**Problem 1:** Determine whether the vector field $\vec{F}$ given by

$$\vec{F} = \langle y - e^{x+y}, x - e^{x+y} + 1, \frac{1}{z}\rangle \tag{1}$$

is a conservative vector field. If $\vec{F}$ is conservative, find a potential function $\varphi$.

---

**Solution:** We see that

$$\begin{aligned}
&\vec{F} = \langle m, n, p\rangle, \quad \text{with} \qquad (2)\\
&m(x,y,z) = y - e^{x+y}, \quad n(x,y,z) = x - e^{x+y} + 1, \quad p(x,y,z) = \frac{1}{z}.\\
&\text{Since } \frac{\partial m}{\partial y} = 1 - e^{x+y} = \frac{\partial n}{\partial x}, \quad \frac{\partial n}{\partial z} = 0 = \frac{\partial p}{\partial y}, \quad \frac{\partial m}{\partial z} = 0 = \frac{\partial p}{\partial x},
\end{aligned}$$

we see that $\vec{F}$ is a conservative vector field. We will now find the potential function $\varphi$ for $\vec{F}$. We recall that

$$\langle m, n, p\rangle = \vec{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle. \tag{3}$$

We will now handle the 3 scalar differential equations that arise from (3) in order to find $\varphi$ (but only up to a constant).

$$\begin{aligned}
&\varphi_x(x,y,z) = m(x,y,z) = y - e^{x+y} \qquad (4)\\
\to&\varphi(x,y,z) = \int (y - e^{x+y})dx = xy - e^{x+y} + h(y,z)\\
&x - e^{x+y} + 1 = n(x,y,z) = \varphi_y(x,y,z) = x - e^{x+y} + h_y(y,z)\\
\to&h_y(y,z) = 1 \to h(y,z) = \int 1dy = y + g(z)\\
\to&\varphi(x,y,z) = xy - e^{x+y} + y + g(z)\\
&\frac{1}{z} = p(x,y,z) = \varphi_z(x,y,z) = g_z(z) = g'(z)\\
\to&g(z) = \int \frac{1}{z}dz = \ln|z| + C\\
\to&\boxed{\varphi(x,y,z) = xy - e^{x+y} + y + \ln|z| + C}.
\end{aligned}$$

**Problem 2:** Consider the vector field $\vec{F} = \langle x, -y\rangle$ and the curve $C$ which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.

