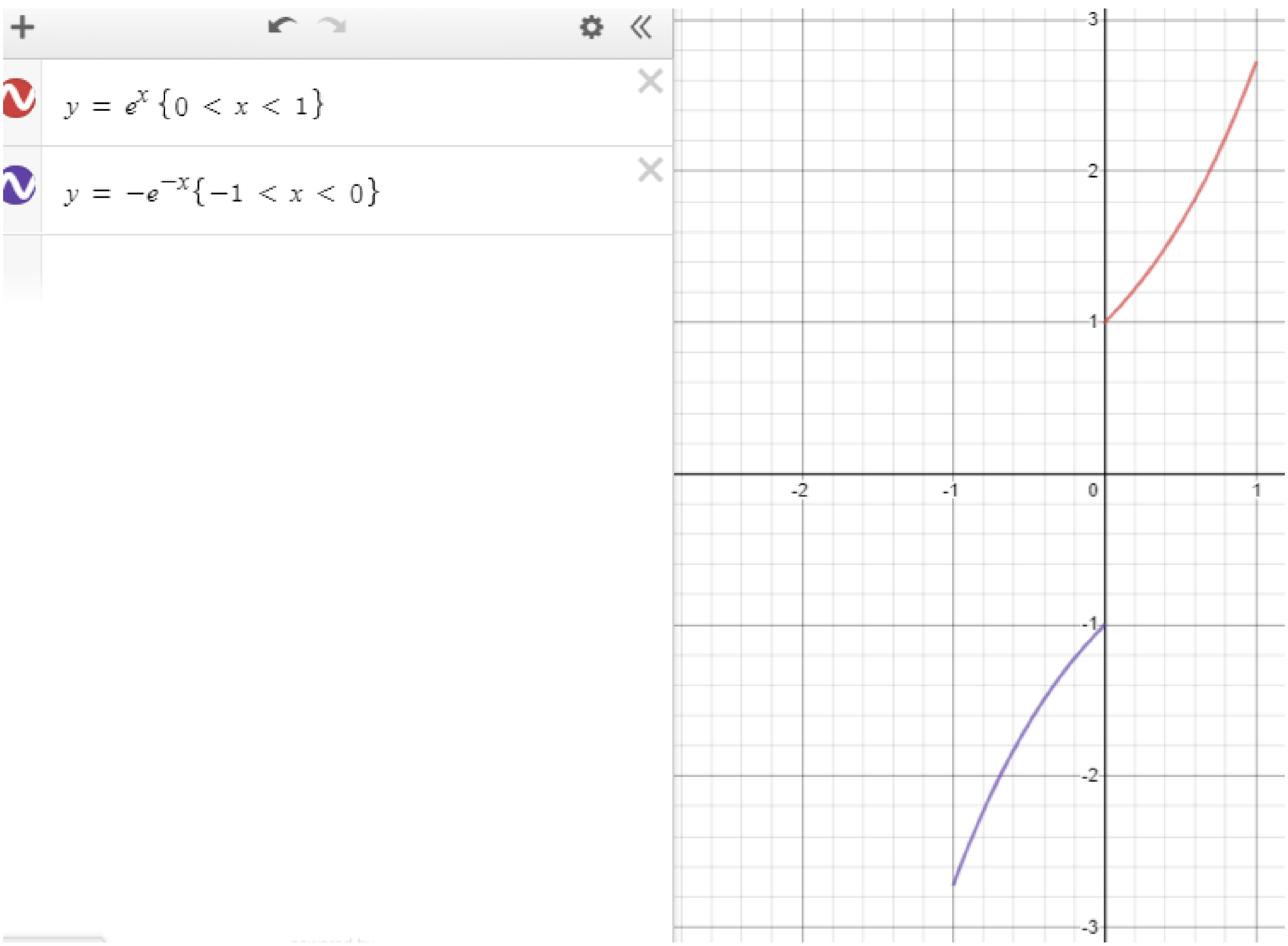
Problem 12.6: Find the Fourier sine series for

$$f(x) = e^x, \quad 0 < x < 1. \tag{1696}$$

Solution: The fourier sine series of $f(x)$ is just the fourier series of $g(x)$, the odd 2-periodic extension of $f(x)$, which is the 2-periodic function defined by the formula

$$g(x) = \begin{cases} f(x) & \text{if} \quad 0 < x < 1 \\ -f(-x) & \text{if} \ -1 < x < 0 \end{cases}. \tag{1697}$$

