Problem 2: An idealized two-dimensional ocean is modeled by the square region $R = [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. with boundary $\mathcal{C}$. Consider the stream function $\Psi(x, y) = 4\cos(x)\cos(y)$ defined on R as shown in the figure below.

$\Psi(\frac{\pi}{4}, \frac{\pi}{4}) = 2$

$(\cos(0), \sin(0)) = (1, 0)$

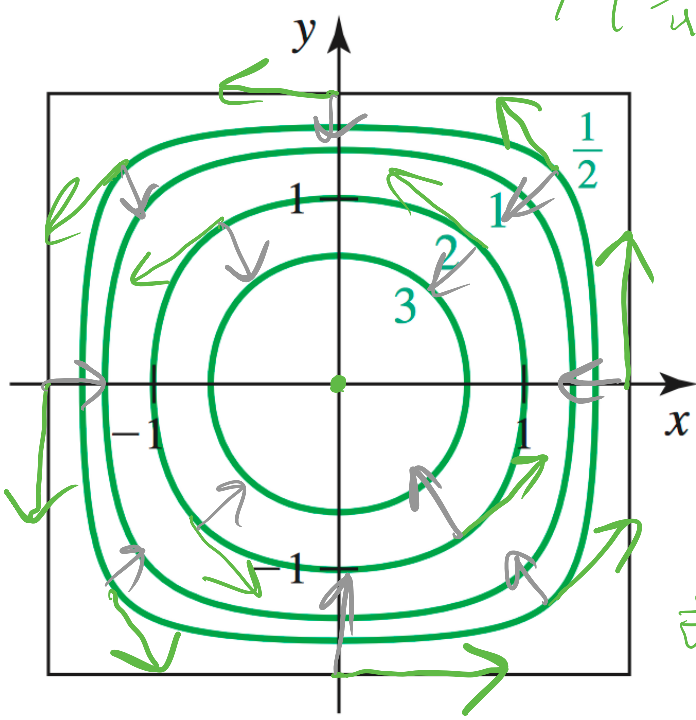

Figure 1: Some level curves of the stream function $\Psi(x, y)$.

$\vec{\nabla} \times \Psi \perp \vec{\nabla}\Psi$, so $\vec{\nabla} \times \Psi$ is tangent to level curves

(a) The horizontal (east-west) component of the velocity is $u = \Psi_y$ and the vertical (north-south) component of the velocity is $v = -\Psi_x$. Sketch a few representative velocity vectors and show that the flow is counterclockwise around the region.

(b) Is the velocity field source free? Explain.

(c) Is the velocity field irrotational? Explain.

(d) Find the total outward flux across $\mathcal{C}$.

(e) Find the circulation on $\mathcal{C}$ assuming counterclockwise orientation.

$\vec{\nabla}\Psi \cdot (\vec{\nabla} \times \Psi) = \Psi_x \cdot \Psi_y + \Psi_y(-\Psi_x) = 0$

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a) $\vec{F} = \langle u, v \rangle = \vec{\nabla} \times \Psi = \langle \Psi_y, -\Psi_x \rangle$

(Compare to $\vec{\nabla}\Psi = \langle \Psi_x, \Psi_y \rangle$)

$\Psi(x, y) = 4\cos(x)\cos(y) \rightarrow$

$\Psi_x(x, y) = -4\sin(x)\cos(y)$, $\Psi_y = -4\cos(x)\sin(y)$

$\to \vec{F} = \langle -4\cos(x)\sin(y), 4\sin(x)\cos(y) \rangle$.

(x,y)	$\vec{F}(x,y)$
$(0,0)$	$\langle 0,0 \rangle$
$(\pi/4, \pi/4)$	$\langle -2, 2 \rangle$
$(-\frac{\pi}{4}, \frac{\pi}{4})$	$\langle -2, -2 \rangle$
$(\frac{\pi}{4}, -\frac{\pi}{4})$	$\langle 2, 2 \rangle$
$(-\frac{\pi}{4}, -\frac{\pi}{4})$	$\langle 2, -2 \rangle$

$\cos(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$

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b) $\vec{F} = \langle u, v \rangle$ is source free

if and only if

$$\mathrm{Div}(\vec{F}) = \vec{\nabla}\cdot\vec{F} = u_x + v_y = 0.$$

we now see that for $\vec{F} = \langle \underbrace{\psi_y}_{u}, \underbrace{-\psi_x}_{v} \rangle$

$$\to u_x + v_y = (\psi_y)_x + (-\psi_x)_y = \psi_{yx} - \psi_{xy} \underset{\substack{\uparrow \\ \text{Clairut's Thm}}}{=} 0$$

So $\boxed{\vec{F} \text{ is source free.}}$

c) $\vec{F} = \langle u, v \rangle$ is irrotational if and only if

$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = v_x - u_y = 0.$

$$\vec{F} = \langle \underbrace{-4\cos(x)\sin(y)}_{u}, \underbrace{4\sin(x)\cos(y)}_{v} \rangle.$$

$$v_x - u_y = \left[4\cos(x)\cos(y) - (-4\cos(x)\cos(y)) \right]$$

$$= 8\cos(x)\cos(y) \neq 0.$$

(not identically 0)

$\rightarrow \boxed{\vec{F} \text{ is NOT irrotational}}.$

If $\vec{F} = \vec{\nabla}\psi = \langle \underbrace{\psi_x}_{u}, \underbrace{\psi_y}_{v} \rangle$, then

$$v_x - u_y = (\psi_y)_x - (\psi_x)_y = \psi_{yx} - \psi_{xy} \underset{\uparrow}{=} 0$$

Clairut's Thm,

So $\vec{F}$ is irrotational.

d) $\text{Flux}(C) = \int_C \vec{F} \cdot \hat{n}\, ds \underset{\uparrow}{=} \iint_R \text{div}(\vec{F})\, dA$

(Flux form of Green's Thm)

$= \iint_R 0\, dA = \boxed{0}$

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e) $\text{Circulation}(C) = \int_C \vec{F} \cdot \hat{T}\, ds \underset{\uparrow}{=} \iint_R \text{curl}(\vec{F})\, dA$

(circulation form of green's Thm)

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos(x)\cos(y)\, dx\, dy$$

$$= 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\, dx\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(y)\, dy\right)$$

$$= 8\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\, dx\right)^2$$

$$= 8\left(\sin(x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right)^2$$

$$= 8(1-(-1))^2 = 8 \cdot 2^2 = \boxed{32}$$