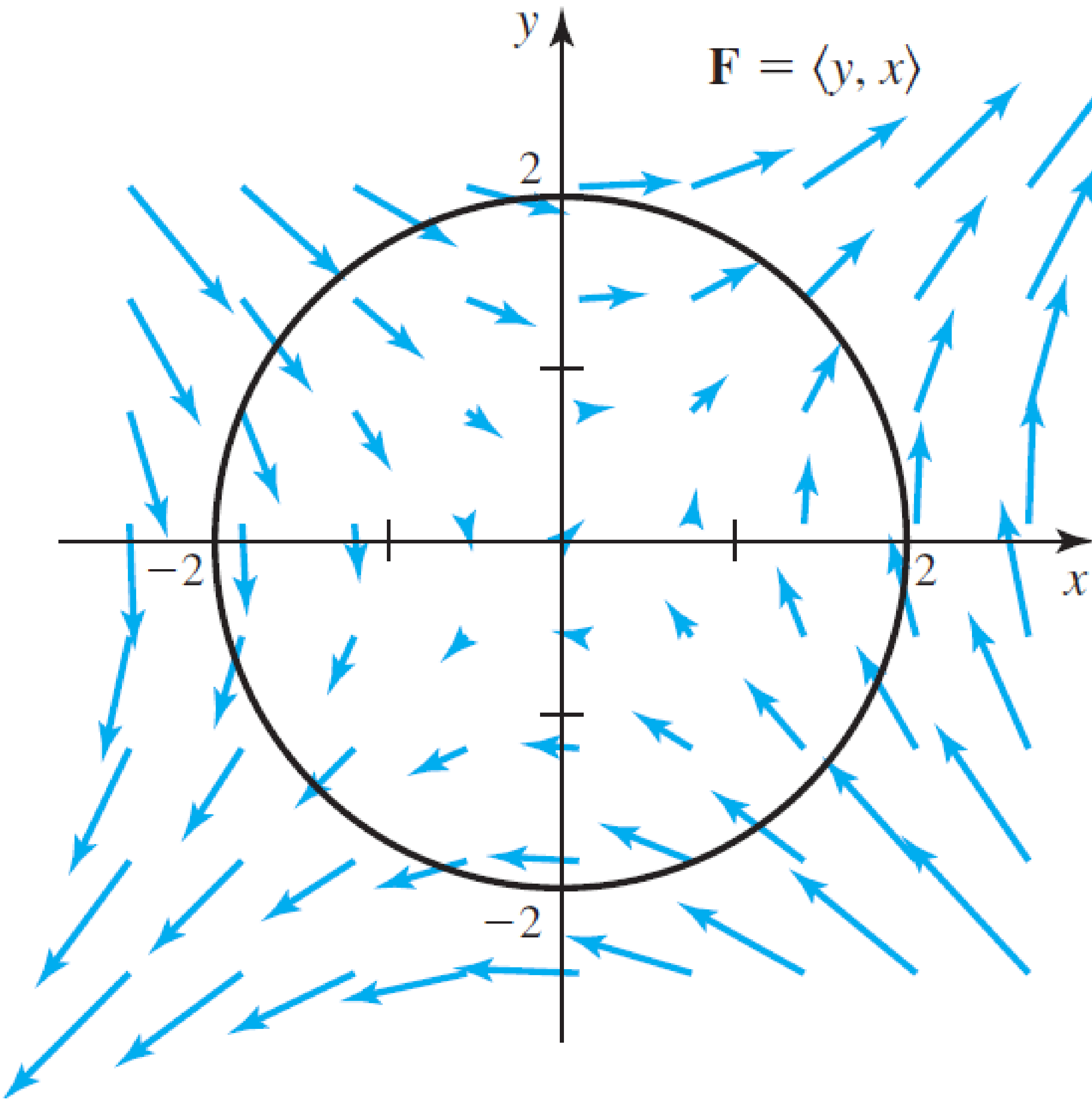
Problem 1(Repeated from last week)**:** Consider the flow field $\mathbf{F} = \langle y, x\rangle$ shown in the figure below.



(a) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$, $0 \leq t \leq \frac{\pi}{2}$.

(b) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$, $\frac{\pi}{2} \leq t \leq \pi$.

(c) Explain why the flux across the quarter circle in the third quadrant equals the flux computed in part **a**.