Below is a graph of $g(x)$ restricted to the interval $(-1, 1)$. The red portion of the graph is also the graph of $f(x)$.



Since $g(x)$ is an odd function (by construction, this will always be the case) the fourier series of $g(x)$ will not have any cosine terms in it. We see that for any $n \geq 1$, we have

$$b_n = \frac{1}{1}\int_{-1}^{1} g(x)\sin(\frac{2n\pi x}{2})dx \overset{\text{by oddness}}{=} \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)dx \tag{1698}$$

$$= 2\int_0^1 e^x \sin(n\pi x)dx = 2\int_0^1 \frac{e^{(1+n\pi i)x} - e^{(1-n\pi i)x}}{2i}dx \tag{1699}$$

$$= -i\int_0^1 (e^{(1+n\pi i)x} - e^{(1-n\pi i)x})dx = -i\left(\frac{e^{(1+n\pi i)x}}{1+n\pi i} - \frac{e^{(1-n\pi i)x}}{1-n\pi i}\right)\Big|_0^1 \tag{1700}$$

$$= \left(\frac{e^{1+n\pi i}}{1+n\pi i} - \frac{e^{1-n\pi i}}{1-n\pi i}\right) - \left(\frac{e^0}{1+n\pi i} - \frac{e^0}{1-n\pi i}\right) \tag{1701}$$

$$= \left(\frac{e(\cos(n\pi) + i\sin(n\pi))}{1+n\pi i} - \frac{e(\cos(n\pi) + i\sin(-n\pi))}{1-n\pi i}\right) - \left(\frac{1}{1+n\pi i} - \frac{1}{1-n\pi i}\right) \tag{1702}$$

$$= \frac{e(-1)^n - 1}{1+n\pi i} - \frac{e(-1)^n - 1}{1-n\pi i} = \frac{2e(-1)^n - 2}{1+n^2\pi^2} \tag{1703}$$

$$\to \boxed{f(x) \sim \sum_{n=1}^{\infty} \frac{2e(-1)^n - 2}{1+n^2\pi^2}\sin(n\pi x)}. \tag{1704}$$