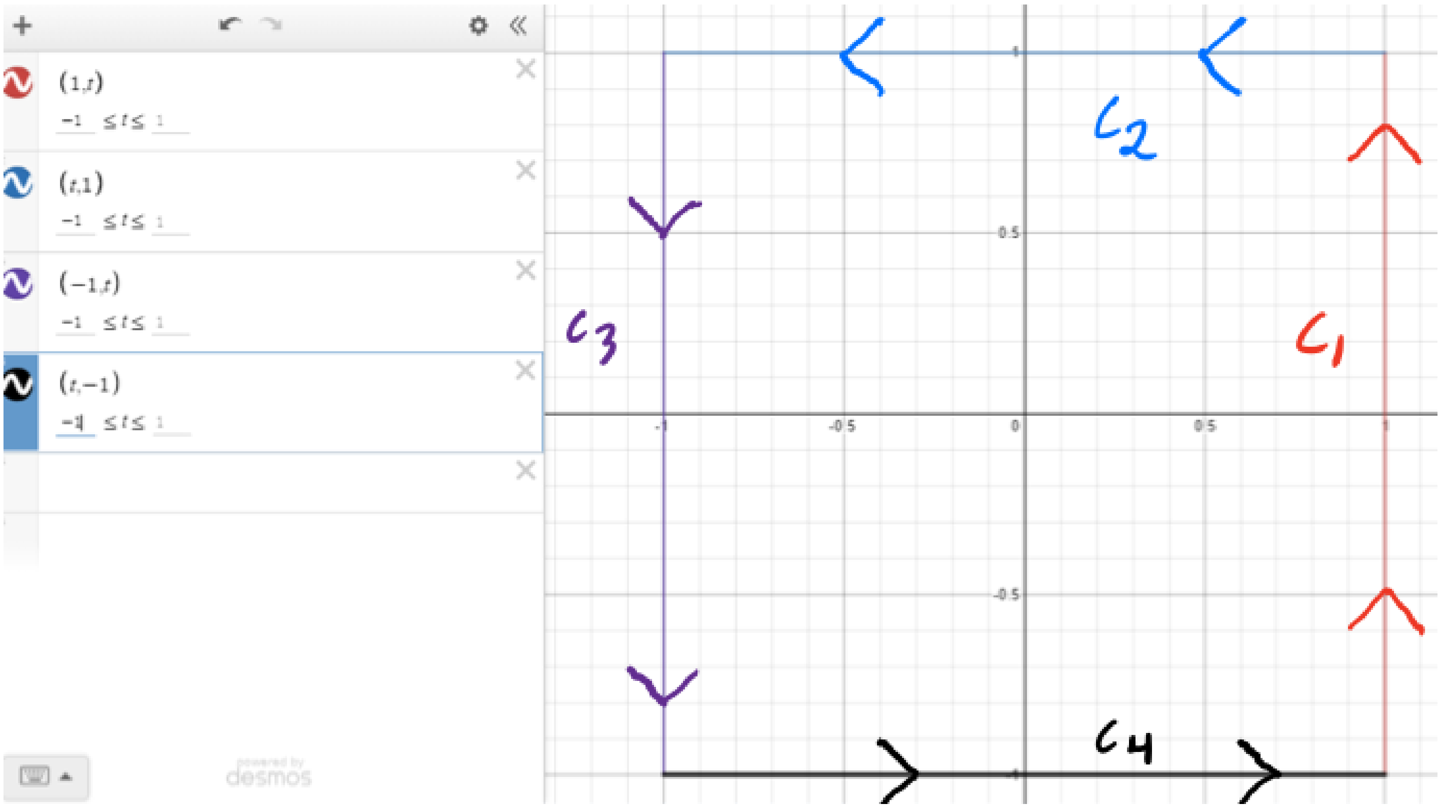


Figure 1: The curve $C$.

(a) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by finding a parametrization $\vec{r}(t)$ for the curve $C$.

(b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by using the Fundamental Theorem for Line Integrals.

---

**Solution to (a):** Letting $C_1, C_2, C_3$, and $C_4$ be as in Figure 1, we see that

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r} + \int_{C_4}\vec{F}\cdot d\vec{r}. \quad (5)$$

Since

$$\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{-1}^{1}\langle 1, -t\rangle\cdot\langle 0,1\rangle dt = \int_{-1}^{1} -tdt = -\frac{1}{2}t^2\Big|_{-1}^{1} = 0, \quad (6)$$

$$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{1}^{-1}\langle t, -1\rangle\cdot\langle 1,0\rangle dt = \int_{1}^{-1} tdt = \frac{1}{2}t^2\Big|_{1}^{-1} = 0, \quad (7)$$

$$\int_{C_3}\vec{F}\cdot d\vec{r} = \int_{1}^{-1}\langle -1, -t\rangle\cdot\langle 0,1\rangle dt = \int_{1}^{-1} -tdt = -\frac{1}{2}t^2\Big|_{1}^{-1} = 0, \quad (8)$$

$$\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle t, 1\rangle \cdot \langle 1, 0\rangle dt = \int_{-1}^{1} t dt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0, \tag{9}$$

we see that

$$\int_C \vec{F} \cdot d\vec{r} = 0 + 0 + 0 + 0 = \boxed{0}. \tag{10}$$

**Solution to (b):** Since

$$\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}(-y), \tag{11}$$

we see that $\vec{F} = \langle x, -y\rangle$ is a conservative vector field. We now have 2 ways in which to finish the problem.

