11:30 am → pg 6
12:40 pm → pg 10

Problem 1: Determine whether the vector field $\vec{F}$ given by

$$\vec{F} = \langle \underbrace{y - e^{x+y}}_{m}, \underbrace{x - e^{x+y} + 1}_{n}, \underbrace{\frac{1}{z}}_{p} \rangle \tag{1}$$

is a conservative vector field. If $\vec{F}$ is conservative, then find a potential function φ.

$\vec{F}$ is conservative b/c

$$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}, \quad \frac{\partial m}{\partial z} = \frac{\partial p}{\partial x}, \text{ and } \frac{\partial n}{\partial z} = \frac{\partial p}{\partial y}$$

we want $\varphi(x,y,z)$ s.t.

$\varphi + C \to \vec{\nabla}(\varphi + C) = \vec{\nabla}\varphi$

$$\langle \varphi_x, \varphi_y, \varphi_z \rangle = \vec{\nabla}\varphi = \vec{F} = \langle m, n, p \rangle$$

$e^x e^y$

$$\varphi = \int \varphi_x \, dx = \int m \, dx = \int (y - e^{x+y}) \, dx$$

$$= xy - e^{x+y} + h(y,z)$$

$$x - e^{x+y} + 1 = n = \varphi_y = \frac{\partial}{\partial y}\left(xy - e^{x+y} + h(y,z)\right)$$

$$= x - e^{x+y} + h_y(y,z)$$

$$\to \quad h_y(y,z) = 1$$

$$\to \quad h(y,z) = \int h_y(y,z)\,dy = \int 1\,dy$$

$$= y + g(z)$$

$$\to \quad \varphi = xy - e^{x+y} + y + g(z)$$

$$\frac{1}{z} = p = \varphi_z = g_z(z)$$

$$\to \quad g(z) = \int \frac{1}{z}\,dz = \ln|z| + C$$

$$\to \quad \boxed{\varphi(x,y,z) = xy - e^{x+y} + y + \ln|z| + C}$$

Problem 2: Consider the vector field $\vec{F} = \langle x, -y \rangle$ and the curve C which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.