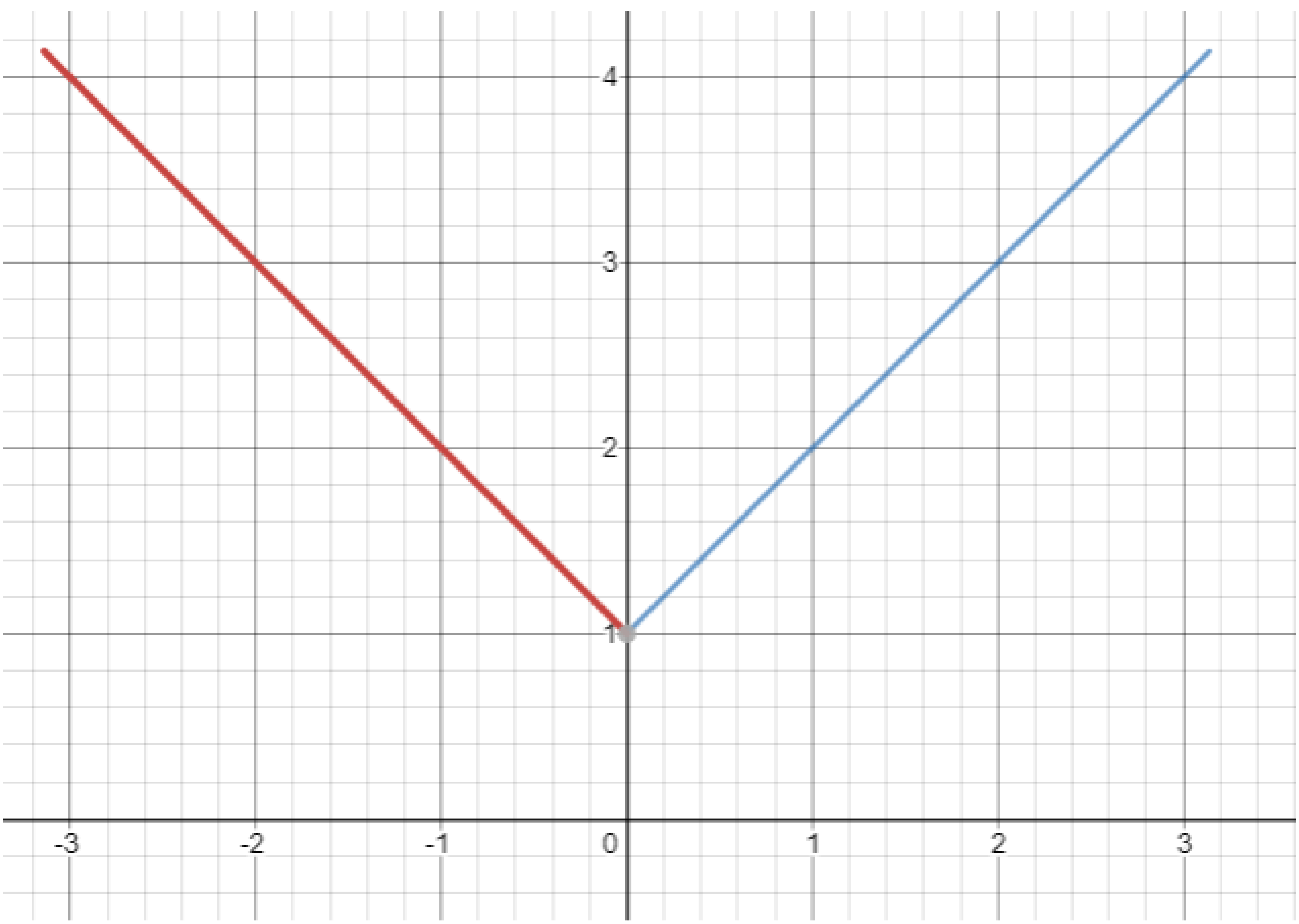
Problem 12.7: Find the Fourier cosine series for

$$f(x) = 1 + x, \quad 0 < x < \pi. \tag{1705}$$

Solution: The fourier cosine series of $f(x)$ is just the fourier series of $g(x)$, the even 2π-periodic extension of $f(x)$, which is the 2π-periodic function defined by the formula

$$g(x) = \begin{cases} f(x) & \text{if} \quad 0 < x < \pi \\ f(-x) & \text{if} \ -\pi < x < 0 \end{cases}. \tag{1706}$$

Below is a graph of $g(x)$ restricted to the interval $(-\pi, \pi)$. The blue portion of the graph is also the graph of $f(x)$.