(d) Explain why the flux across the quarter circle in the fourth quadrant equals the flux computed in part **b**.

(e) What is the outward flux across the full circle?

Solution to (a): We begin by calculating the unit normal vector $\hat{n}(t)$ at any point on the circle (as opposed to only on the first quadrant). We see that

$$\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle \to |\mathbf{r}'(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} = 2 \tag{1}$$

$$\to \hat{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -2\sin(t), 2\cos(t)\rangle}{2} = \langle -\sin(t), \cos(t)\rangle \tag{2}$$

$$\to \hat{n}(t) = \hat{T}(t) \times \hat{k} = \langle \cos(t), -(-\sin(t))\rangle = \langle \cos(t), \sin(t)\rangle. \tag{3}$$

We now are able to calculate the desired flux as

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_0^{\frac{\pi}{2}} \mathbf{F}(2\cos(t), 2\sin(t))\cdot\langle\cos(t),\sin(t)\rangle \underbrace{2dt}_{ds} \tag{4}$$

$$= \int_0^{\frac{\pi}{2}} \langle 2\sin(t), 2\cos(t)\rangle \cdot \langle 2\cos(t), 2\sin(t)\rangle dt \tag{5}$$

$$= \int_0^{\frac{\pi}{2}} (4\sin(t)\cos(t) + 4\cos(t)\sin(t))dt \tag{6}$$

$$= \int_0^{\frac{\pi}{2}} 8\sin(t)\cos(t)dt = \int_0^{\frac{\pi}{2}} 4\sin(2t)dt = -2\cos(2t)\Big|_0^{\frac{\pi}{2}} = \boxed{4}. \tag{7}$$

Solution to (b): Since we have already found $\hat{n}(t)$ in part **a**, we proceed directly to the calculation of the flux, which is also similar to the calculation that we did in part **a**.

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(2\cos(t), 2\sin(t))\cdot\langle\cos(t),\sin(t)\rangle 2dt \tag{8}$$

$$= \int_{\frac{\pi}{2}}^{\pi} 8\sin(t)\cos(t)dt = -2\cos(2t)\Big|_{\frac{\pi}{2}}^{\pi} = \boxed{-4}. \tag{9}$$

Solution to (c): The symmetry in the given picture shows us that the flux through the circle in quadrant 1 is the same as the flux through the circle in quadrant 3. To be more detailed, we can observe that the map $(x, y) \mapsto (-x, -y)$ will send the first quadrant to the third quadrant, and the map $\theta \mapsto \theta + \pi$ (which is basically the same map) also maps the first quadrant to the third quadrant. It follows that for each $0 \le t \le \frac{\pi}{2}$ (remembering that t is essentially the angle θ in this situation) we have

$$\mathbf{F}(\mathbf{r}(t+\pi)) = \mathbf{F}(-\mathbf{r}(t)) = -\mathbf{F}(\mathbf{r}(t)), \text{ and} \tag{10}$$

$$\hat{n}(t+\pi) = -\hat{n}(t), \text{ so} \tag{11}$$

$$\text{Flux(Third Quadrant)} = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}\cdot\hat{n}ds = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}(\mathbf{r}(t))\cdot\hat{n}ds = \tag{12}$$

$$= \int_{0}^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t+\pi))\cdot\hat{n}(t+\pi)ds = \int_{0}^{\frac{\pi}{2}} (-\mathbf{F}(\mathbf{r}(t)))\cdot(-\hat{n}(t))ds \tag{13}$$

$$= \int_{0}^{\frac{\pi}{2}} \mathbf{F}\cdot\hat{n}(t)ds = \text{Flux(First Quadrant)}. \tag{14}$$

Solution to (d): Once again the symmetry in the given picture shows us that the flux through the circle in quadrant 2 is the same as the flux through the circle in quadrant 4. To be more detailed, we perform calculations similar to those of part (c) to see that

$$\text{Flux(Fourth Quadrant)} = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}\cdot\hat{n}ds = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\hat{n}ds \tag{15}$$

$$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(\mathbf{r}(t+\pi))\cdot\hat{n}(t+\pi)ds = \int_{\frac{\pi}{2}}^{\pi} (-\mathbf{F}(\mathbf{r}(t)))\cdot(-\hat{n}(t))ds \tag{16}$$

$$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}\cdot\hat{n}(t)ds = \text{Flux(Second Quadrant)}. \tag{17}$$