**Finish 1:** Since $\vec{F}$ is a conservative vector field and $C$ is a (simple, piecewise smooth, oriented) closed curve, and $\vec{F}$ is continuous on $C$ and its interior, we see that

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}. \tag{12}$$

**Finish 2:** We now want to find a potential function $\varphi(x, y)$ for $\vec{F}$. Since

$$\langle \varphi_x, \varphi_y\rangle = \nabla\varphi = \vec{F} = \langle x, -y\rangle, \tag{13}$$

we see that

$$\varphi_x(x, y) = x \to \varphi(x, y) = \int x dx = \frac{1}{2}x^2 + g(y) \to \tag{14}$$

$$\begin{aligned} g'(y) &= \varphi_y(x, y) = -y \\ \to g(y) &= -\frac{1}{2}y^2 + C \to \varphi(x, y) = \frac{1}{2}(x^2 - y^2) + C. \end{aligned} \tag{15}$$

Now let $P$ be any point on the curve $C$. For example, we may take $P = (1, 1)$. Since the curve $C$ can be seen as starting at $P$ and ending at $P$, the Fundamental Theorem for Line Integrals tells us that

$$\int_C \vec{F} \cdot d\vec{r} = \varphi((1, 1)) - \varphi((1, 1)) = \boxed{0}. \tag{16}$$

**Remark:** We see that in Finish 2, we did not even need to determine what the function $\varphi$ was in order to conclude that the final answer is 0.

**Problem 3:** Evaluate

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 5 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r}, \tag{17}$$

where $C$ is the curve that is shown in the picture below.