Since $g(x)$ is an even function (by construction, this will always be the case) the fourier series of $g(x)$ will not have any sine terms in it. We see that for any $n \geq 1$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos\left(\frac{2\pi n x}{2\pi}\right)dx \overset{\text{by evenness}}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \tag{1707}$$

$$= \frac{2}{\pi}\int_0^{\pi}(1+x)\cos(nx)dx = \frac{2}{\pi}\cdot(1+x)\frac{\sin(nx)}{n}\Big|_{x=0}^{\pi} - \frac{2}{\pi}\int_0^{\pi} 1\cdot\frac{\sin(nx)}{n}dx \tag{1708}$$

$$= 0 - \frac{2}{\pi}\left(\frac{-\cos(nx)}{n^2}\Big|_{x=0}^{\pi}\right) = \frac{2\cos(n\pi)-2}{\pi n^2} = \begin{cases} 0 & \text{if n is even} \\ \frac{-4}{\pi n^2} & \text{if n is odd} \end{cases}. \tag{1709}$$

Similarly, we see that

$$a_0 \overset{*}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(1+x)dx \tag{1710}$$

$$\frac{(1+x)^2}{2\pi}\Big|_{x=0}^{\pi} = \frac{(\pi+1)^2-1}{2\pi} = \frac{\pi}{2}+1. \tag{1711}$$

Putting everything together, we see that

$$\boxed{f(x) \sim \frac{\pi}{2}+1+\sum_{n=0}^{\infty} -\frac{4}{\pi(2n+1)^2}\cos((2n+1)x)}. \tag{1712}$$

Problem 12.8: Determine the function to which the Fourrier series of

$$f(x) = |x|, \quad -\pi < x < \pi \tag{1713}$$

converges pointwise.

Note: The graphs for this problem do not have open circles at individual points at which the function is undefined. Luckily, the precise definition of $f(x)$ or its periodic extension at these endpoints does not change the final answer to this question.

Solution: We begin by examining a graph of $f(x)$ and a graph of $g(x)$, the 2π-periodic extension of $f(x)$.

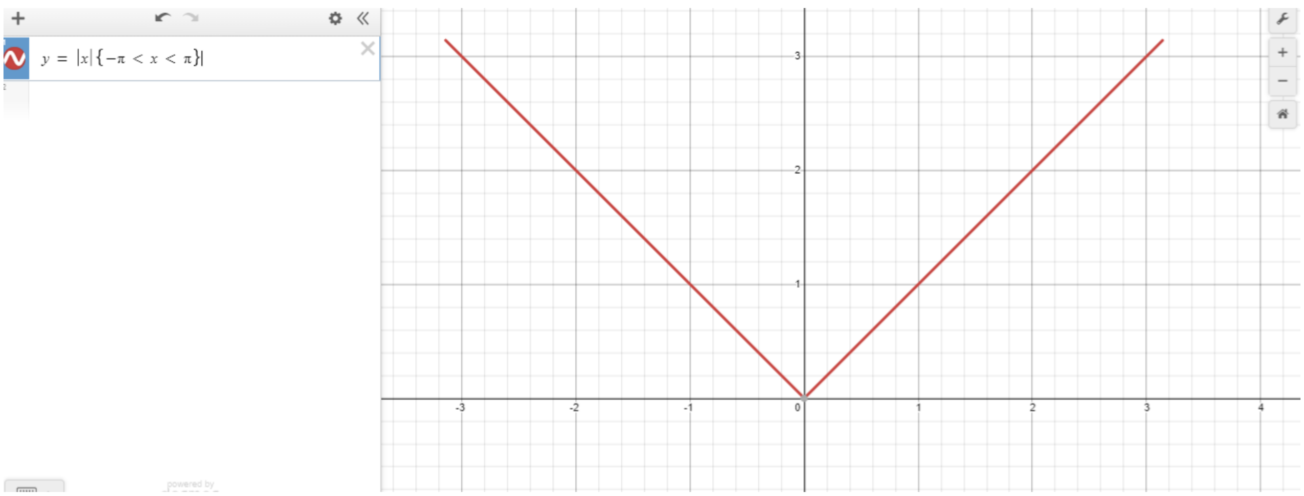


FIGURE 70. Graph of f(x).