Solution to (e): We could calculate the total flux directly, but to make use of parts **a**–**d**, we observe that

$$\text{Total Flux} = \int_{0}^{2\pi} \mathbf{F}\cdot\hat{n}ds \tag{18}$$

$$= \underbrace{\int_{0}^{\frac{\pi}{2}} \mathbf{F}\cdot\hat{n}ds}_{\text{Q1 Flux}} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \mathbf{F}\cdot\hat{n}ds}_{\text{Q2 Flux}} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}\cdot\hat{n}ds}_{\text{Q3 Flux}} + \underbrace{\int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}\cdot\hat{n}ds}_{\text{Q4 Flux}} \tag{19}$$

$$= 4 + (-4) + 4 + (-4) = \boxed{0}. \tag{20}$$

Problem 2: An idealized two-dimensional ocean is modeled by the square region $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. with boundary $\mathcal{C}$. Consider the stream function $\Psi(x, y) = 4\cos(x)\cos(y)$ defined on R. Some of the level curves of Ψ are shown in the figure below.

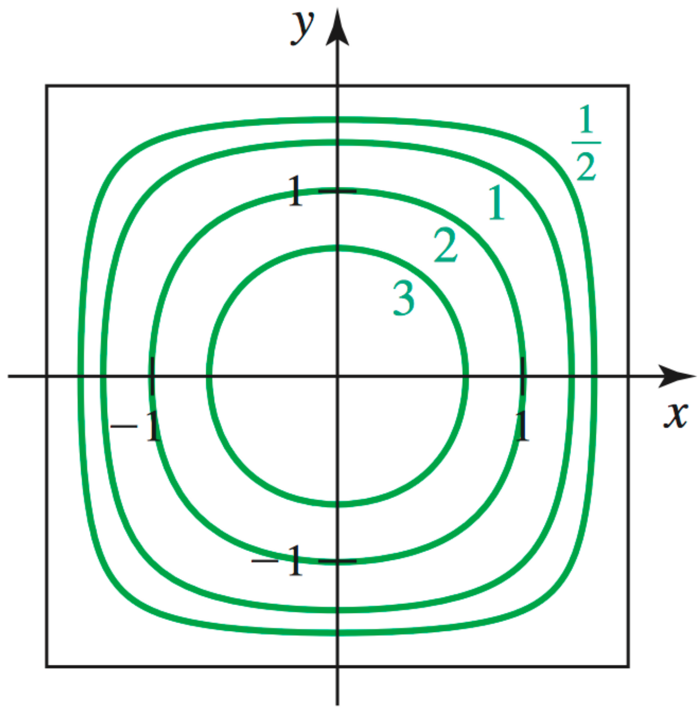


Figure 1: Some level curves of the stream function $\Psi(x, y)$.

(a) The horizontal (east-west) component of the velocity is $u = \Psi_y$ and the vertical (north-south) component of the velocity is $v = -\Psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.

(b) Is the velocity field source free? Explain.

(c) Is the velocity field irrotational? Explain.

(d) Find the total outward flux across $\mathcal{C}$.

(e) Find the circulation on $\mathcal{C}$ assuming counterclockwise orientation.

Solution to part (a): We see that

$$u(x, y) = \Psi_y(x, y) = -4\cos(x)\sin(y), \text{ and} \tag{21}$$

$$v(x, y) = -\Psi_x(x, y) = 4\sin(x)\cos(y), \tag{22}$$

so the velocity field $\vec{F} = \vec{F}(x, y)$ is given by

$$\vec{F}(x, y) = \langle u(x, y), v(x, y)\rangle = \langle -4\cos(x)\sin(y), 4\sin(x)\cos(y)\rangle. \tag{23}$$

Solution to part (b): We see that the divergence of $\vec{F}$ is given by

$$\text{Div}(\vec{F}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 4\sin(x)\sin(y) - 4\sin(x)\sin(y) = 0, \tag{24}$$

so the $\boxed{\text{velocity field } \vec{F} \text{ is source free.}}$ In fact, we can show that any vector field $\vec{F} = \langle f, g\rangle = \langle \Psi_y, -\Psi_x\rangle$ that arises from a stream function Ψ is source free. It suffices to observe that

$$\text{Div}(\vec{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = (\Psi_y)_x + (-\Psi_x)_y = \Psi_{yx} - \Psi_{xy} = 0. \tag{25}$$

This should be compared to the fact that any vector field $\vec{F} = \langle \varphi_x, \varphi_y\rangle$ coming from a potential function φ is conservative/irrotational.