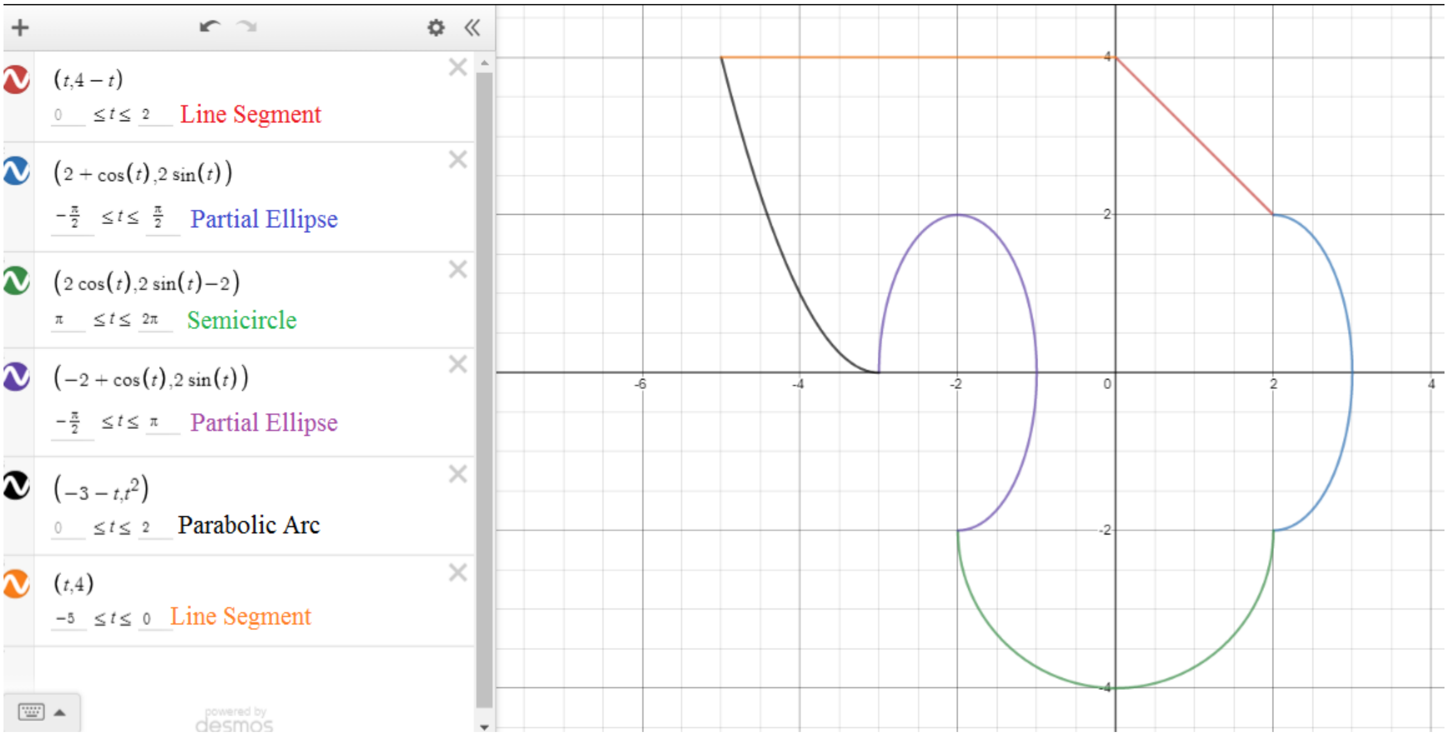


Figure 2: The curve $C$.

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**Solution:** Letting

$$m(x,y,z) = \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 5 + x))) - 1, \text{ and} \tag{18}$$

$$n(x,y,z) = y^3 + 2 + e^{y^2}, \text{ we see that} \tag{19}$$

$$\vec{F} := \langle m, n \rangle, \text{ satisfies} \tag{20}$$

$$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \tag{21}$$

so $\vec{F}$ is a conservative vector field. We also see that

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 5 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}. \tag{22}$$

Since $\vec{F}$ is conservative and $C$ is a (simple, piecewise smooth, oriented) closed curve, and $\vec{F}$ is continuous on $C$ and its interior, we see that

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}. \tag{23}$$

**Challenge for the brave:** Letting $C$ once again denote the curve in figure 2, evaluate

$$\int_C \langle y, 0 \rangle \cdot d\vec{r}. \tag{24}$$

**Problem 4:** Let $\vec{F}$ be the vector field

$$\vec{F} = \langle f(x,y), g(x,y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle.$$

It is a rotational vector field with the graph below

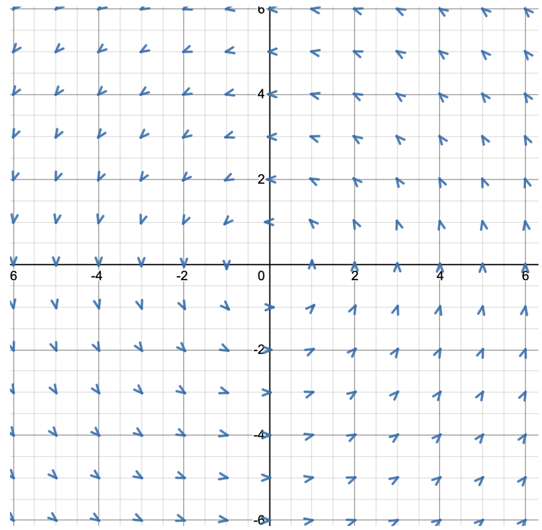


Figure 3: vector field $\vec{F}$

(a) Find the domain $R$ of $\vec{F}$.

(b) Is the domain $R$ connected? Is $R$ simply connected?

(c) Show that $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$.

(d) Let $C_a$ be the parametrized circle $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$, $0 \leq t < 2\pi$ of radius $a > 0$. Show that the integral

$$\int_{C_a} \vec{F}\cdot d\vec{r} = 2\pi.$$

(e) Is $\vec{F}$ a conservative vector field on $R$? If so, please explain. Otherwise, please explain why it doesn't contradict the result in (c).

(f) Let $R_1$ be the region $R_1 = \{1 \leq x \leq 2, 1 \leq y \leq 2\}$. Is $\vec{F}$ a conservative vector field on $R_1$? Please explain.