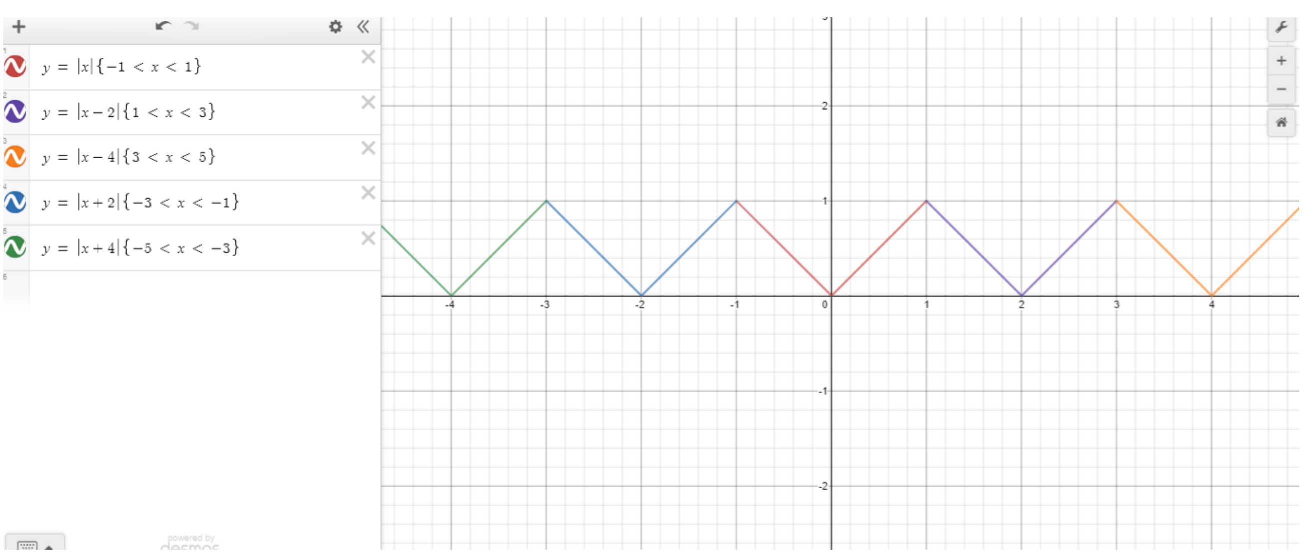


FIGURE 71. Graph of $g(x)$.

We see that if we define $g(n\pi) = 1$ for every odd integer n (since these are precisely the points at which $g(x)$ is currently undefined), then $g(x)$ is a continuous function whose derivative is piecewise continuous. It follows from Theorem 6.3.3 (stated below) that the Fourrier series of $f(x)$ converges pointwise (actually, uniformly) to $g(x)$ (after declaring that $g(n) = 1$ for every odd integer n).

Theorem 6.3.3 (Page 504): Let f (or g in this problem) be a continuous function on $(-\infty, \infty)$ and periodic of period $2L$. If f' is piecewise continuous on $[-L, L]$, then the Fourrier series of f converges uniformly to f on $[-L, L]$ and hence on any interval. That is, for each $\epsilon > 0$, there exists an integer N_0 (that depends on ϵ) such that

$$\left|f(x) - \left[\frac{a_0}{2} + \sum_{n=1}^{N}\left\{a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})\right\}\right]\right| < \epsilon, \tag{1714}$$

for all $N \geq N_0$, and all $x \in (-\infty, \infty)$.

Remark: The astute reader will notice that Theorem 6.3.3 actually gives us more than what the problem originally asked for since uniform convergence is better than pointwise convergence.

Problem 12.9: Determine the function to which the Fourrier series of

(1715) $$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ x^2 & \text{if } 0 < x < \pi \end{cases}$$

converges pointwise.

Note: The graphs for this problem do not have open circles at individual points at which the function is undefined. Luckily, the precise definition of $f(x)$ or its periodic extension at these endpoints does not change the final answer to this question.

Solution: We begin by examining a graph of $f(x)$ and a graph of $g(x)$, the 2π-periodic extension of $f(x)$.