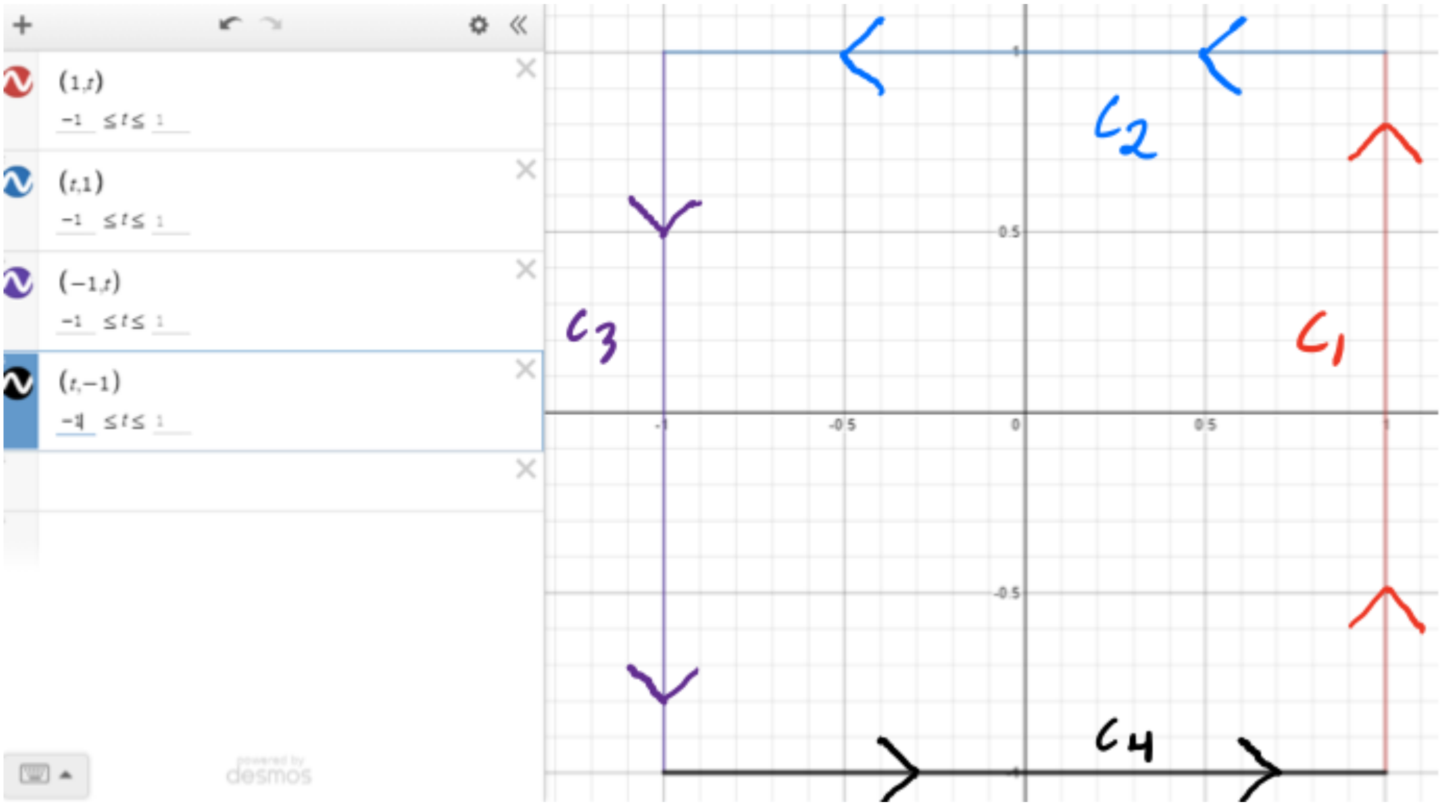


Figure 1: The curve C.

(a) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by finding a parameterization $\vec{r}(t)$ for the curve C.

(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ by using the Fundamental Theorem for Line Integrals.

B) We need to check that

$$\vec{F} = \langle m(x,y), n(x,y) \rangle = \langle x, -y \rangle$$

is conservative.

$$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \rightarrow \vec{F} \text{ is conservative.}$$

I.e. $\exists\ \varphi(x,y)$ s.t.

$\vec{F} = \vec{\nabla}\varphi$.

Since C is a simple, piecewise smooth, closed curve, the FToLI tells us that (and $\vec{F}$ is continuous on C and its interior)

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

Since C is a simple, piecewise smooth, closed curve, the FToLI tells us that

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{\nabla}\varphi\cdot d\vec{r} = \varphi(\vec{r}(B)) - \varphi(\vec{r}(A))$$

where $\vec{r}(t)$, $A \le t \le B$ is a pametrization of C.

Since C is closed $\vec{r}(A) = \vec{r}(B)$, so

$$\varphi(\vec{r}(B)) - \varphi(\vec{r}(A)) = 0.$$

Problem 1: Determine whether the vector field $\vec{F}$ given by

$$\vec{F} = \langle \underbrace{y - e^{x+y}}_{m}, \underbrace{x - e^{x+y} + 1}_{n}, \underbrace{\frac{1}{z}}_{p} \rangle \tag{1}$$

is a conservative vector field. If $\vec{F}$ is conservative, then find a potential function φ.

$\vec{F}$ is conservative b/c

$$\frac{\partial m}{\partial y} = \frac{\partial n}{\partial x}, \quad \frac{\partial m}{\partial z} = \frac{\partial p}{\partial x}, \text{ and } \frac{\partial n}{\partial z} = \frac{\partial p}{\partial y}.$$

The potential $\varphi(x,y,z)$ satisfies

$$\langle \varphi_x, \varphi_y, \varphi_z \rangle = \nabla \varphi = \vec{F} = \langle m, n, p \rangle$$

$$y - e^{x+y} = m = \varphi_x \to \varphi = \int \varphi_x \, dx = \int (y - e^{x+y}) \, dx$$

$$= xy - e^{x+y} + h(y,z)$$

$$x - e^{x+y} + 1 = n = \varphi_y = \frac{\partial}{\partial y}\left(xy - e^{x+y} + h(y,z)\right)$$

$$= x - e^{x+y} + h_y(y,z) \to 1 = h_y(y,z)$$

$$\to h(y,z) = \int h_y \, dy = \int 1 \, dy = y + g(z)$$

$$\rightarrow \varphi = xy - e^{x+y} + y + g(z)$$

$$\frac{1}{z} = p = \varphi_z = \frac{\partial}{\partial z}\left(xy - e^{x+y} + y + g(z)\right)$$