Solution to part (c): We see that the curl of $\vec{F}$ is given by

$$\begin{aligned}\text{Curl}(F) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= 4\cos(x)\cos(y) - (-4\cos(x)\cos(y)) \\ &= 8\cos(x)\cos(y) \not\equiv 0,\end{aligned} \tag{26}$$

so the $\boxed{\text{velocity field } \vec{F} \text{ is not irrotational.}}$

Solution to part (d): Using the flux form of Green's Theorem we see that

$$\int_{\mathcal{C}} \vec{F}\cdot\hat{n}ds = \iint_R \text{Div}(\vec{F})dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0dxdy = \boxed{0}. \tag{27}$$

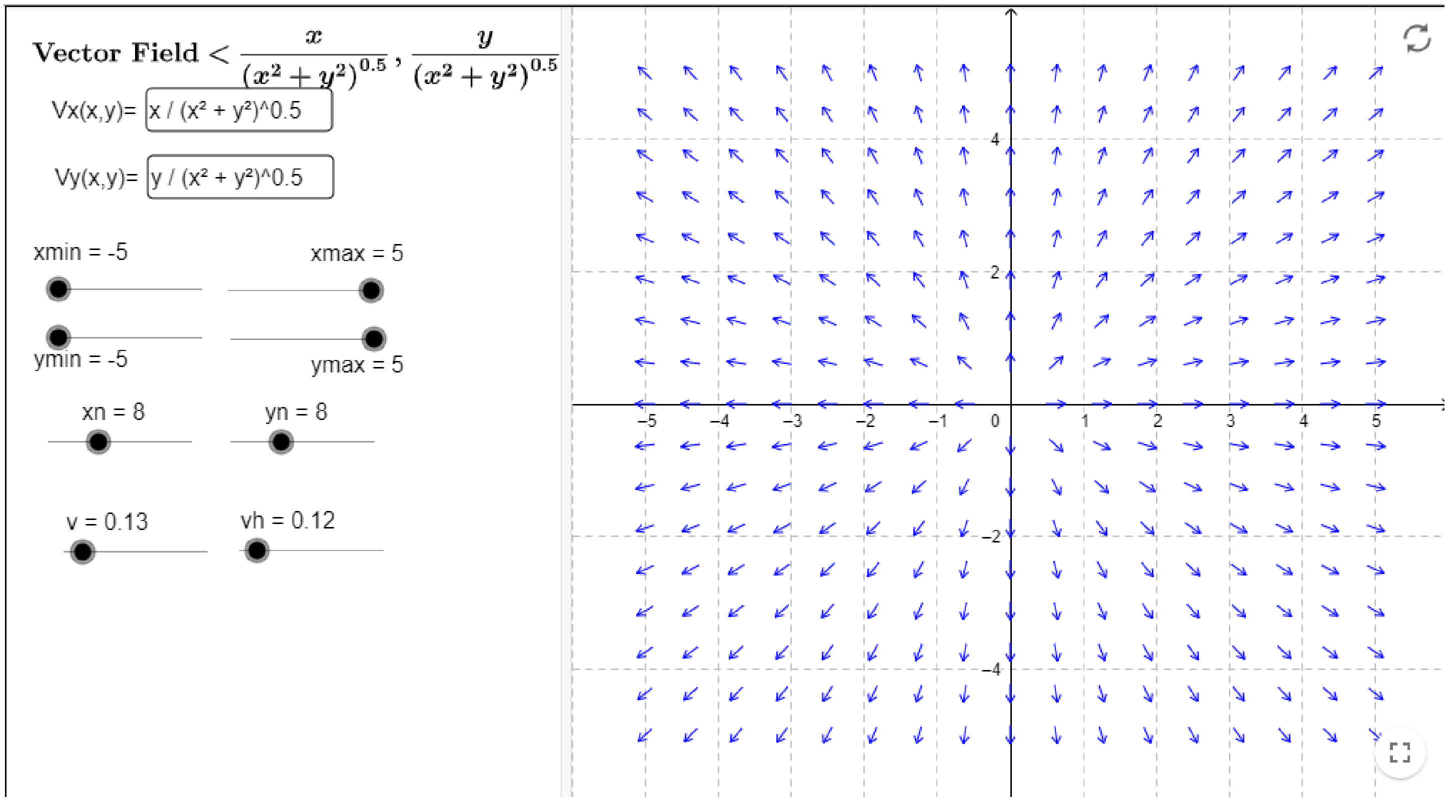
Solution to part (e): Using the circulation form of Green's Theorem we see that