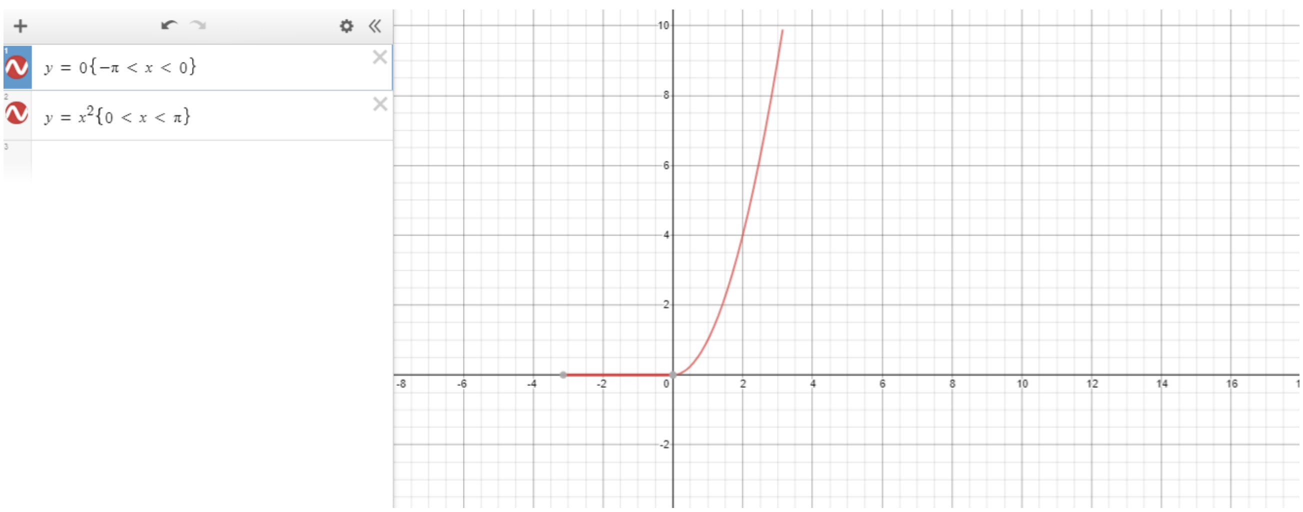


FIGURE 72. Graph of f(x).

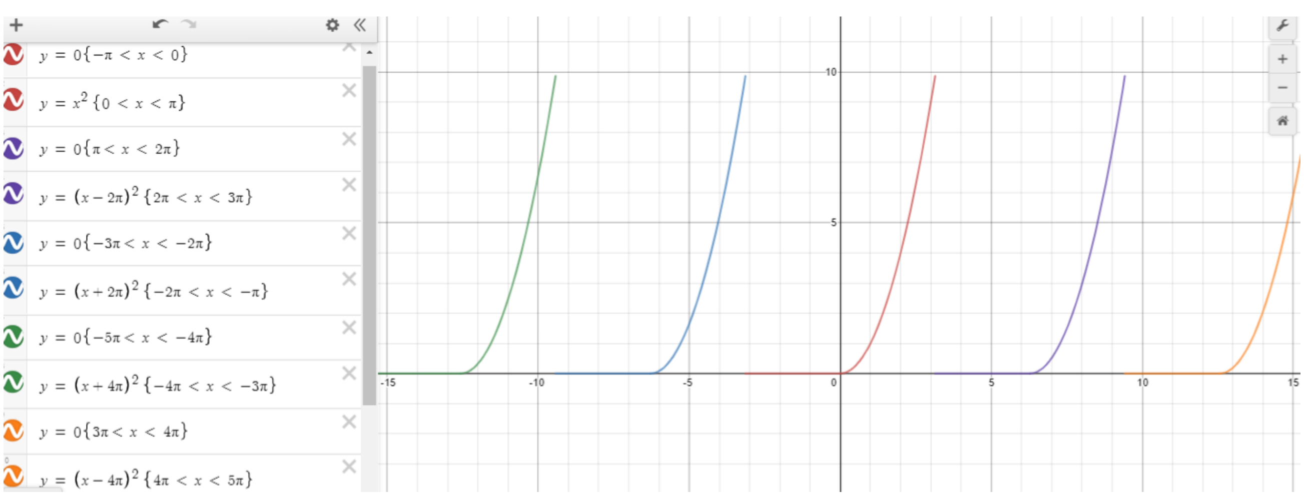


FIGURE 73. Graph of $g(x)$.