$$= g_z(z) \rightarrow g(z) = \int g_z(z)\,dz$$

$$= \int \frac{1}{z}\,dz = \ln|z| + C$$

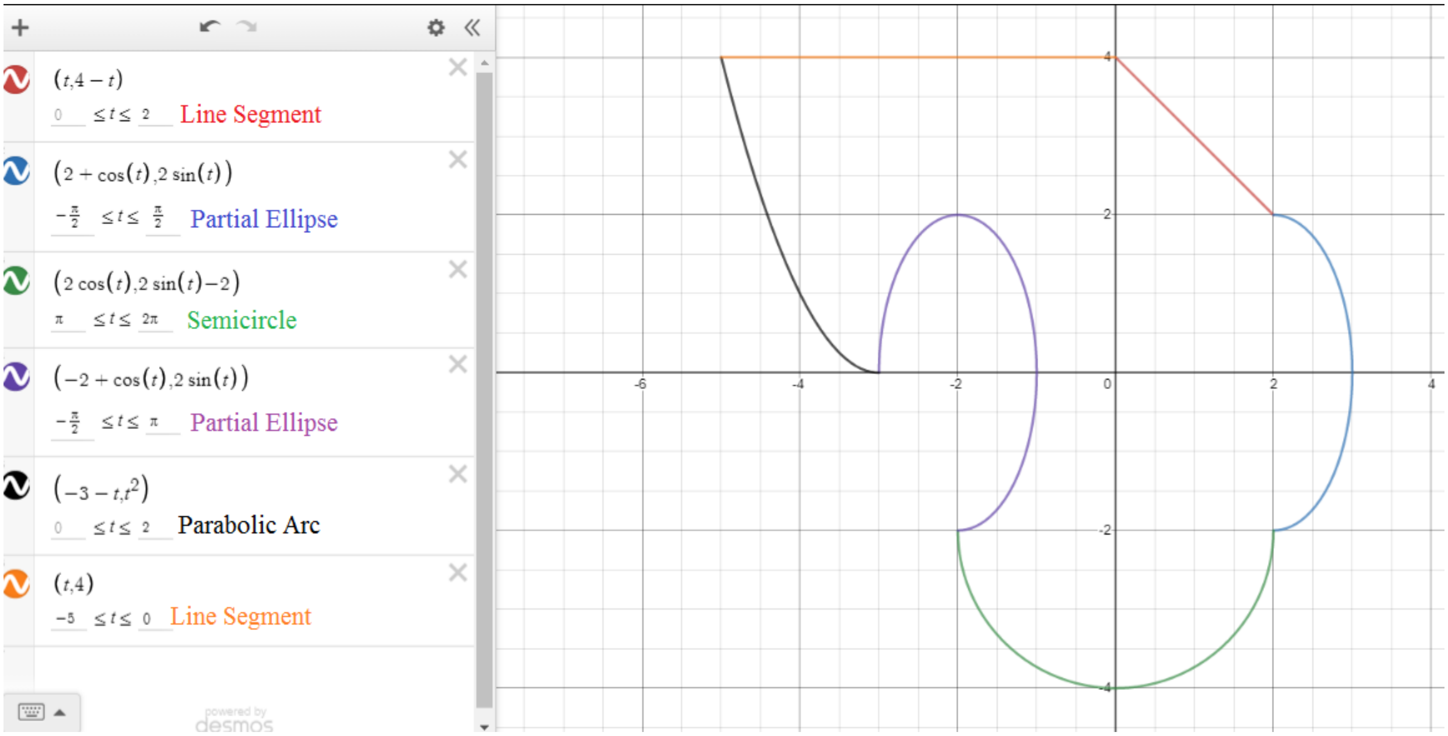
$$\rightarrow \boxed{\varphi(x,y,z) = xy - e^{x+y} + y + \ln|z| + C}$$

Problem 3: Evaluate

$\int e^{y^2} dy$ cannot be evaluated by hand.

$$\int_C \langle \overbrace{\sqrt[4]{x+6}+\ln(\ln(\ln(e^{e^e}+4+x)))-1}^{m}, \underbrace{y^3+2+e^{y^2}}_{n} \rangle \cdot d\vec{r}, \tag{2}$$

where C is the curve that is shown in the picture below.



We need to check that $\vec{F} = \langle m, n \rangle$ is conservative.

$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \rightarrow \vec{F}$ is conservative.

Because $\vec{F}$ is a conservative vector field, and C is a simple, piecewise smooth, closed curve, and $\vec{F}$ is continuous on C and inside of C, the FToLI tells us that

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

Since $\vec{F}$ is conservative, it has a potential function $\varphi(x,y)$ s.t. $\vec{\nabla}\varphi = \vec{F}$, By the FToLI

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla}\varphi \cdot d\vec{r} = \varphi(\vec{r}(B)) - \varphi(\vec{r}(A)),$$

where $\vec{r}(t)$, $A \leq t \leq B$ is a parametrization of C.

$\vec{r}(B) = \vec{r}(A)$ since C is closed,

so $\varphi(\vec{r}(B)) - \varphi(\vec{r}(A)) = 0$.