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \iint_R \text{Curl}(\vec{F})dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos(x)\cos(y)dxdy \tag{28}$$

$$= 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(y)dy\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)dx\right) = 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)dx\right)^2 \tag{29}$$

$$= 8\left(\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)^2 = 8\cdot 2^2 = \boxed{32}. \tag{30}$$

Problem 3: Consider the radial field $\vec{F}(x,y) = \frac{\langle x,y\rangle}{\sqrt{x^2+y^2}} = \frac{\vec{r}}{|\vec{r}|}$ shown below.



(a) Explain why the conditions of Green's Theorem do not apply to $\vec{F}$ on a region R containing the origin.

(b) Let R be the unit disk centered at the origin and compute

$$\iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA. \tag{31}$$

(c) Evaluate the line integral in the flux form of Green's Theorem applied to the region R and the vector field $\vec{F}$.

(d) Do the results of parts (b) and (c) agree? Explain.

Solution to part (a): We see that for

$$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \text{ and } g(x,y) = \frac{y}{\sqrt{x^2+y^2}}, \tag{32}$$

we have $\vec{F} = \langle f, g\rangle$. One of the conditions of Green's Theorem (flux form and circulation form) is that f and g have continuous first partial derivatives in R. Since neither of f and g are continuous at $(0,0)$, their first partial derivatives don't even exist at $(0,0)$, so they are not continuous. It follows that the conditions of Green's Theorem are not satisfied if $(0,0) \in R$.

Solution to part (b): We see that

$$\frac{\partial f}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}+x\left(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}\cdot 2x\right)=\frac{y^2}{\sqrt{x^2+y^2}^3}, \text{ and} \tag{33}$$

$$\frac{\partial g}{\partial y}=\frac{1}{\sqrt{x^2+y^2}}+y\left(-\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}\cdot 2y\right)=\frac{x^2}{\sqrt{x^2+y^2}^3}. \tag{34}$$

It follows that

$$\iint_R (\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y})dA=\iint_R\left(\frac{y^2}{\sqrt{x^2+y^2}^3}+\frac{x^2}{\sqrt{x^2+y^2}^3}\right)dA \tag{35}$$

$$=\iint_R\frac{1}{\sqrt{x^2+y^2}}dA=\int_0^{2\pi}\int_0^1\frac{1}{r}rdrd\theta=\int_0^{2\pi}\int_0^1 drd\theta=\boxed{2\pi}. \tag{36}$$

Solution to part (c): We recall that

$$\vec{r}(t)=\langle\cos(t),\sin(t)\rangle, 0\le t\le 2\pi \tag{37}$$

is the parameterization by arclength of the unit circle. In this case we may naturally identify $\vec{r}(t)$ with the radial vector $\vec{r}$, so we will do so by abuse of notation. Furthermore, recalling that $\hat{n}$ is the *outward* unit normal vector, we see that

$$\hat{n}(t)=\langle\cos(t),\sin(t)\rangle=\vec{r}(t), 0\le t\le 2\pi. \tag{38}$$

It follows that

$$\int_{\mathcal{C}}\vec{F}\cdot\hat{n}ds=\int_{\mathcal{C}}\frac{\vec{r}}{|\vec{r}|}\cdot(\vec{r}(t))ds=\int_0^{2\pi}\frac{|\vec{r}(t)|^2}{|\vec{r}(t)|}dt=\int_0^{2\pi}\frac{1^2}{1}dt=\boxed{2\pi}. \tag{39}$$

Solution to part (d): Even though the conditions of Green's Theorem do not apply, the answers to parts (b) and (c) are the same. This shows that the conditions of Green's Theorem are sufficient conditions but not necessary conditions to attain the result of Green's Theorem.