---

**Solution to part (a):** The domain of $\vec{F}$ consists of all points in $\mathbb{R}^2$ at which $\vec{F}$ is defined. We see that the only time that $\vec{F}$ is undefined is when $x^2+y^2=0$, as we cannot divide by 0, but $x^2+y^2=0$ is only satisfed by $(x,y)=(0,0)$, so the domain of $\vec{F}$ is $\boxed{R = \mathbb{R}^2 \setminus \{(0,0)\}}$.

**Solution to part (b):** The domain of $R$ is connected since it is actually path connected[1].

[1] Being path connected is a stronger condition than just being connected, but you probably won't study the difference between the 2 notions unless you go on to take a course in real analysis or topology.

Given any 2 points in $R$, there exists a path consisting of either 1 or 2 straight line segments that connects the 2 points.

**Solution to part (c):** We see that

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = -\frac{x}{(x^2+y^2)^2}\cdot 2x + \frac{1}{x^2+y^2} \tag{25}$$

$$= -\frac{2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \text{ and} \tag{26}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = -\frac{-y}{(x^2+y^2)^2}\cdot 2y + \frac{-1}{x^2+y^2} \tag{27}$$

$$= \frac{2y^2}{(x^2+y^2)^2} - \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial f}{\partial x}. \tag{28}$$

**Solution to part (d):** We see that

$$\int_{C_a} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)dt = \int_0^{2\pi} \vec{F}(a\cos(t), a\sin(t))\cdot\langle -a\sin(t), a\cos(t)\rangle dt \tag{29}$$

$$= \int_0^{2\pi} \left\langle \frac{-a\sin(t)}{(a\cos(t))^2+(a\sin(t))^2}, \frac{a\cos(t)}{(a\cos(t))^2+(a\sin(t))^2}\right\rangle\cdot\langle -a\sin(t), a\cos(t)\rangle dt \tag{30}$$

$$= \int_0^{2\pi} \langle -\sin(t), \cos(t)\rangle\cdot\langle -\sin(t), \cos(t)\rangle dt = \int_0^{2\pi} (\sin^2(t)+\cos^2(t))dt \tag{31}$$

$$= \int_0^{2\pi} 1dt = \boxed{2\pi}. \tag{32}$$

**Solution to part (e):** Since $C_a$ is a closed loop inside of $R$ for any radius $a > 0$, and $\int_{C_a} \vec{F}\cdot d\vec{r} = 2\pi \neq 0$, we see (Theorem 15.6 on page 1118) that $\vec{F}$ is not a conservative vector field on $R$. Our calcuations in part (c) cannot be used alongside Theorem 15.3 (on page 1113) to conclude that the vector field $\vec{F}$ is conservative, because Theorem 15.3 requires that the vector field $\vec{F}$ be defined on a simply connected region $D$.

**Solution to part (f):** Since the region $R_1$ is simply connected (it has no holes) and $\vec{F}$ is continuous on $R_1$, we may use the result of part (c) to conclude that the vector field $\vec{F}$ is conservative on the region $R_1$.

**Problem 5:** Evaluate the line integral $\int_C \nabla\phi \cdot d\vec{r}$ for $\phi(x,y) = xy$ and $C : \vec{r}(t) = \langle \cos(t), \sin(t)\rangle$, for $0 \le t \le \pi$ in two ways.

(a) Use a parametric description of $C$ and evaluate the integral directly;

(b) Use the Fundamental Theorem for line integrals.