Problem 4: Let $\vec{F}$ be the vector field

$$\vec{F} = \langle f(x,y), g(x,y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle.$$

It is a rotational vector field with the graph below

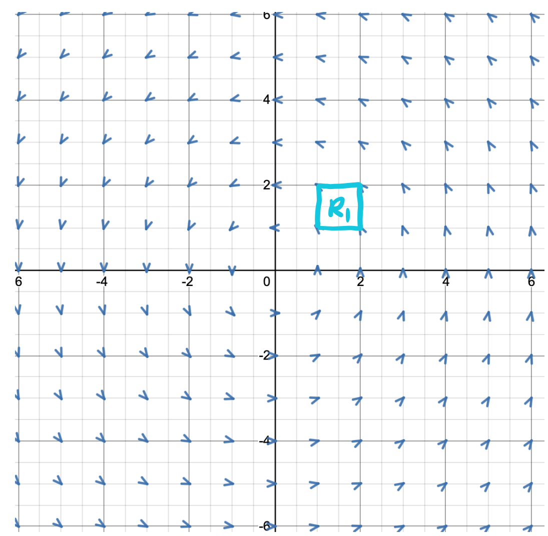


Figure 2: vector field $\vec{F}$

(a) Find the domain R of $\vec{F}$.

(b) Is the domain R connected? Is R simply connected?

(c) Show that $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$.

(d) Let C_a be the parameterized circle $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle$, $0 \le t < 2\pi$ of radius $a > 0$. Show that the integral

$$\int_{C_a} \vec{F}\cdot d\vec{r} = 2\pi.$$

(e) Is $\vec{F}$ a conservative vector field on R? If so, please explain. Otherwise, please explain why it doesn't contradict the result in (3).

(f) Let R_1 be the region $R_1 = \{1 \le x \le 2, 1 \le y \le 2\}$. Is $\vec{F}$ a conservative vector field on R_1? Please explain.

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a) $\vec{F}$ is defined as long as we don't divide by 0. $x^2+y^2=0$ if and only if $x=y=0$, so the domain of $\vec{F}$ is $\mathbb{R}^2 \setminus \{(0,0)\} = \{(x,y) \mid x\neq 0 \text{ or } y \neq 0\}$.

b) R is connected but not simply connected because "it has a hole".

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c) $\vec{F} = \langle f(x,y), g(x,y)\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle.$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial y}(-y)\cdot \frac{1}{x^2+y^2} + (-y)\frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$= -1\cdot\frac{1}{x^2+y^2} + (-y)\,\frac{-1}{(x^2+y^2)^2}\cdot\frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{-1}{x^2+y^2} + \frac{y}{(x^2+y^2)^2}\cdot 2y$$

$$= \frac{-(x^2+y^2)}{(x^2+y^2)^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial x}(x)\cdot\frac{1}{x^2+y^2} + x\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= 1\cdot\frac{1}{x^2+y^2} + x\cdot\frac{-1}{(x^2+y^2)^2}\cdot\frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} + \frac{-x}{(x^2+y^2)^2}\cdot 2x$$

$$= \frac{x^2+y^2}{(x^2+y^2)^2} + \frac{-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

d) $\vec{r}(t) = \langle a\cos(t), a\sin(t)\rangle, \ 0 \le t < 2\pi$

$\vec{r}'(t) = \langle -a\sin(t), a\cos(t)\rangle$

$$\int_{C_a} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}'(t)\,dt$$

$$= \int_0^{2\pi} \vec{F}(a\cos(t), a\sin(t)) \cdot \langle -a\sin(t), a\cos(t)\rangle dt$$

$$= \int_0^{2\pi} \left(\left\langle \frac{-a\sin(t)}{a^2\cos^2(t)+a^2\sin^2(t)}, \frac{a\cos(t)}{a^2\cos^2(t)+a^2\sin^2(t)} \right\rangle \cdot \langle -a\sin(t), a\cos(t)\rangle \right) dt$$

$$= \int_0^{2\pi} \left\langle \frac{-a\sin(t)}{a^2}, \frac{a\cos(t)}{a^2} \right\rangle \cdot \langle -a\sin(t), a\cos(t) \rangle \, dt$$

$$= \int_0^{2\pi} \left(\frac{a^2 \sin^2(t)}{a^2} + \frac{a^2 \cos^2(t)}{a^2} \right) dt$$

$$= \int_0^{2\pi} 1 \, dt = \boxed{2\pi}$$

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e) $\vec{F}$ is not conservative on R b/c we found a (many) closed curve $C \subseteq R$ for which $\int_C \vec{F} \cdot d\vec{r} \neq 0$.