We apply Theorem 6.3.2 (stated below) in order to find the answer.

Theorem 6.3.2 (Page 503): If f and f' are piecewise continuous on $[-L, L]$, then for any $x \in (-L, L)$,

$$\underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right\}}_{\text{Fourrier series of f(x)}} = \frac{1}{2}[f(x^+) + f(x^-)]. \tag{1716}$$

For $x = \pm L$, the series converges to $\frac{1}{2}[f(-L^+) + f(L^-)]$.

Noting that $L = \pi$ in this problem, let us first determine the function that the Fourrier series of $f(x)$ converges pointwise to on $[-\pi, \pi]$. We see that on $(-\pi, 0) \cup (0, \pi)$, $f(x)$ is continuous, so the Fourrier series of $f(x)$ converges pointwise to $f(x)$ for every $x \in (-\pi, 0) \cup (0, \pi)$. Since $f(0^-) = f(0^+) = 0$, we see that the Fourrier series of $f(x)$ converges to 0 when $x = 0$. Since $f(-\pi^+) = 0$ and $f(\pi^-) = \pi^2$, we see that the Fourrier series of $f(x)$ converges to $\frac{1}{2}\pi^2$ when $x = \pm\pi$. Recalling that the Fourrier series of $f(x)$ is 2π-periodic, we first define $g(n\pi) = \frac{1}{2}\pi^2$ whenever n is an odd integer and $g(n\pi) = 0$ whenever n is an even integer (so that we may give a definition to $g(x)$ in the places that it is currently undefined), and then we see that the Fourrier series of $f(x)$ converges to $g(x)$.

Problem 12.10: Find the solution $u(x,t)$ to the heat flow problem

(1717) $$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

(1718) $$\mu(0,t) = \mu(L,t) = 0, \quad t > 0$$

(1719) $$u(x,0) = f(x), \quad 0 < x < L,$$

with $\beta = 5, L = \pi$, and the initial value function

(1720) $$f(x) = 1 - \cos(2x).$$

Solution: We know that a general solution to the heat flow problem is

(1721) $$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} c_n e^{-5n^2 t} \sin(nx).$$

From equation (1719), we see that

(1722) $$1 - \cos(2x) = u(x,0) = \sum_{n=1}^{\infty} c_n e^{-5n^2 \cdot 0} \sin(nx) = \sum_{n=1}^{\infty} c_n \sin(nx),$$

So we have to compute the fourier sine series of $1 - \cos(x)$[18]. Before doing so, we recall the following helpful trigonometric identity.

(1723) $$\sin(n+m) + \sin(n-m) = 2\sin(n)\cos(m).$$

We see that for $n \geq 1$, we have

(1724) $$c_n = \frac{2}{L} \int_0^L f(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} (1 - \cos(2x)) \sin(nx) dx$$

[18] Sometimes the function $f(x)$ is a sum of sine functions, such as $f(x) = 2\sin(3x) - \pi \sin(4x)$. In cases such as these, we are (luckily) already given the fourrier sine series of $f(x)$! We see that $c_3 = 2$, $c_4 = -\pi$, and $c_n = 0$ for all other $n \geq 1$.