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**Solution to (a):** We see that $\nabla\phi(x,y) = \langle y, x\rangle$, so

$$\int_C \nabla\phi \cdot d\vec{r} = \int_0^\pi \nabla\phi(\vec{r}(t)) \cdot \vec{r}\,'(t)dt \tag{33}$$

$$\int_0^\pi \nabla\phi(\cos(t), \sin(t)) \cdot \langle -\sin(t), \cos(t)\rangle dt \tag{34}$$

$$= \int_0^\pi \langle \sin(t), \cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle dt \tag{35}$$

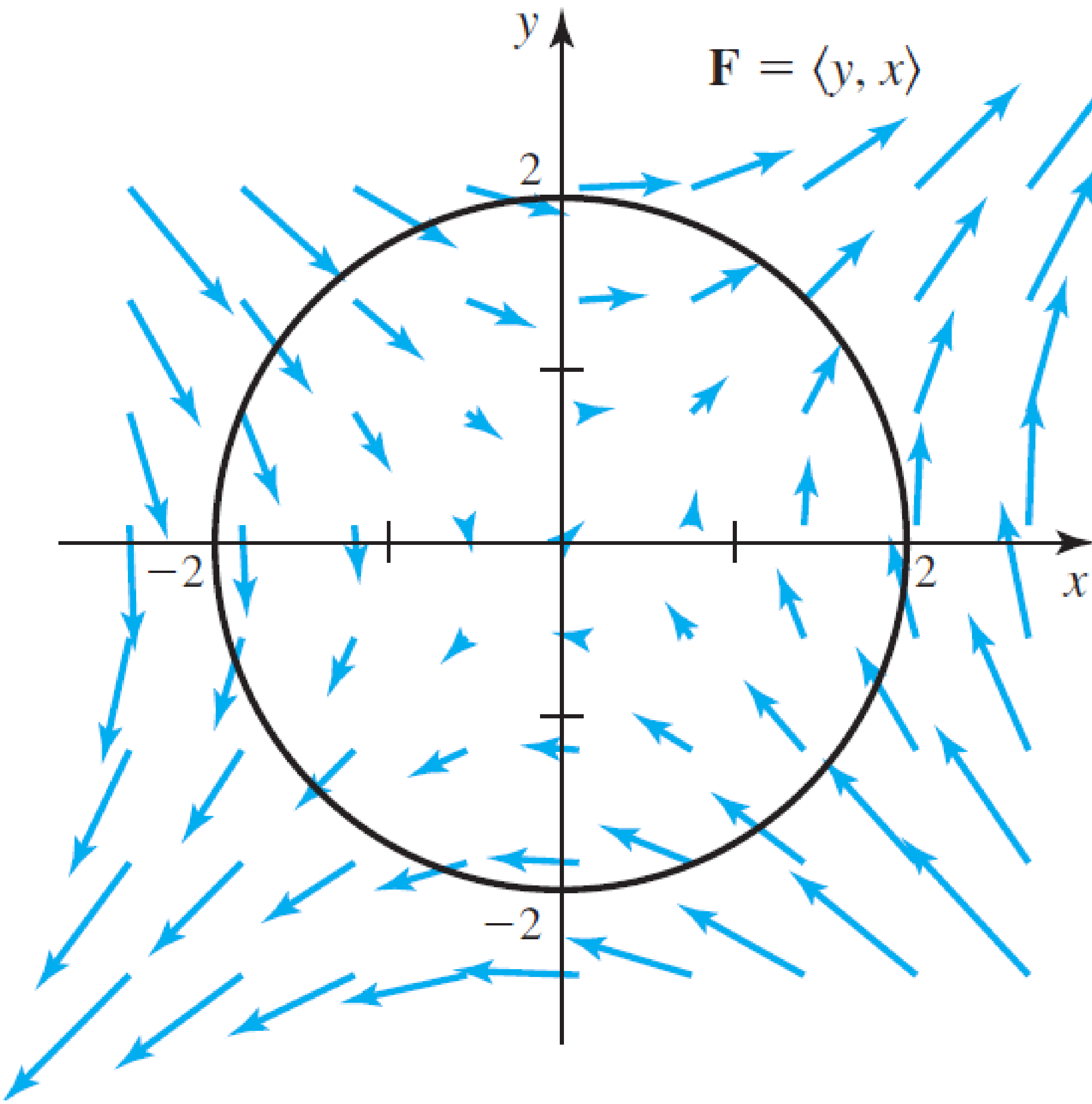
$$= \int_0^\pi (-\sin^2(t) + \cos^2(t))dt = \int_0^\pi \cos(2t)dt = \frac{1}{2}\sin(2t)\Big|_0^\pi = \boxed{0}. \tag{36}$$

**Solution to (b):** We see that

$$\int_C \nabla\phi \cdot d\vec{r} = \int_0^\pi \nabla\phi(\vec{r}(t)) \cdot \vec{r}\,'(t)dt \tag{37}$$

$$= \phi(\vec{r}(\pi)) - \phi(\vec{r}(0)) = \phi(-1,0) - \phi(1,0) = 0 - 0 = \boxed{0}. \tag{38}$$

**Problem 6:** Consider the flow field $\mathbf{F} = \langle y, x \rangle$ shown in the figure below.




(a) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $0 \leq t \leq \frac{\pi}{2}$.

(b) Compute the outward flux across the quarter circle C:$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$, $\frac{\pi}{2} \leq t \leq \pi$.

(c) Explain why the flux across the quarter circle in the third quadrant equals the flux computed in part (a).

(d) Explain why the flux across the quarter circle in the fourth quadrant equals the flux computed in part (b).

(e) What is the outward flux across the full circle?

**Solution to (a):** We begin by calculating the unit normal vector $\hat{n}(t)$ at any point on the circle (as opposed to only on the first quadrant). We see that

$$\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t)\rangle \rightarrow |\mathbf{r}'(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} = 2 \tag{39}$$

$$\rightarrow \hat{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle -2\sin(t), 2\cos(t)\rangle}{2} = \langle -\sin(t), \cos(t)\rangle \tag{40}$$

$$\rightarrow \hat{n}(t) = \hat{T}(t) \times \hat{k} = \langle \cos(t), -(-\sin(t))\rangle = \langle \cos(t), \sin(t)\rangle. \tag{41}$$

We now are able to calculate the desired flux as

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_0^{\frac{\pi}{2}} \mathbf{F}(2\cos(t), 2\sin(t))\cdot\langle\cos(t), \sin(t)\rangle \underbrace{2dt}_{ds} \tag{42}$$

$$= \int_0^{\frac{\pi}{2}} \langle 2\sin(t), 2\cos(t)\rangle \cdot \langle 2\cos(t), 2\sin(t)\rangle dt \tag{43}$$

$$= \int_0^{\frac{\pi}{2}} (4\sin(t)\cos(t) + 4\cos(t)\sin(t))dt \tag{44}$$

$$= \int_0^{\frac{\pi}{2}} 8\sin(t)\cos(t)dt = \int_0^{\frac{\pi}{2}} 4\sin(2t)dt = -2\cos(2t)\Big|_0^{\frac{\pi}{2}} = \boxed{4}. \tag{45}$$

**Solution to (b):** Since we have already found $\hat{n}(t)$ in part **a**, we proceed directly to the calculation of the flux, which is also similar to the calculation that we did in part **a**.

$$\text{Flux}(C) = \int_C \mathbf{F}\cdot\hat{n}ds = \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(2\cos(t), 2\sin(t))\cdot\langle\cos(t), \sin(t)\rangle 2dt \tag{46}$$

$$= \int_{\frac{\pi}{2}}^{\pi} 8\sin(t)\cos(t)dt = -2\cos(2t)\Big|_{\frac{\pi}{2}}^{\pi} = \boxed{-4}. \tag{47}$$