$$= \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx - \frac{2}{\pi}\int_0^{\pi}\sin(nx)\cos(2x)dx \tag{1725}$$

$$\overset{\text{by (1723)}}{=} \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\Big|_{x=0}^{\pi}\right) - \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}(\sin((n+2)x)+\sin((n-2)x))dx \tag{1726}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)x)}{n+2} + \frac{-\cos((n-2)x)}{n-2}\Big|_{x=0}^{\pi}\right) \tag{1727}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)\pi)+1}{n+2} + \frac{-\cos((n-2)\pi)+1}{n-2}\right) \tag{1728}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos(n\pi)+1}{n+2} + \frac{-\cos(n\pi)+1}{n-2}\right) \tag{1729}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2}{n} - (\frac{1}{n+2} + \frac{1}{n-2})\right) \tag{1730}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2(n+2)(n-2) - n(n-2) - n(n+2)}{n(n+2)(n-2)}\right) \tag{1731}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{-4}{n^3-4n}\right) = \frac{4\cos(n\pi)-4}{L(n^3-4n)} \tag{1732}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{8}{(n^3-4n)\pi} & \text{if n is odd} \end{cases}. \tag{1733}$$

It follows that our solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{8}{((2n-1)^3 - 4(2n-1))\pi} e^{-5(2n-1)^2 t} \sin((2n-1)x). \tag{1734}$$

Problem 12.11: Formally solve the vibrating string problem

(1735) $$\frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

(1736) $$u(0,t) = u(L,t) = 0, \quad t > 0,$$

(1737) $$u(x,0) = f(x), \quad 0 \leq x \leq L,$$

(1738) $$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \leq x \leq L,$$

with $\alpha = 4$, $L = \pi$, and the initial value functions

(1739) $$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx),$$

(1740) $$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

Solution: We know that a general solution of the vibrating string problem is

(1741) $$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi\alpha}{L}t) + b_n \sin(\frac{n\pi\alpha}{L}t)\right] \sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx).$$

From equation (1737), we see that

(1742) $$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx) = f(x) = u(x,0)$$

(1743) $$= \sum_{n=1}^{\infty} [a_n \cos(4n \cdot 0) + b_n \sin(4n \cdot 0)] \sin(nx)$$

$$= \sum_{n=1}^{\infty} [a_n \cdot 1 + b_n \cdot 0] \sin(nx) = \sum_{n=1}^{\infty} a_n \sin(nx), \tag{1744}$$

so $a_n = \frac{1}{n^2}$ for every $n \geq 1$. Next, from equation (1738), we see that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = g(x) = \frac{\partial u}{\partial t}(x, 0) \tag{1745}$$

$$= \frac{\partial}{\partial t} \sum_{n=1}^{\infty} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx) \Big|_{t=0} \tag{1746}$$

$$= \sum_{n=1}^{\infty} \frac{\partial}{\partial t} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx) \Big|_{t=0} \tag{1747}$$

$$= \sum_{n=1}^{\infty} [-4na_n \sin(4nt) + 4nb_n \cos(4nt)] \sin(nx) \Big|_{t=0} \tag{1748}$$

$$= \sum_{n=1}^{\infty} [-4na_n \sin(4n \cdot 0) + 4nb_n \cos(4n \cdot 0)] \sin(nx) \tag{1749}$$

$$= \sum_{n=1}^{\infty} [-4na_n \cdot 0 + 4nb_n \cdot 1] \sin(nx) = \sum_{n=1}^{\infty} 4nb_n \sin(nx). \tag{1750}$$

The conclusion of equations (1745) − (1750) is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = \sum_{n=1}^{\infty} 4nb_n \sin(nx), \tag{1751}$$

which shows us that

$$\frac{(-1)^{n+1}}{n} = 4nb_n \to b_n = \frac{(-1)^{n+1}}{4n^2} \text{ for all } n \geq 1. \tag{1752}$$

It follows that our solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[\frac{1}{n^2}\cos(4nt) + \frac{(-1)^{n+1}}{4n^2}\sin(4nt)\right]\sin(nx)}. \tag{1753}$$