**Solution to (c):** The symmetry in the given picture shows us that the flux through the circle in quadrant 1 is the same as the flux through the circle in quadrant 3. To be more detailed, we can observe that the map $(x, y) \mapsto (-x, -y)$ will send the first quadrant to the third quadrant, and the map $\theta \mapsto \theta + \pi$ (which is basically the same map) also maps the first quadrant to the third quadrant. It follows that for each $0 \le t \le \frac{\pi}{2}$ (remembering that $t$ is essentially the angle $\theta$ in this situation) we have

$$\mathbf{F}(\mathbf{r}(t+\pi)) = \mathbf{F}(-\mathbf{r}(t)) = -\mathbf{F}(\mathbf{r}(t)), \text{ and} \tag{48}$$

$$\hat{n}(t+\pi) = -\hat{n}(t), \text{ so} \tag{49}$$

$$\text{Flux(Third Quadrant)} = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}\cdot\hat{n}ds = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}(\mathbf{r}(t))\cdot\hat{n}ds = \tag{50}$$

$$= \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{r}(t+\pi))\cdot\hat{n}(t+\pi)ds = \int_0^{\frac{\pi}{2}} (-\mathbf{F}(\mathbf{r}(t)))\cdot(-\hat{n}(t))ds \tag{51}$$

$$= \int_0^{\frac{\pi}{2}} \mathbf{F}\cdot\hat{n}(t)ds = \text{Flux(First Quadrant)}. \tag{52}$$

**Solution to (d):** Once again the symmetry in the given picture shows us that the flux through the circle in quadrant 2 is the same as the flux through the circle in quadrant 4. To be more detailed, we perform calculations similar to those of part (c) to see that

$$\text{Flux(Fourth Quadrant)} = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}\cdot\hat{n}ds = \int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\hat{n}ds \tag{53}$$

$$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}(\mathbf{r}(t+\pi))\cdot\hat{n}(t+\pi)ds = \int_{\frac{\pi}{2}}^{\pi} (-\mathbf{F}(\mathbf{r}(t)))\cdot(-\hat{n}(t))ds \tag{54}$$

$$= \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}\cdot\hat{n}(t)ds = \text{Flux(Second Quadrant)}. \tag{55}$$

**Solution to (e):** We could calculate the total flux directly, but to make use of parts **a**–**d**, we observe that

$$\text{Total Flux} = \int_0^{2\pi} \mathbf{F}\cdot\hat{n}ds \tag{56}$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} \mathbf{F}\cdot\hat{n}ds}_{\text{Q1 Flux}} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \mathbf{F}\cdot\hat{n}ds}_{\text{Q2 Flux}} + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}\cdot\hat{n}ds}_{\text{Q3 Flux}} + \underbrace{\int_{\frac{3\pi}{2}}^{2\pi} \mathbf{F}\cdot\hat{n}ds}_{\text{Q4 Flux}} \tag{57}$$

$$= 4 + (-4) + 4 + (-4) = \boxed{0}. \tag{58}$$

**Problem 7:** Compute the circulation of $\vec{F} = \langle y - x, x\rangle$ on the curve $\mathcal{C}$ which is given by $\vec{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$ for $0 \le t \le 2\pi$.

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**Solution:** We see that

$$\text{Circulation} = \int_C \vec{F} \cdot \hat{T} ds = \int_C \vec{F} \cdot \vec{r}\,'(t) dt \tag{59}$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \langle -2\sin(t), 2\cos(t)\rangle dt \tag{60}$$

$$= \int_0^{2\pi} \langle\langle \underbrace{2\sin(t)}_{y} - \underbrace{2\cos(t)}_{x}, \underbrace{2\cos(t)}_{x}\rangle \cdot \langle -2\sin(t), 2\cos(t)\rangle dt \tag{61}$$

$$= \int_0^{2\pi} \left(-4\sin^2(t) + 4\cos(t)\sin(t) + 4\cos^2(t)\right) dt \tag{62}$$

$$=^{2} \int_0^{2\pi} \left(4\cos(2t) + 2\sin(2t)\right) dt \tag{63}$$

$$= 2\sin(2t) - \cos(2t)\Big|_0^{2\pi} = \boxed{0}. \tag{64}$$

[2] $\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1 = 1 - 2\sin^2(t)$ and $\sin(2t) = 2\sin(t)\cos(t)$.